

**New insights into smile, mispricing and
value at risk: the hyperbolic model**

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Abstract

We investigate a new basic model for asset pricing, the hyperbolic model, which allows an almost perfect statistical fit of stock return data. After a brief introduction into the theory supported by an appendix we use also secondary market data to compare the hyperbolic model to the classical Black-Scholes model. We study implicit volatilities, the smile effect and the pricing performance. Exploiting the full power of the hyperbolic model, we construct an option value process from a statistical point of view by estimating the implicit risk-neutral density function from option data. Finally we present some new value-at-risk calculations leading to new perspectives to cope with model risk.

I Introduction

There is little doubt that the Black-Scholes model has become the standard in the finance industry and is applied on a large scale in everyday trading operations. On the other side its deficiencies have become a standard topic in research. Given the vast literature where refinements and improvements of Black-Scholes theory are discussed, we want to make some pointed historic remarks.

Osborne (1959) was the first to rediscover the normal distribution and consequently the Brownian motion as a model for stock returns after the ingenious and nowadays well-known work of Bachelier (1900). Strange enough the astrophysicist Osborne read his paper, which appeared in “Operations Research”, before the U.S. Naval Research Laboratory Solid State Seminar, a seminar certainly not in the very focus of attention of scientists interested in finance at that time (see Bernstein (1992)). Although Osborne concluded log-normal behaviour of stock returns it was Samuelson (1965), who introduced the geometric or in his words “economic” Brownian motion, giving the price process an exponential form. It is this price process which is the crucial part in the Black-Scholes reasoning in option valuation. Already in the early 60’s Mandelbrot (1963) and Fama (1965) began the search for alternative models. Although deviation from normality is well-known, until nowadays many of the improved models are still based on Brownian motion as the driving process. Let us underline that distributional assumptions are not the only direction where improvements aim at. Merton (1976) added Poisson jumps to the geometric Brownian motion model. Dependence structures and existence of moments among other facts have been investigated thoroughly (see Pagan (1996) for an extensive survey). It is important to note that with respect to option pricing these new models lead to intricate problems. On the one hand it is impossible to take care of all kinds of deficiencies within one particular model. On the other hand the numerical problems which arise are often substantial.

Our aim is to present some new empirical results concerning the valuation of contingent claims based on the hyperbolic model which was introduced in Eberlein, Keller (1995). This hyperbolic model allows an almost perfect fit of stock return data. In addition it is in a certain way opposite to the Brownian world, since its paths are purely discontinuous. If one looks at real stock price movements on the intraday scale it is exactly this discontinuous behaviour what one observes. Although the hyperbolic Lévy motion is not so easy to handle as Brownian motion, a closed option pricing formula could be derived in the paper mentioned above. It can be evaluated efficiently from the numerical point of view.¹ Thus given the optimal approximation of the

¹Readers interested in a test can access our Hyperbolic Option Calculator under <http://www.fdm.uni-freiburg.de/UK/>

return distribution we want to show empirically how far the improvements of its option pricing behavior reach. This gives an answer to the question if distributional assumptions are the most important source of the well-known option pricing deviations, the holes in Black-Scholes (see Black (1990)). We study the smile effect and pricing performance by the model. Finally we present some new value at risk calculations.

II The hyperbolic density

The hyperbolic distribution was introduced by Barndorff-Nielsen (1977) for modelling the grain size distribution of wind blown sand. The name of the hyperbolic distribution derives from the fact that its log-density is a hyperbola. Recall, that the log-density of the normal distribution is a parabola. Hence the former distribution provides the possibility of modelling the well-known heavier tails of return distributions. Its density is given by

$$f_{(\alpha,\beta,\delta,\mu)}(x) = \frac{\sqrt{\alpha^2 - \beta^2}}{2\alpha\delta K_1(\delta\sqrt{\alpha^2 - \beta^2})} e^{-\alpha\sqrt{\delta^2 + (x-\mu)^2} + \beta(x-\mu)}, \quad (1)$$

where K_1 denotes the modified Bessel function of the third kind with index 1. It has four parameters $\mu \in \mathbb{R}$, $\delta > 0$ and $0 \leq |\beta| < \alpha$. Roughly spoken α and β determine the shape, where β is responsible for skewness; δ and μ are scale and location parameters. Given a sample of independent observations these parameters can be estimated by maximum likelihood methods (see Blæsild, Sørensen (1992)).

