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Implied Trinomial Trees of the Volatility Smile

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SUMMARY

In options markets where there is a significant or persistent volatility smile, implied tree models can ensure the consistency of exotic options prices with the market prices of liquid standard options.

Implied trees can be constructed in a variety of ways. Implied binomial trees are minimal: they have just enough parameters – node prices and transition probabilities – to fit the smile. In this paper we show how to build implied trinomial tree models of the volatility smile. Trinomial trees have inherently more parameters than binomial trees. We can use these additional parameters to conveniently choose the "state space" of all node prices in the trinomial tree, and let only the transition probabilities be constrained by market options prices. This freedom of state space provides a flexibility that is sometimes advantageous in matching trees to smiles.

A judicially chosen state space is needed to obtain a reasonable fit to the smile. We discuss a simple method for building "skewed" state spaces which fit typical index option smiles rather well.

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INTRODUCTION

Binomial trees are perhaps the most commonly used machinery for options pricing. A standard Cox-Ross-Rubinstein (CRR) binomial tree [1979] consists of a set of nodes, representing possible future stock prices, with a constant logarithmic spacing between these nodes. This spacing is a measure of the future stock price volatility, itself assumed to be constant in the CRR framework. In the continuous limit, a CRR tree with an "infinite" number of time steps to expiration represents a continuous risk-neutral evolution of the stock price with constant volatility. Option prices computed using the CRR tree will converge to the Black-Scholes continuous-time results [1973] in this limit.

The constancy of volatility in the Black-Scholes theory and its corresponding binomial framework cannot easily be reconciled with the observed structure of implied volatilities for traded options. In most index options markets, the implied Black-Scholes volatilities vary with both strike and expiration. This variation, known as the implied volatility "smile," is currently a significant and persistent feature of most global index option markets. But the constant local volatility assumption in the Black-Scholes theory and the CRR tree leads to the absence of a volatility smile, at least as long as market frictions are ignored.

Implied tree theories extend the Black-Scholes theory to make it consistent with the shape of the smile[1]. They achieve this consistency by extracting an implied evolution for the stock price in equilibrium from market prices of liquid standard options on the underlying stock. Implemented discretely, implied binomial trees are constructed so that local volatility (or spacing) varies from node to node, making the tree "flexible," so that market prices of all standard options can be matched. There is a unique implied binomial tree that fits option prices in any market[2], because binomial trees have the minimal number of parameters, just enough to match the smile.

The uniqueness of implied binomial trees is usually desirable. Sometimes, however, it becomes disadvantageous, because the uniqueness leaves little room for compromise or adjustment when the mechanism of matching tree parameters to market options prices runs into difficulties. These difficulties can arise from inconsistent and/or arbitrage-violating market options prices, which make a consistent tree

1. See Derman and Kani [1994].

2. Strictly speaking, the implied binomial tree is unique up to a user-specified central tree trunk. In the continuous limit where there are an infinite number of nodes at each time step, this choice becomes irrelevant.

impossible. Or, difficulties can arise in a more qualitative sense that, even though the constructed tree is consistent, its local volatility and probability distributions are jagged and "implausible." In these cases, one prefers to obtain trees whose local distributions are more plausible, even though they may not match every single market options price[3]. One way to make implied tree structures more flexible is to consider using implied trinomial (or multinomial) trees. These trees inherently possess more parameters than binomial trees. In the continuous limit, these parameters are superfluous and have no effect, and many different multinomial trees can all converge to the same continuous evolution process. But, for a finite number of tree periods, the parameters in any particular tree can be tuned to impose plausible smoothness constraints on the local distributions, and some trees may be more appropriate than others.

The use of trinomial trees for building implied models has been suggested by Dupire [1994]. The extra parameters in trinomial trees give us the freedom to choose the price (that is, the location in "price space") at each node in the tree. This freedom to pre-specify the "state space" can be quite advantageous if used judiciously. After an appropriate choice of the state space has been made, the transition probabilities can be iteratively calculated to ensure that all European standard options (with strikes and maturities coinciding with the tree nodes) will have theoretical values which match their market prices. We will show the equations to perform the iteration are very similar to those derived for implied binomial trees.

IMPLIED THEORIES

Implied theories assume that the stock (or index) price follows a process whose instantaneous (local) volatility $\sigma(S, t)$ varies only with spot price and time[4]. Under this assumption, since all uncertainty in the local volatility is derived from uncertainty in the stock price, we can hedge options using the stock, and so, as in the traditional Black-

3. To be honest, we point out that no one really knows all the market options prices needed to find the price and transition probabilities at every tree node. In practice, there are market quotes for a discrete set of commonly traded strikes and expirations, and market participants interpolate or extrapolate implied volatilities to other points.

4. This process is an extension of constant volatility lognormal process, and is described by the stochastic differential equation:

$$\frac{dS}{S} = r(t)dt + \sigma(S, t)dZ$$

where dZ is the standard Wiener process, $r(t)$ is the risk-free rate of interest at time t, and $\sigma(S, t)$ is the local volatility assumed to depend only on the future time t and future spot price S.

Scholes theory, valuation remains preference-free. To do this we need to know the functional form of the local volatility function. It has been shown by Derman and Kani [1994], and separately by Dupire [1994] and Rubinstein [1994], that in principle we can determine the local volatility function directly from the market prices of liquidly-traded options.

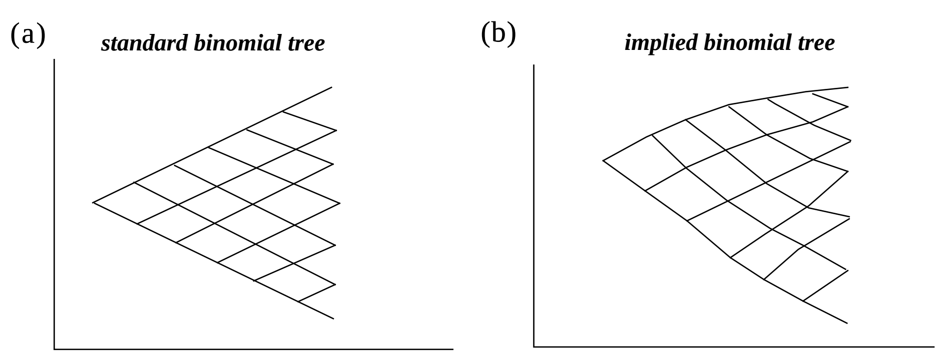
Once the local volatility function is determined, all future evolution of the stock price S is known. We can price any option using this evolution process, secure in the knowledge that our pricing model is completely consistent with the prices of all liquid options with the same underlier. Therefore, implied theories provide us with a method for moving directly from the market option prices to the underlying equilibrium price process.

Implied Binomial Trees

An implied binomial tree is a discrete version of a continuous evolution process that fits current options prices, in much the same way as the standard CRR binomial tree is a discrete version of the Black-Scholes constant volatility process[5]. Figure 1 shows schematic representation of an implied binomial tree compared to a standard CRR tree. The node spacing is constant throughout a standard CRR tree, whereas in an implied binomial tree it varies with market level and time, as specified by the local volatility function $\sigma(S, t)$.