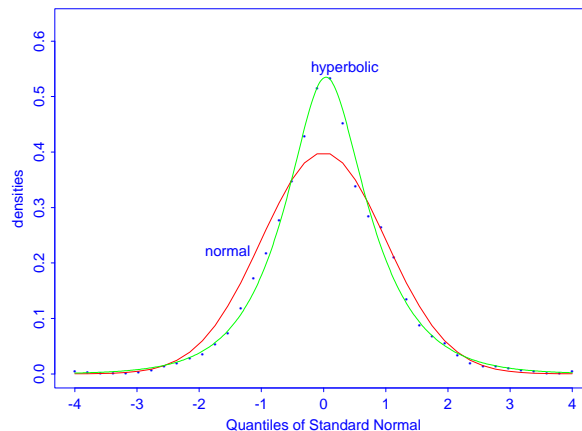


Figure 1: *Fitted densities for Thyssen (parameters of the fitted hyperbolic distribution: $\hat{\alpha} = 107.6$, $\hat{\beta} = 2.10$, $\hat{\delta} = 0.0066$, $\hat{\mu} = 0.0003$).*

In Eberlein, Keller (1995) the hyperbolic distribution proved to provide an excellent fit for German stock returns. In the latter study the authors

gave also some empirical evidence that the hyperbolic model is superior to the well-known α -stable model. We consider log-returns, i.e. given a sample $\{S_i \mid i \leq n\}$ of prices, the dividend corrected returns are defined by $R_i = \log((S_i + d_i)/S_{i-1})$, where d_i is the dividend payment at time i . Figure 1 shows the fit for the empirical density of Thyssen, the largest steel-producing corporation in Germany. The data set consists of daily closing prices from January 1, 1988 to May 24, 1994 resulting in 1598 observations for the log-returns. The points in the figure represent the empirical density of log-returns. The other curve is the fitted normal distribution.

Similar fits are obtained for other blue chips of the German market and the DAX, the German stock index itself. Also US stock market data can be modelled quite well. For example, we provide the fit for the index of the New York stock exchange, the so-called NYSE-composite. This index represents all the common stocks listed at the NYSE. The empirical and the fitted hyperbolic and normal densities are shown in figure 2. The quantile-quantile

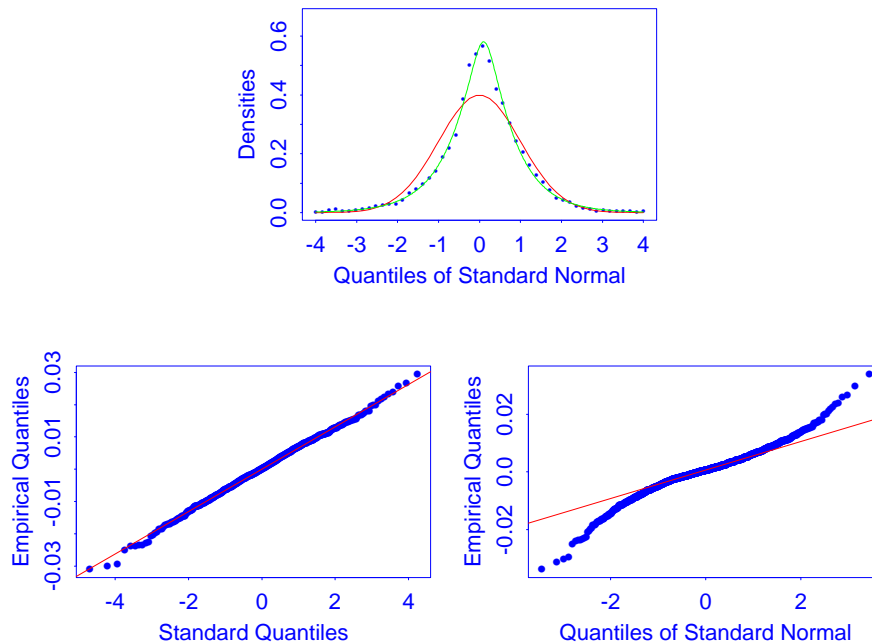


Figure 2: *Fitted densities for the NYSE-composite (parameters of the fitted hyperbolic distribution: $\hat{\alpha} = 225.0$, $\hat{\beta} = -5.80$, $\hat{\delta} = 0.0015$, $\hat{\mu} = 0.0006$).*

plots below show the perfect fit and the deviation from normality. The data series consists of 1748 daily closes from January 2, 1990 to November 11, 1996.

III Modelling financial assets

As pointed out above the price process S_t underlying the Black-Scholes (1973) model is given by the geometric Brownian motion. The latter is often formulated as the solution of the following linear stochastic differential equation:

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \quad (2)$$

where W_t is a standard Brownian motion. Equation (2) describes the local dynamics of the price process in terms of drift and driving process W_t . There is a vast literature on generalisations of this model, see Bakshi, Cao, Chen (1997) for a state of the art evaluation and comparison of these models. The most general form of this equation applied to financial data is given by

$$dS_t = a(t, S_t)dt + b(t, S_t)dW_t + \lambda_t S_t dN_t, \quad (3)$$

where N_t is a standard Poisson process incorporating the jumps. Now the equation is no longer linear. This leads to severe technical problems concerning the estimation of the processes a and b . Sometimes b is formulated as the solution of another stochastic differential equation which is driven by another independent Brownian motion. As a result one gets stochastic volatility models.

These generalisations of the basic model (2) aim at the structure of the equation describing its dynamics. Although Merton (1976) added Poisson jumps to the model the main source of its randomness is still Brownian motion. The empirical results of the preceding section led us to substitute Brownian motion by an appropriate alternative process, the hyperbolic Lévy motion discussed below. Because we want to reshape the basic model we will provide in the following sections an extensive empirical comparison of both basic models.

The hyperbolic Lévy motion is a pure jump process, hence equation (2) attains the following form

$$dS_t = \mu S_{t-} dt + \sigma S_{t-} dX_t, \quad (4)$$

where X_t is a standardized hyperbolic Lévy motion and S_{t-} denotes limits from the left. The solution of this equation is the well-known Doléans-Dade or stochastic exponential given by

$$S_t = S_0 \exp(\mu t + \sigma X_t) \prod_{s \leq t} (1 + \sigma \Delta X_s) e^{-\sigma \Delta X_s}. \quad (5)$$

Here $\Delta X_s = X_s - X_{s-}$ denotes the jump at time s if there is one (see e.g. Jacod, Shiryaev (1987), pp. 58-61). In contrast to the geometric Brownian motion the process given by (5) is not suitable for modelling price paths of financial assets, because it has negative values with positive probability.

One way to circumvent this problem is to truncate the negative jumps of X_t in order to satisfy the condition $\Delta X_t > -\sigma^{-1}$, which is necessary for S_t being positive. Note that the Poisson process does not violate this condition because of its simple jump structure. However, it is the large jumps which are responsible for the empirically observed heavy tails. Our empirical analysis of financial data led us to consider the following reformulation of the basic model:

$$dS_t = \mu S_{t-} dt + \sigma S_{t-} dX_t + S_{t-} (e^{\sigma \Delta X_t} - 1 - \sigma \Delta X_t), \quad (6)$$

where the jumps now enter explicitly into the dynamics by the last term. This is of course motivated by Ito's formula which leads to the following solution

$$S_t = S_0 \exp(\mu t + \sigma X_t). \quad (7)$$

This process is positive and has hyperbolic log-returns. In the next section we provide a detailed description of the hyperbolic model.

IV The hyperbolic model

According to Barndorff-Nielsen, Halgreen (1977) hyperbolic distributions are infinitely divisible. As described in Eberlein, Keller (1995) this implies that they generate a Lévy process $X = (X_t)_{t \geq 0}$, i.e. a process with stationary and independent increments, such that the distribution of X_1 is given by the density (1). Recall, that both, Brownian motion and the Poisson process are Lévy processes. For an introduction into the theory of Lévy processes see e.g. Protter (1992). We call this process $(X_t)_{t \geq 0}$ the hyperbolic Lévy motion depending on the four parameters $(\alpha, \beta, \delta, \mu)$. Note, that for each choice of these parameters this is a different process. The moment generating function of X_1 is given by (see appendix A)

$$M(u) = e^{\mu u} \frac{\sqrt{\alpha^2 - \beta^2}}{K_1(\delta \sqrt{\alpha^2 - \beta^2})} \frac{K_1(\delta \sqrt{\alpha^2 - (\beta + u)^2})}{\sqrt{\alpha^2 - (\beta + u)^2}}, \quad |\beta + u| < \alpha. \quad (8)$$