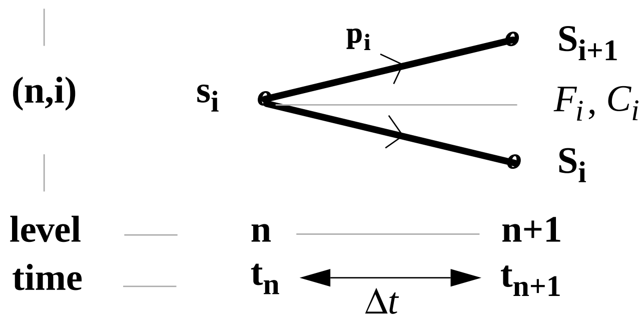
Derman and Kani [1994] show how to construct the implied binomial tree inductively. Figure 2 depicts the parameters to be determined in moving from level n to level $n+1$ starting from an already known

FIGURE 1. Schematic representation of (a) standard CRR binomial tree, (b) implied binomial tree.



5. For a review of implied binomial tree models, also see Chriss [1996a].

FIGURE 2. The implied binomial tree is constructed inductively using forward prices and interpolated option prices at each tree node.



representative stock price s_i at time t_n. All node prices and transition probabilities up to the n^{th} level of the tree at time t_n are assumed to be known at this stage. We want to determine the node prices at the $(n+1)^{th}$ level of the tree at time t_{n+1} along with transition probabilities for moving from level n to level $n+1$ of the tree. There are $2n+1$ unknown parameters; $n+1$ node values S_i and n transition probabilities p, to be determined from the known values of n forward contracts F_i whose delivery date is t_{n+1} with delivery price S_i , and n options C_i expiring at t_{n+1} with strike S_i. Derman and Kani [1994] provide the detailed algorithm that fixes the unknown parameters from the known prices[6].

There is one free parameter because the number of unknowns exceeds the number of knowns by one. This free parameter allows an arbitrary choice for the central node at each level of the tree. For example, in a CRR-style implied binomial tree, we choose the central node of an odd-numbered tree level to have the same value as today's spot price. Another alternative is to choose the price of the central node to grow at the forward interest rate[7]. In the continuous limit of a tree with infinite levels, all trees become identical and any European standard option valued on the tree has a price that matches its market price.

Aside from the choice of the central trunk, the implied binomial tree is uniquely determined from forward and option prices. As mentioned

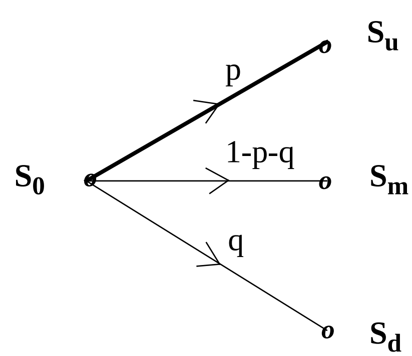
6. The extension of this result to American options is discussed in Chriss [1996b].
7. See Barle and Cakici [1995].

above, sometimes we desire more flexibility in setting up the theory discretely. The need for flexibility reflects the common-sense feeling that, to be considered plausible, the local volatilities, transition probabilities and probability distributions generated by the implied tree should vary as smoothly as possible with market level and time across the tree. This is particularly important when the available options market prices are inaccurate because they reflect bids made at an earlier market level, or are inefficient because of various market frictions that may not be included in the model. In these cases we would prefer to start by using "smooth" trees for valuing and hedging complex options. One way to introduce more flexibility is to use trinomial (or higher multinomial) tree structures for building implied tree models, as we will discuss in the remainder of this paper.

TRINOMIAL TREES

Trinomial trees provide another discrete representation of stock price movement, analogous to binomial trees[8]. Figure 3 illustrates a single time step in a trinomial tree. The stock price at the beginning of the time step is S_0. During this time step the stock price can move to one of three nodes: with probability p to the *up* node, value S_u; with probability q to the *down* node, value S_d; and with probability $1 - p - q$ to the middle node, value S_m. At the end of the time step, there are five unknown parameters: the two probabilities p and q, and the three node prices S_u, S_m and S_d.

FIGURE 3. In a single time step of a trinomial tree the stock price can move to one of three possible future values, each with its respective probability. The three transition probabilities sum to one.



8. We remind the reader that both trinomial and binomial trees approach the same continuous time theory as the number of periods in each is allowed to grow without limit. Despite their limiting similarity, one kind of tree may sometimes be more *convenient* than another.

In a *risk-neutral* trinomial tree the expected value of the stock at the end of the period must be its known forward price $F_0 = S_0 e^{(r-\delta)\Delta t}$, where δ is the dividend yield. Therefore:

$$pS_u + qS_d + (1-p-q)S_m = F_0 \quad \text{(EQ 1)}$$

If the stock price volatility during this time period is σ, then the node prices and transition probabilities satisfy:

$$p(S_u - F_0)^2 + q(S_d - F_0)^2 + (1-p-q)(S_m - F_0)^2 = F_0^2\sigma^2\Delta t + O(\Delta t) \quad \text{(EQ 2)}$$

where $O(\Delta t)$ denotes terms of higher order than Δt. Different discretizations of risk-neutral trinomial trees have different higher order terms in Equation 2.

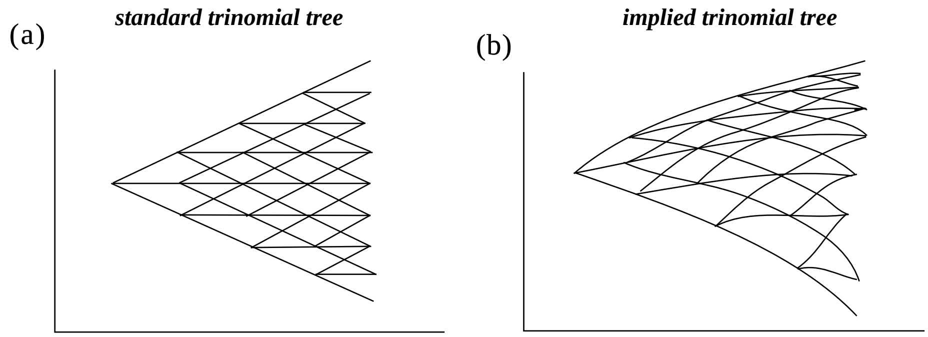
Of the five parameters needed to fix the whole tree, Equations 1 and 2 provide only two constraints, and so we have three more parameters than are necessary to satisfy them. By contrast, for implied binomial trees, all unknown parameters were determined by the constraints. As a result, we can construct many “economically equivalent” trinomial trees which, in the limit as the time spacing Δt becomes very small, represent the same continuous theory. Appendix A discusses a few different ways for building constant volatility trinomial trees. When volatilities are not constant, a common method is to choose the stock prices at every node and attempt to satisfy the two constraints through the choice of the transition probabilities. This method of initially choosing the *state space* of prices for the trinomial tree, and then solving for the transition probabilities, is familiar in most applications of the finite-difference method. We must make a judicious choice of the state space in order to insure that the transition probabilities remain between *0* and *1*, a necessary condition for the discrete world represented by the tree to preclude arbitrage.

Implied Trinomial Trees

Figure 4 gives schematic representations for both standard and implied trinomial trees.

Standard trinomial trees represent a constant volatility world and are constructed out of a regular mesh. The implied trinomial tree has an irregular mesh conforming to the variation of local volatility with level and time across the tree. To fix the nodes and probabilities in an implied trinomial tree we need the forward prices and option prices corresponding to strikes and expiration at all tree nodes. In contrast to the construction of an implied binomial tree, here we have total freedom over the choice of the state space of an implied trinomial tree. In choosing a state space, we eliminate three of the five

FIGURE 4. Schematic representations of (a) standard trinomial tree, (b) implied trinomial tree



unknown parameters corresponding to the evolution of each node (see Figure 3), leaving only the transition probabilities to solve for. We will construct our implied trinomial tree model in two stages. First, we judiciously choose a state space, i.e specify the position of every tree node. Next, knowing the location of every node, we use market forward and option prices to calculate the transition probabilities between the nodes.