For the variance of the process we obtain

$$\text{Var} X_t = t \delta^2 k(\zeta, \alpha, \beta), \quad (9)$$

where k is a complicated expression in terms of various modified Bessel functions (see appendix A, where one also finds the characteristic function of the process and the corresponding densities). $\zeta = \delta \sqrt{\alpha^2 - \beta^2}$ and β/α are scale- and location-invariant parameters. They determine the shape, i.e. skewness and kurtosis (see Barndorff-Nielsen et al. (1985), p. 78). These shape parameters $(\zeta, \beta/\alpha)$ are estimated from longer time series, e.g. three or six years daily data. They are kept fixed while rescaling the distribution when we calculate implicit volatilities and option prices in later sections.

Given the explicit form of the moment generating function we show in appendix B that the hyperbolic Lévy motion is a purely discontinuous process, i.e. it changes its values by jumps only. Looking at the fluctuations of stock prices on a microscale (intraday scale), confirms that processes with jumps are the natural choice. Moreover, we capture the risk of sudden larger price changes by using a jump process as model. As pointed out above the model for stock prices which produces hyperbolic returns along time intervals of length 1 is given by (7). Let us emphasize that contrary to the classical modelling by a geometric Brownian motion we get individual price processes for each stock corresponding to its individual shape. In fact the shape parameters may be viewed as a label for the financial asset. This adds to an increase in accuracy. As mentioned above the hyperbolic parameters can be estimated from financial data using maximum likelihood methods (see section 3 in Eberlein, Keller (1995) for more details). By this we also obtain an implicit estimate of the volatility of the process according to (26).

In the diffusion context one usually derives a partial differential equation for the option price by replicating the price path. Pure jump processes (excluding the Poisson model) lead to incomplete models in general. Hence the above approach is bound to fail.

The modern martingale approach to option pricing uses equivalent martingale measures leading to arbitrage free prices (see Delbaen, Schachermayer (1994)). These are given as expectations under those measures. We follow Eberlein, Keller (1995) to choose Esscher transforms for pricing. By this approach we obtain a closed option pricing formula which proved to explain some of the well-known deficiencies of Black-Scholes prices. The Esscher equivalent martingale measure P^θ is given by $dP^\theta = e^{\theta X_t - t \log M(\theta)} dP$, where θ is the solution of $r = \log M(\theta + 1) - \log M(\theta)$ and r is the constant interest rate (see appendix C). The latter equation ensures that the discounted price process is in fact a P^θ -martingale. Consequently, the price of an option with time to expiration T and payoff function $H(S_T)$ is given by $e^{-rT} E^\theta H(S_T)$. In particular, for a call option with strike K whose payoff is $H(S_T) = (S_T - K)_+$ we obtain for this expectation under P^θ

$$C_{\text{Hyp}} = S_0 \int_{\gamma}^{\infty} f_T(x; \theta + 1) dx - e^{-rT} K \int_{\gamma}^{\infty} f_T(x; \theta) dx \quad (10)$$

where $\gamma = \ln(K/S_0)$ and $f_t(\cdot; \theta)$ is the density of the distribution of X_t under the risk-neutral measure. Note, that the hyperbolic call price is the usual weighted difference of S_0 and $e^{-rT}K$, where the weights are given by certain probabilities. This is very much the same as in the Black-Scholes case, in fact the Black-Scholes formula can also be derived by an Esscher approach. The hyperbolic price can be computed in real time employing fast Fourier transformation and numerical integration. In Eberlein, Keller (1995) one can find some examples of hyperbolic option prices compared to Black-Scholes prices. We summarize this comparison in figure 3. The