CONSTRUCTING THE IMPLIED TRINOMIAL TREE

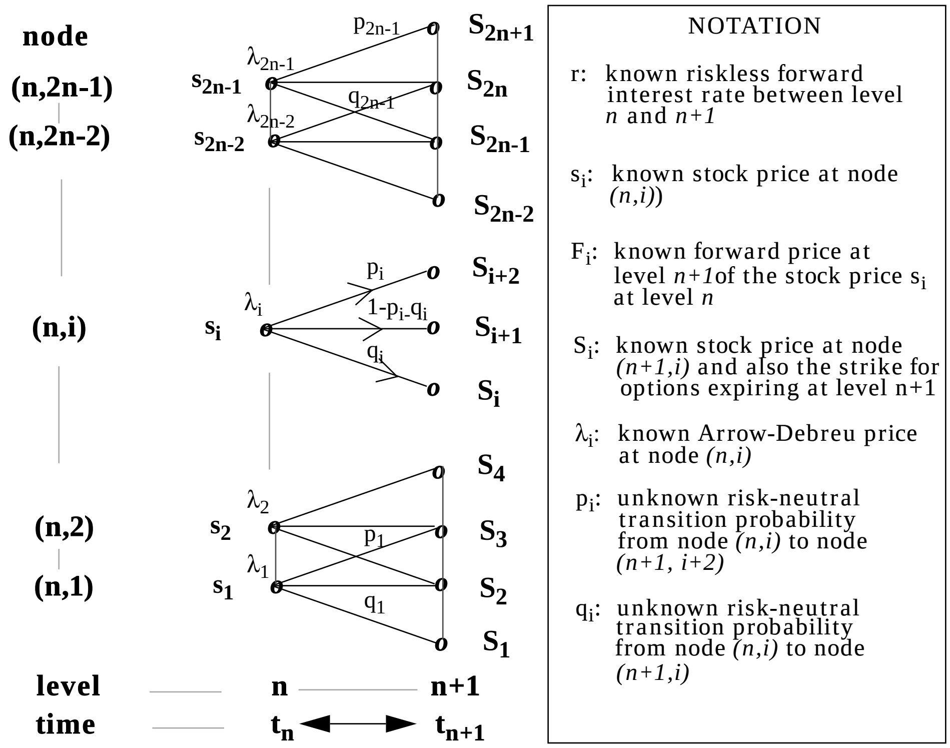
Suppose that we have already fixed the state space of the implied trinomial tree. Figure 5 shows the n^{th} and $(n+1)^{th}$ levels of the tree. We will use induction to infer the transition probabilities p_i and q_i for all tree nodes (n,i) at each tree level n. Our notation and treatment follows the Derman and Kani [1994] binomial tree construction.

Since the implied trinomial tree is risk-neutral, the expected value of the stock at the node (n,i) at the later time t_{n+1}, must be the known forward price of that node. This gives one relationship between the unknown transition probabilities and known stock and forward prices:

$$p_i S_{i+2} + (1 - p_i - q_i) S_{i+1} + q_i S_i = F_i \qquad \textbf{(EQ 3)}$$

Let $C(S_{i+1},t_{n+1})$ and $P(S_{i+1},t_{n+1})$ respectively denote today's market price for a standard call and put option struck at S_{i+1} and expiring at time t_{n+1}. We obtain the values of these options by interpolating the smile surface at various strike and time points corresponding to the implied tree nodes. The trinomial tree value of a call option struck at K and expiring at t_{n+1} is the sum over all nodes $(n+1, j)$ of the dis-

FIGURE 5. Computing the transition probabilities from nth to (n+1)th level of an implied trinomial tree, assuming that the position of all the nodes is already fixed.



counted probability of reaching that node multiplied by the call pay-off there. Hence

$$C(K, t_{n+1}) = e^{-r\Delta t} \sum_j \{\lambda_{j-2} p_{j-2} + \lambda_{j-1}(1 - p_{j-1} - q_{j-1}) + \lambda_j q_j\} max(S_j - K, 0) \qquad \textbf{(EQ 4)}$$

If we set the strike K to the value S_{i+1}, the stock price at the node $(n+1,i+1)$, then we can rearrange the terms and use Equation 3 to write the call price in terms of known Arrow-Debreu prices, known stock prices, known forward prices, and a contribution from up-transition probability p_i to the first in-the-money node:

$$e^{r\Delta t} C(S_{i+1}, t_{n+1}) = \lambda_i p_i (S_{i+2} - S_{i+1}) + \sum_{j=i+1}^{2n} \lambda_j (F_j - S_{i+1}) \qquad \textbf{(EQ 5)}$$

The only unknown in Equation 5 is the transition probability p_i, since the stock prices are already fixed by the choice of state space, and the option price $C(S_{i+1}, t_{n+1})$ and the forward prices F_i are known from the smile. We can solve this equation for p_i:

$$p_i = \frac{e^{r\Delta t}C(S_{i+1}, t_{n+1}) - \sum_{j=i+1}^{2n} \lambda_j (F_j - S_{i+1})}{\lambda_i (S_{i+2} - S_{i+1})} \quad \text{(EQ 6)}$$

Using Equation 3 we can solve for the down transition probability q_i:

$$q_i = \frac{F_i - p_i(S_{i+2} - S_{i+1}) - S_{i+1}}{S_i - S_{i+1}} \quad \text{(EQ 7)}$$

We use put option prices $P(S_{i+1},t_{n+1})$ to determine the transition probabilities from all the nodes below (and including) the center node $(n+1,n)$ at time t_n. This leads to the equation

$$q_i = \frac{e^{r\Delta t}P(S_{i+1}, t_{n+1}) - \sum_{j=0}^{i-1} \lambda_j (S_{i+1} - F_j)}{\lambda_i (S_{i+1} - S_i)} \quad \text{(EQ 8)}$$

for q_i and, using Equation 3, the following equation for p_i:

$$p_i = \frac{F_i + q_i(S_{i+1} - S_i) - S_{i+1}}{S_{i+2} - S_{i+1}} \quad \text{(EQ 9)}$$

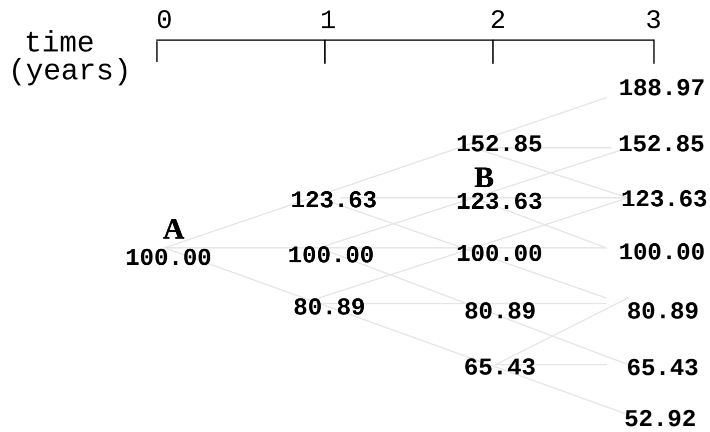
We can now use Equation 2 to find the local volatility σ at each node.