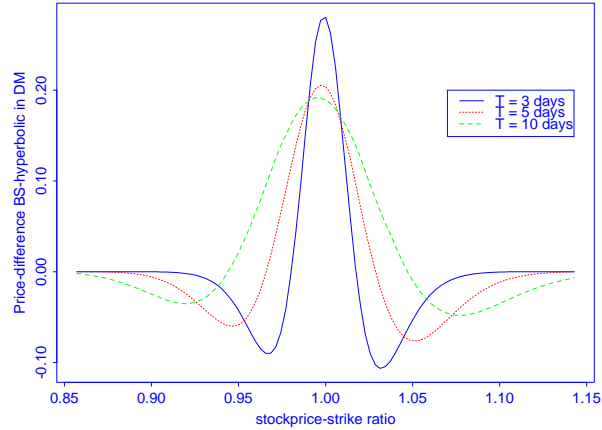


Figure 3: *Comparison of option prices.*

curves show the difference Black-Scholes price minus the hyperbolic price as a function of the stockprice-strike ratio, i.e. as a function of moneyness. The three curves represent different time intervals to expiration. Compared to prices derived from the more accurate hyperbolic model, Black-Scholes prices are too high at the money, but they are too low in and out of the money. Option prices which are too low arise from the underestimation of the risk of bigger stock price jumps in the Black-Scholes model. Not only close to expiration, where most of the trading takes place, but also for longer periods of time we obtain a significant deviation from Black-Scholes prices. Figure 4 provides an overview of the price differences of the two models for all times to maturity from $T = 1, \dots, 30$ days.

Recall, that the location or drift parameter μ enters our pricing formula, as we are working in the incomplete setting. Hence the risk-preference of the trader given by μ (or more precisely by the underlying probability measure generated by the hyperbolic distribution with parameter μ) has an effect on the hyperbolic price. This is not the case for the Black-Scholes model and results in the striking price difference given in figure 4.

V Implicit volatilities

In view of the empirical results concerning stock returns given above we want to show now, how this translates into option pricing. Brownian motion is the basic building block for all price models given by (3). As we want to replace the Black-Scholes model by the hyperbolic model a comparison with respect to option pricing has to focus on these basic models. It is the following question we want to pursue: is the hyperbolic model an appropriate candidate for modelling financial assets. Finally various generalizations may

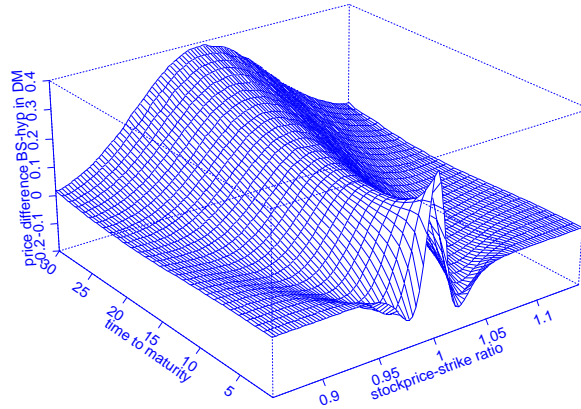


Figure 4: 3-dim comparison of both models: BS minus hyperbolic prices.

very well lead to the same level of sophistication, that has been reached for the classical model.

The study is based on intraday option and stock market data of Bayer, Daimler Benz, Deutsche Bank, Siemens and Thyssen from July 1992 to September 1994. The option data set contains all trades reported by the Deutsche Terminbörse (DTB) during the period above. Per month there are 736 to 7924 observed option and 434 to 2049 observed stock prices.

The data is processed in the following way. At first we assign to each option price the corresponding intraday stock price. The electronic DTB has longer business hours than the stock exchange in Frankfurt. Following Rubinstein (1985) and Clewlow, Xu (1994) we remove all option quotes without stock trading in the preceding 20 minutes. This leads to a removal of approximately half of the option quotes.

The time to maturity is calculated on the basis of actual trading days. This means that the days with trading at the exchanges in Frankfurt during the lifetime of each option were counted. In contrast to Cox, Rubinstein (1985) we use the resulting trading time scale for option pricing and variance estimation. This guarantees comparability between the implicit volatilities produced by contingent claims and the empirical (historical) volatilities of the stock.

Dividend payments reduce the price of the stock. Following Kolb (1995) we correct the share value by subtracting the discounted dividend, i.e. $S^- = S_0 - d \exp(-rt)$. d denotes the amount of the dividend payment which is made t trading days after the option trade and r the riskless daily interest rate. On the German market dividends are paid only once a year. Hence, we had to correct just 18% of the values. For the interest rate we used the Frankfurt interbank offered rate (FIBOR) on a monthly basis with

different maturities (1, 3, 6 months). Hence the substantial changes in the term structure in the years from 1992 to 1994 were taken into account. Finally, option prices must satisfy some simple arbitrage relationships (see Cox, Rubinstein (1985)). If an option quote violates these bounds it is removed. Note, that most of the trading takes place at the money and with a short time to maturity ($T = 1, \dots, 50$ trading days). Therefore, one should pay particular attention to this region.

Writing the actual market price on the left side of the equation and the Black-Scholes option pricing formula on the right side and solving for the volatility parameter σ yields the Black-Scholes implicit volatility $\sigma_{\text{Imp.BS}}$. This is the volatility assumed by the traders. According to the model the volatility should be constant for different stockprice-strike ratios $\rho = S^-/K$. Figure 5 shows that in reality it depends on the stockprice-strike ratio and

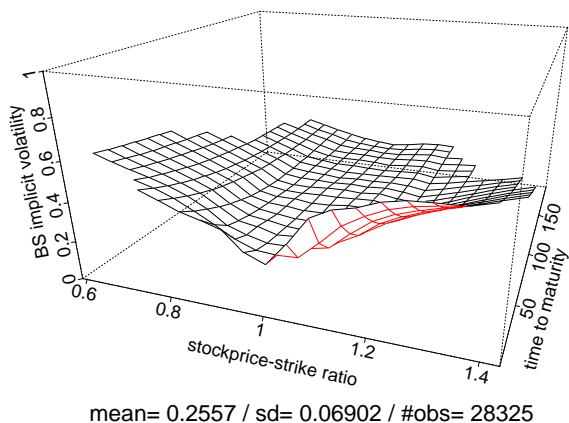


Figure 5: *Black-Scholes implicit volatilities of Thyssen calls from July 1, 1992 to September 18, 1994.*

on time to maturity. The curves are smoothed by the LOESS algorithm (see Cleveland, Grosse, Shyu (1993)). In this approach it is only assumed that the implicit volatilities could be fitted locally by a polynomial of first or second order. We do not want to make global assumptions regarding the behaviour of volatility.

Typically the implicit volatility is higher in the money and out of the money. This effect is called the smile because the shape of the curve resembles a smiling face. The smile is decreasing with time to maturity and has its minimum for $\rho \approx 1$. This leads to a positive strangle volatility. Moreover, smiles are frequently asymmetric leading to non-zero costs for risk reversal strategies (see McCauley, Melick (1996)). Looking at data sets corresponding to different time periods the implicit volatility follows always the same

pattern but of course the smile is more regular for longer observation periods. The pattern repeats for all analysed share values and for the option on the DAX future given in figure 6. Below the interdependence of implicit volatility and stockprice-strike ratio respectively time to maturity is plotted separately for Thyssen calls.

VI Reduction of the smile

One of the first who systematically and empirically studies alternative but now outdated option pricing formulas was Rubinstein (1985). None of the models he examined correct all the observed deficiencies of the Black-Scholes model. Therefore he proposed to build a composite model or to correlate the bias of the option prices to macroeconomic variables.

A widely proposed approach to improve modelling of asset prices is the introduction of a stochastic volatility (see Hull, White (1987), Scott (1987), Melino, Turnbull (1990), Bates (1996)). Heston (1993a) showed that the prices of these models may give the characteristic W-shape and skewness in comparison to Black-Scholes prices. However, these models do not explain the empirically observed magnitude of the smile effect (see Scott (1987), Wiggins (1987), Taylor, Xu (1993), Clewlow, Xu (1993), Bates (1997)).

Processes with jumps and mixed jump-diffusion processes were introduced by Merton (1976), Naik, Lee (1990), Madan, Milne (1991), Heston (1993b), Bates (1991) especially to improve the tail behaviour of the stochastic processes. On the one hand Ball, Torous (1985) observed only a small difference of the jump model prices to Black-Scholes option prices. On the other hand Shastri, Wethyavivorn (1987) showed that a jump-diffusion model explains partially the smile effect. Bates (1997) discovered some pricing improvements for his stochastic volatility/jump-diffusion model. But the “smirks” of implicit volatilities that were reported for the S&P future and index options (see also Derman, Kani (1994) and Longstaff (1995)) show a different pattern than the smile effect we observed for individual German stock and DAX future options.

Dupire (1994), Derman, Kani (1994) and Rubinstein (1994) proposed to compute an implied tree as a model for the stock price. They make the assumption that the risk-neutral density exists and for quoted option prices they compute the parameter values for the binomial or trinomial process. Because this approach does not start with an empirical analysis of the underlying asset process, it is not comparable to the option pricing methods, described above.