A DETAILED EXAMPLE

We illustrate the construction of an implied trinomial tree. We assume that the current index level is 100, the dividend yield is 5% per annum and the annually compounded riskless interest rate is 10% for all maturities. We also assume that implied volatility of an at-the-money European call is 15%, for all expirations, and that implied volatility increases (decreases) 0.5 percentage points with every 10 point drop (rise) in the strike price. To keep our example simple, we choose the state space of our implied trinomial model to coincide with nodes of a 3-year, 3-period, 15% (constant) volatility CRR-type, trinomial tree, as shown in Figure 6.

The method used to construct this state space is described in Figure 14(a) in Appendix A. The first node, at time $t_0 = 0$, is labeled A in Figure 6 and it has a price $S_A = 100$, equal to today's spot price. All the central nodes $(i+1,i)$ in this tree also have the same price as this node. The next three nodes, at time $t_1 = 1$, have prices $S_1 = 80.89$,

FIGURE 6. State space of a trinomial tree with constant volatility of 15%. The method described in diagram (a) of Figure 14 (in Appendix A) is used to construct this state space.



$S_2 = 100$ and $S_3 = 123.63$, respectively. These values are found using the equation $S_{1,3} = S\exp(\mp\sigma\sqrt{2\Delta t})$, as discussed in Appendix A.

We can determine the up and down transition probabilities p_A and q_A, corresponding to node A, using Equations 8 and 9 with $e^{r\Delta t} = 1.1$ and $\lambda_A = 1.0$. Then

$$q_A = \frac{1.1 \times P(S_2, t_1) - \Sigma}{1.0 \times (100 - 80.89)}$$

$P(S_2,t_1)$ is the calculated value of a put option, struck at $S_2 = 100$ and expiring one year from now. From the smile, the implied volatility of this option is 15%. We calculate its price using a constant volatility discrete trinomial tree with the same state space, and find it to be *\$3.091*. Also the summation term Σ in the numerator is zero in this case because there are no nodes with price higher than S_3 at time t_1. Combining these we find $q_A = 0.178$.

The one-period forward price corresponding to node A is $F_A = Se^{(r-\delta)\Delta t} = 104.50$. Equation 9 then gives the value of p_A:

$$p_A = \frac{104.5 + 0.178 \times (100 - 80.89) - 100.00}{123.63 - 100.00} = 0.334$$

Since probabilities add to one, the middle transition probability is equal to $1 - p_A - q_A = 0.488$.

The Arrow-Debreu prices corresponding to each of the three nodes at time t_1 are defined to be the (total) discounted probabilities that the stock price reaches at that node. For these nodes the Arrow-Debreu prices turn out to be just the transition probabilities divided by $e^{r\Delta t}$. The implied local volatility at node A is calculated using Equation 2:

$$\sigma_A = \sqrt{\frac{0.334 \times (123.63 - 104.5)^2 + 0.488(100 - 104.5)^2 + 0.178 \times (80.89 - 104.5)^2}{104.5^2 \times 1}}$$

$$= 14.6\ \%$$

The difference between the 14.6% implied local volatility and the 15% implied volatility assumed for this option arises from the higher order terms in Equation 2, and will vanish as the time spacing approaches zero.

As another example let us look at node **B** in year *2* of Figure 6. The stock price at this node is $S_B = 123.63$, whose forward price one period later is $F_B = 129.19$. From this node, the stock can move to one of three future nodes at time $t_3 = 3$, with prices $S_4 = 100$, $S_5 = 123.63$ and $S_6 = 152.85$. We can apply Equations 5 and 6 to find the up and down transition probabilities from this node. Then using $\lambda_B = 0.292$ we find

$$p_B = \frac{1.1 \times C(S_5, t_3) - \Sigma}{0.292 \times (152.85 - 123.63)}$$

The value of a call option, struck at *123.63* and expiring at year *3* is $C(S_5,t_3) = \$4.947$, corresponding to the implied volatility of *13.81%* interpolated from the smile. There is a single node above node B whose forward price $F_5 = 159.73$ contributes to the summation term Σ, giving $\Sigma = 0.0825\ x\ (159.73\text{-}123.63) = 2.978$. Putting this back into the above equation we find $p_B = 0.289$. The down transition probability q_B is then calculated as

$$q_B = \frac{129.19 - 0.289 \times (152.86 - 123.63) - 123.63}{100 - 123.63} = 0.122$$

Also from Equation 2 we find the implied local volatility at this node is $\sigma_B = 13.6\%$.

What Can Go Wrong?

The transition probabilities in Equations 6-9 for any node must lie between *0* and *1*, otherwise the implied tree allows riskless arbitrages which are inconsistent with rational options prices. For implied trinomial trees, there are two possible causes for negative probability at a node. First, the forward price F_i of a node *(n,i)* may fall outside the range S_i to S_{i+2} as illustrated in Figure 8. In that case the forward condition of Equation 3 cannot be satisfied with all transition probabilities lying between *0* and *1*.

FIGURE 7. The up- and down- transition probability trees, Arrow-Debreu tree and local volatility tree

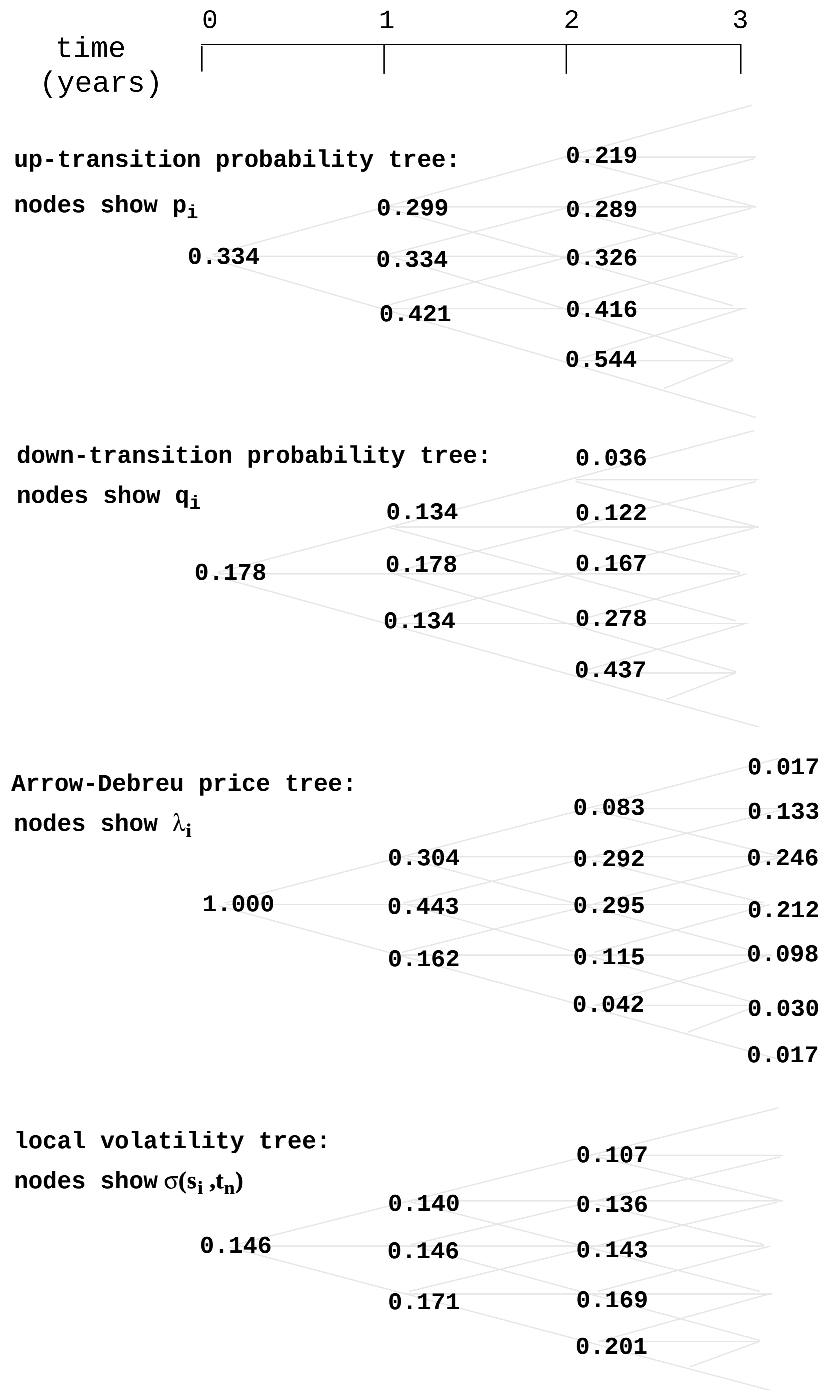
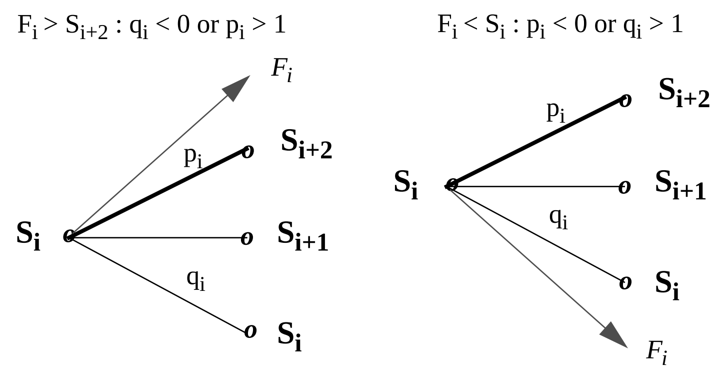


FIGURE 8. There will be negative transition probabilities for any node whose forward price does not lie between S_i and S_{i+2}.

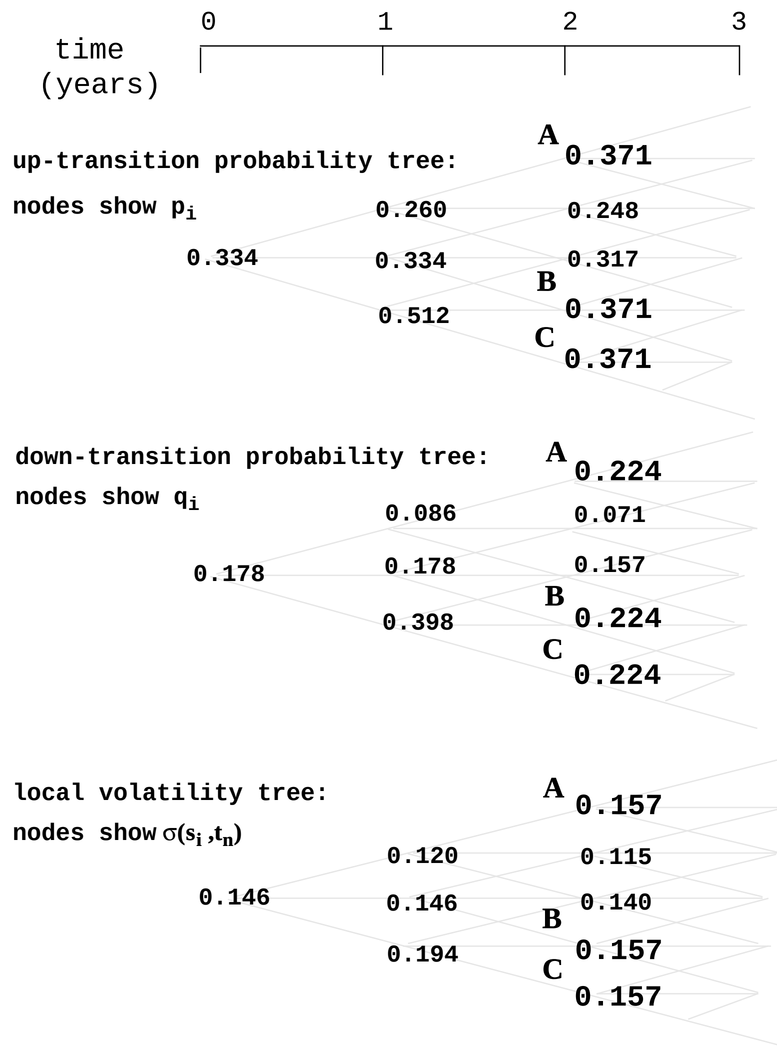


However, it is usually not difficult to avoid these types of situations by making an appropriate choice for the state space of the trinomial tree. *We must make sure that our choice of state space does not allow any violations of the forward price condition, as shown in Figure 8.*

The second reason for having negative probabilities relates to the magnitude of local volatility at an implied tree node. For instance, an especially large (small) value for the call option $C(S_{i+1},t_{n+1})$ in Equation 6 would imply a large (small) value of local volatility at s_i. Having fixed the position of all the nodes, it may not be possible to obtain such extreme values of local volatility with probabilities between 0 and 1. In these cases we must overwrite the option price which produces the unacceptable probabilities, and replace it with another option price of our choice. In doing so we must maintain the forward condition at every node of the tree. This is always possible when we are working with a state space in which the forward price violations of Figure 8 do not occur, as our next example illustrates.

For our second example, we assume that the implied volatility of an at-the-money European call is 15% and that implied volatility increases (decreases) 1 percentage point with every 10 point drop (rise) in the strike price. This skew is twice as steep as in our previous example. Using the same state space (i.e the 15% constant-volatility CRR-type trinomial tree) as was used before in Figure 6, we find negative transition probabilities at nodes A, B and C of Figure 9.

FIGURE 9. **For the skew of 1 percentage point for every 10 strike points, the 15% constant-volatility trinomial tree has negative probabilities at nodes A, B and C. These probabilities, shown in larger type, have been overwritten while maintaining the forward price condition.**



There are an infinite number of ways to overwrite negative probabilities with numbers between *0* and *1* which satisfy the forward condition. For example, since we work with state spaces where the forward price condition $S_i < F_i < S_{i+2}$ holds at every tree node, we can always choose the value of middle transition probability to be zero (essentially reducing the node to a binomial node) and set the up and down transition probabilities to $p_i = (F_i - S_i)/(S_{i+2} - S_i)$ and $q_i = 1 - p_i$ respectively.

In Figure 9 we have used an alternative method of overwriting where, if $S_{i+1} < F_i < S_{i+2}$, we set

$$p_i = \frac{1}{2}\left[\frac{F_i - S_{i+1}}{S_{i+2} - S_{i+1}} + \frac{F_i - S_i}{S_{i+2} - S_i}\right] \quad \text{and} \quad q_i = \frac{1}{2}\left[\frac{S_{i+2} - F_i}{S_{i+2} - S_i}\right]$$

and, if $S_i < F_i < S_{i+1}$, we set

$$p_i = \frac{1}{2}\left(\frac{F_i - S_i}{S_{i+2} - S_i}\right) \text{ and } q_i = \frac{1}{2}\left[\frac{S_{i+2} - F_i}{S_{i+2} - S_i} + \frac{S_{i+1} - F_i}{S_{i+1} - S_i}\right]$$

In either case the middle probability is equal to $1 - p_i - q_i$.

Constructing the State Space for the Implied Trinomial Tree

Regular state spaces with uniform mesh sizes are usually adequate for constructing implied trinomial tree models when implied volatility varies slowly with strike and expiration. But if volatility varies significantly with strike or time to expiration, it may be necessary to choose a state space whose mesh size (or node spacing) changes significantly with time and stock level.

Figure 10 shows a more appropriate choice of state space in which the negative probabilities in the above example do not occur and there is no need for overwrites. This state space is “skewed” with spacing between the nodes at the same time point decreasing with stock level. This helps the state space fit the market’s negative volatility skew better. The results are shown in Figure 11.

The implied trinomial tree model constructed using this skewed state space has no negative probabilities, fits the option market prices accurately, and generates reasonably smooth values for local volatility at different stock and time points.

One way to construct trinomial state spaces with proper skew and term structure is to build it in the following two stages:

- First, assume all interest rates (and dividends) are zero and build a regular trinomial lattice with constant time spacing Δt and logarithmic level spacing ΔS. Any constant volatility trinomial tree corresponding to a typical market implied volatility (see Appendix A) is an example of this type of lattice. Then modify Δt at different times and, subsequently, ΔS at different stock levels until the lattice captures the basic term-and skew- structures of local volatility in the market.

FIGURE 10. A skewed choice for the state space of the implied trinomial tree model for the second example.

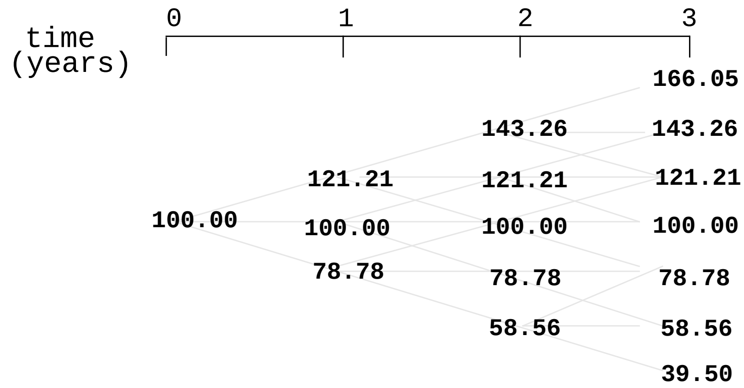
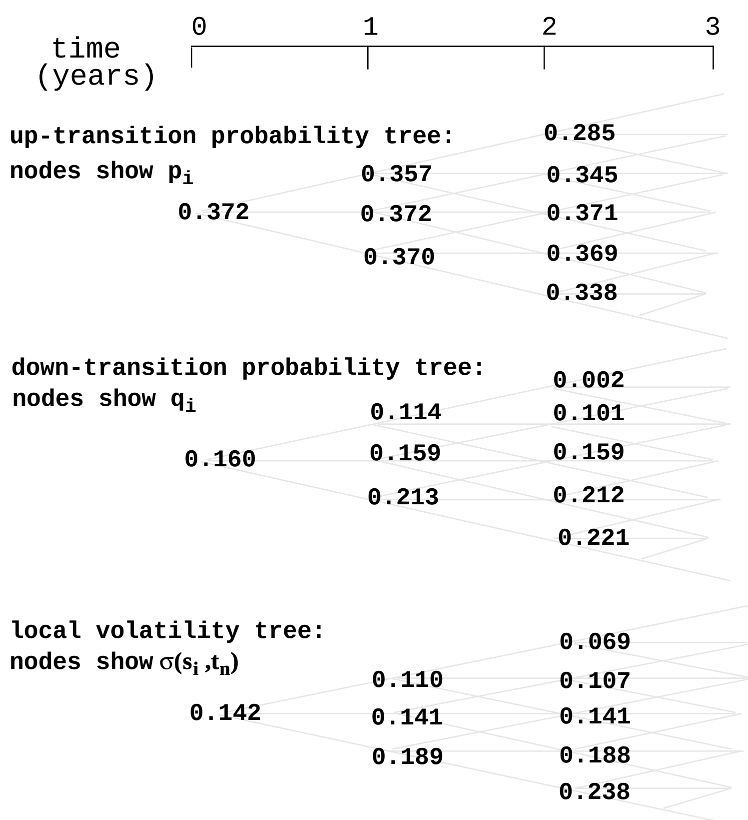


FIGURE 11. For the skew of 1 percentage point for every 10 strike points, the skewed trinomial tree has no negative probabilities. The resulting local volatilities at different nodes are reasonably smooth.

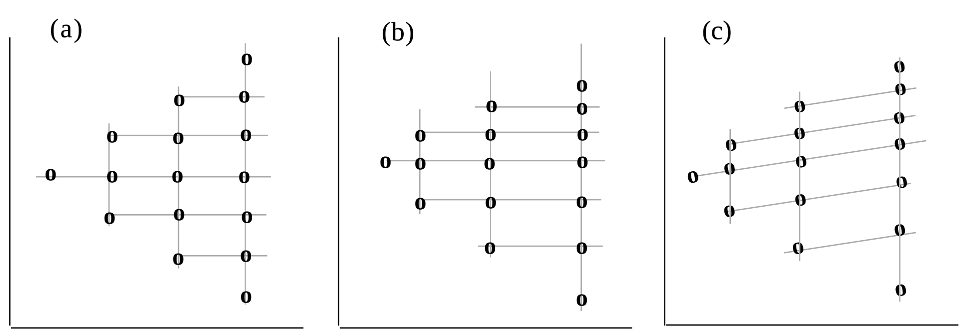


- Next, if there are forward price violations, in the sense of Figure 8, in any of the nodes, grow all the node prices along the forward curve[9] by multiplying all zero-rate node prices at time t_i by the growth factor $e^{(r-\delta)t_i}$. This effectively removes all forward price violations.

Figure 12 shows the steps described above. Figure 12(a) depicts a regular state space. In the state space of Figure 12(b) the time steps increase with time and price steps decrease with stock price. In Figure 12(c) the forward growth factor has been applied to all the nodes to ensure that no forward price violations remain. The resulting state space in this figure is more suitable for a market with significant (inverted) term-structure and (negative) skew-structure. Some of the details of this type of construction are provided in Appendix B.

We must point out that, for a fixed number of time and stock price levels, it may be impossible to avoid all negative probabilities, no matter what choice we make for the state space. As long as our choice does not violate the forward price condition at any node, we can overwrite the option prices which produce negative probabilities. In this way, even though we give up fitting the option price at some of the implied tree nodes, we fit the forward prices with transition probabilities which lie between *0* and *1* at every node. Generally, the less overwriting we have to do in our implied tree, the better it will fit the smile.

FIGURE 12. A schematic construction for the state space of the implied trinomial tree model: (a) build a regular trinomial lattice with equal time and price steps; (b) modify different time and then price steps in the lattice; (c) grow the lattice along the forward interest rate curve.



9. It may not be necessary to grow the nodes precisely along the forward rate curve. Any sufficiently large growth factor which removes forward price violations of the types described in Figure 8 will be sufficient.

Lastly, in the previous section we described some simple choices for overwriting unacceptable transition probabilities. We may think of other types of overwriting strategies which, for example, may involve keeping local volatilities or distributions as smooth as possible across the tree nodes. One strategy is to try fitting the probabilities to the local volatility at the previous node before applying a more naive overwrite like those discussed in the previous section. Other more complicated strategies require use of optimization over the set of possible overwrites and are more difficult to implement.

APPENDIX A: Constructing Constant-Volatility Trinomial Trees

This appendix provides several methods for constructing constant volatility trinomial trees that can serve as initial state spaces for implied trees. The different methods described here will all converge to the same theory, i.e the constant-volatility Black-Scholes theory, in the continuous limit. In this sense, we can view them as equivalent discretizations of the constant volatility diffusion process. Figure 13 shows two common methods for building binomial trees. There are in general an infinite number of (equivalent) binomial trees, all representing the same discrete constant volatility world. This is due to a freedom in the choice of overall growth of the price at tree nodes (not to be confused with the stock's risk-neutral growth rate). If we multiply all the node prices of a binomial tree by some constant (and reasonably small) growth factor, we will end up with another binomial tree which has different (positive) probabilities but represents the same continuous theory. The familiar Cox-Ross-Rubinstein (CRR) binomial tree has the property that all nodes with same spatial index have the same price. This makes CRR tree look regular in both spatial and temporal directions. The Jarrow-Rudd (JR) binomial tree[10] has the property that all probabilties are equal to 1/2. This property makes the JR tree a natural discretization for the Brownian motion. The JR tree does not grow precisely along the forward risk-free interest rate curve, but we can just as easily construct binomial trees which have this property[11].

FIGURE 13. Two equivalent methods for building constant-volatility binomial tree.

(a) CRR binomial tree

$S_u = Se^{\sigma\sqrt{\Delta t}}$

$S_d = Se^{-\sigma\sqrt{\Delta t}}$

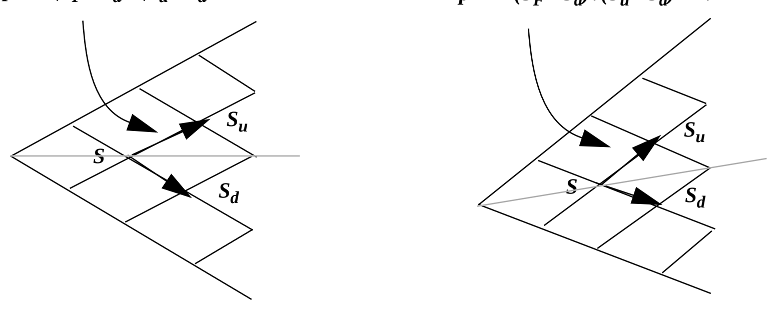
$p = (S_F - S_d)/(S_u - S_d) \mathrel{!=} 1/2$

(b) JR binomial tree

$S_u = Se^{(r - \sigma^2/2)\Delta t + \sigma\sqrt{\Delta t}}$

$S_d = Se^{(r - \sigma^2/2)\Delta t - \sigma\sqrt{\Delta t}}$

$p = (S_F - S_d)/(S_u - S_d) = 1/2$



10. See Jarrow and Rudd [1983].

11. In a recombining constant volatility binomial tree S_u and S_d have the general form: $S_u = Se^{\pi\Delta t + \sigma\sqrt{\Delta t}}$ and $S_d = Se^{\pi\Delta t - \sigma\sqrt{\Delta t}}$, for any reasonable number π.

We have even more freedom when it comes to building constant volatility trinomial tree. Figure 14 illustrates three methods for doing so. The first two are based on the fact that we can view two steps of a binomial tree in combination as a single step of a trinomial tree. Figure 14(a) uses a CRR-type binomial tree do so whereas Figure 14(b) uses a JR-type binomial tree. To construct other kinds of trinomial tree we can apply a variety of criteria, all of which may be equally reasonable[12]. For example, diagram (c) is based on the requirement that all three transition probabilities be equal (to 1/3) for all the tree nodes. Another common choice of probabilities, which we have not described here but is simple to construct, is $p = q = 1/6$.

FIGURE 14. Three equivalent methods for building constant volatility trinomial trees. (a) Combining two steps of a CRR binomial tree. (b) Combining two steps of a JR binomial tree. (c) Equal-probability tree with p = q = 1/3.

(a) Combining two steps of a CRR binomial tree

$$S_u = Se^{\sigma\sqrt{2\Delta t}}$$

$$S_m = S$$

$$Sd = Se^{-\sigma\sqrt{2\Delta t}}$$

$$p = \left(\frac{e^{r\Delta t/2} - e^{-\sigma\sqrt{\Delta t/2}}}{e^{\sigma\sqrt{\Delta t/2}} - e^{-\sigma\sqrt{\Delta t/2}}}\right)^2$$

$$q = \left(\frac{e^{\sigma\sqrt{\Delta t/2}} - e^{r\Delta t/2}}{e^{\sigma\sqrt{\Delta t/2}} - e^{-\sigma\sqrt{\Delta t/2}}}\right)^2$$

(b) Combining two steps of a JR binomial tree

$$S_u = Se^{(r - \sigma^2/2)\Delta t + \sigma\sqrt{2\Delta t}}$$

$$S_m = Se^{(r - \sigma^2/2)\Delta t}$$

$$S_d = Se^{(r - \sigma^2/2)\Delta t - \sigma\sqrt{2\Delta t}}$$

$$p = 1/4$$

$$q = 1/4$$

(c) Equal-probability tree

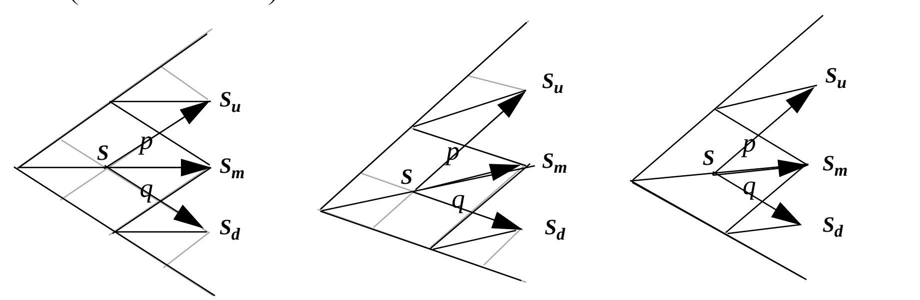
$$S_u = Se^{(r - \sigma^2/2)\Delta t + \sigma\sqrt{3\Delta t/2}}$$

$$S_m = Se^{(r - \sigma^2/2)\Delta t}$$

$$S_d = Se^{(r - \sigma^2/2)\Delta t - \sigma\sqrt{3\Delta t/2}}$$

$$p = 1/3$$

$$q = 1/3$$



12. In a recombining constant volatility trinomial tree S_u, S_m, and S_d have the general form $S_u = Se^{\pi\Delta t + \phi\sigma\sqrt{\Delta t}}$, $S_m = Se^{\pi\Delta t}$ and $S_d = Se^{\pi\Delta t - \phi\sigma\sqrt{\Delta t}}$ for $\phi > 1$ and any reasonable value of π.

APPENDIX B: Constructing Skewed Trinomial State Spaces

If there is significant term- or skew-structure in implied volatilities, we need to build a trinomial state space with irregular mesh size to better accommodate the variations of the local volatilities with time and level. One obstacle in achieving this is the fact that often we do not *a priori* know what the local volatility function looks like. However, there are exceptions. For example, if there is a significant term-structure but little skew-structure in the market then local volatility is mostly a function of time[13]. On the other hand, if there is a large skew-structure but insignificant term-structure in the implied volatilities then we know that local volatility is mostly a function of the level[14].

Assume that interest and dividend rates are zero for now. Consider the term-structure case first. Here local volatility is some function of time $\sigma(t)$. We can introduce the notion of scaled time $\tilde{t}$ as

$$t = c\int_0^{\tilde{t}} \sigma^2(u)du$$

for some constant c. Differentiating both sides of this equation we can write an equivalent non-linear equation describing $\tilde{t}$ in terms of t:

$$\tilde{t} = \frac{1}{c}\int_0^{t} \frac{1}{\sigma^2(\tilde{t}(u))}du$$

Using the scaled time in place of standard time transforms the stock evolution process to a constant volatility (Black-Scholes) process[15]. We can choose the constant c so that the rescaled and physical times coincide at some fixed future time T, e.g. $\tilde{t}(T) = T$. In this case

$$c = T/\left(\int_0^T \sigma^2(u)du\right) = \frac{1}{T}\int_0^T \frac{1}{\sigma^2(\tilde{t}(u))}du$$

13. If $\Sigma = \Sigma(T)$ is the implied volatility for expiration T, then local volatility at time $t=T$ is given by the relation $\sigma^2(T) = d[T\Sigma^2(T)]/dT$.

14. If $\Sigma = \Sigma_0 + b(K-K_0)$ is implied volatility for strike price K, then the local volatility at level K in the vicinity of K_0 is roughly given by the relation $\sigma = \Sigma_0 + 2b(K-K_0)$

15. Define a new stock price variable $\tilde{S}$ by the relation $\tilde{S}(t) = S(\tilde{t})$ and also a new Brownian motion $\tilde{Z}$ by

$$\frac{1}{\sqrt{c}}\tilde{Z}\left(c\int_0^t \sigma^2(u)du\right) = \int_0^t \sigma(u)dZ(u) \quad .$$

Then, using the definition of scaled time we find:

$$\frac{d\tilde{S}(t)}{\tilde{S}(t)} = \frac{d[S(\tilde{t})]}{S(\tilde{t})} = \sigma(\tilde{t})d[Z(\tilde{t})] = \frac{1}{\sqrt{c}}d\left[\tilde{Z}\left(c\int_0^{\tilde{t}} \sigma^2(u)du\right)\right] = \frac{1}{\sqrt{c}}d\tilde{Z}(t)$$

Hence the new stock price variable evolves with constant volatility $\frac{1}{\sqrt{c}}$.

In the discrete-time world of a multinomial tree, with known (equally spaced) time points $t_0 = 0, t_1, \ldots t_n = T$, we want to find unknown scaled times $\tilde{t}_0 = 0, \tilde{t}_1, \ldots \tilde{t}_n = T$ such that $\sigma^2(\tilde{t}_i)\Delta\tilde{t}_i$ is the same constant for all times $\tilde{t}_i$. This guarantees that the tree will recombine. One way to do this is by iteratively searching for solutions of discrete-time analogues of the second set of earlier (non-linear) relations:

$$\tilde{t}_k = \frac{T\sum_{i=1}^{k}\frac{1}{\sigma^2(\tilde{t}_i)}}{\sum_{i=1}^{n}\frac{1}{\sigma^2(\tilde{t}_i)}} \quad ; k = 1, ..n$$

Next consider the skew-structure. Assume that local volatility is some function $\sigma(S)$ of the level. This assumption is roughly valid when implied volatilities have little or no term-structure. We define the scaled stock price $\tilde{S}$ using the equation[16]:

$$\tilde{S} = S_0 \exp\left[c\int_{S_0}^{S}\frac{1}{x\sigma(x)}dx\right]$$

for some constant c. The scaled stock price has a constant volatility equal to c[17]. A reasonable discrete representation of scaled stock price movements can be given by a constant volatility tree. Inverting this equation, we can convert the discrete nodal values of $\tilde{S}$ to discrete values of S. In the resulting S-tree the spacing between nodes varies with the level corresponding to the similar variation in local volatility. We can choose the constant c to represent the at-the-money or some other typical value of local volatility. For any fixed time period, if $\tilde{S}_i$ denote the nodal values of scaled stock price, the corresponding stock price values can be found by solving the discrete version of the above equation. For nodes which lie above the central node this gives:

$$S_k = S_{k-1} + \frac{1}{c}\sigma(S_{k-1})S_{k-1}\log\frac{\tilde{S}_k}{\tilde{S}_{k-1}}$$

16. See Nelson and Ramaswamy [1990].

17. Starting from the stochastic equation $\frac{dS}{S} = \sigma(S)dZ$ and using Ito's lemma

$$d\log\tilde{S} = -\frac{c}{2}(\sigma + \sigma'S)dt + cdZ \ .$$

There is an induced drift from the variation of the local volatility function with level. Starting from a constant volatility state space, the discrete world trinomial (or higher multinomial) implementations can accommodate this drift through the choice of transition probabilities, given small enough drift or short enough time step.

and for those below it gives[18]:

$$S_{k-1} = S_k - \frac{1}{c}\sigma(S_k)S_k \log\frac{\tilde{S}_k}{\tilde{S}_{k-1}}$$

Again we can set the constant c to the at-the-money or any other reasonable value of local volatility[19].

In general, if local volatility is in the form of a product of some function of time and some function of stock price, i.e if local volatility function is "separable", then we can perform the scaling transformations on time and stock price independently. As a result we will obtain state spaces which can accommodate a term-structure with a constant skew-structure superimposed on it. A simple example of this is when local volatility has a Constant Elasticity of Variance (CEV) form:

$$\sigma(S, t) = \sigma_0(t)\left(\frac{S}{S_0}\right)^{\gamma}$$

Here $\sigma_0(t)$ represents the at-the-money term structure and γ is a constant skew or elasticity parameter. Most equity options markets have time-varying skew structures. Despite this, with judicious choices for term-structure and skew parameters and using the procedure outlined above, we can create a state space which fits any particular options market rather well. Non-zero interest rates and dividends can also be incorporated in this state space by growing all the nodes with an appropriate growth factor, as discussed in the main text.

18. To guarantee positivity of stock prices we can use alternative relations: $S_k = S_{k-1}e^{\frac{\sigma(S_k)}{c}\log\frac{\tilde{S}_k}{\tilde{S}_{k-1}}}$ and $S_{k-1} = S_k e^{-\frac{\sigma(S_k)}{c}\log\frac{\tilde{S}_k}{\tilde{S}_{k-1}}}$ for nodes above and below the central nodes respectively. These relations have the further advantage that when volatility is constant (and equal to c) the S- and $\tilde{S}$-trees will be identical.

19. Choosing somewhat larger volatilities increases the spacing between the nodes and often improves the ability of the tree to fit option prices. A similar situation occurs in explicit finite-difference lattices where increasing the price spacing relative to time spacing increases the stability of the solutions.

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