Further models theoretically leading to a reduced smile effect are given without empirical evidence: Platen, Schweizer (1994) introduced a diffusion model starting from a microeconomic equilibrium approach and explain the smile by feedback effects from hedging strategies. Hurst, Platen, Rachev

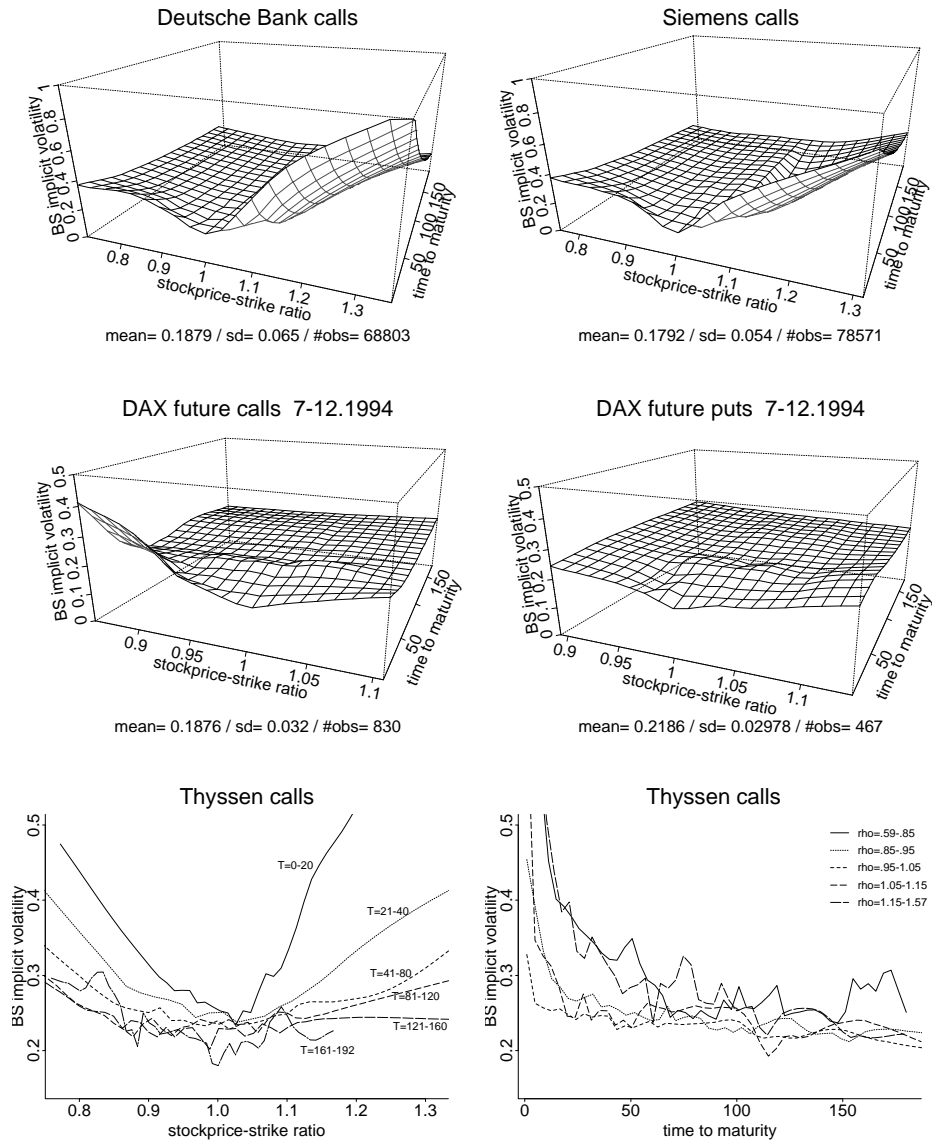


Figure 6: Black-Scholes implicit volatilities of DTB stock options from July 1, 1992 to September 18, 1994 and of DAX futures options from July 1, 1994 to December 29, 1994.

(1995) proposed a logstable asset pricing model to explain the smile.

Moreover, Duan (1995) and Kallsen, Taqqu (1995) hint at a decrease in the smile effect using ARCH-type models.

Now we study to which extent the replacement of the Gaussian model by the hyperbolic one leads to a better option pricing behaviour. For the comparison of the Black-Scholes and the hyperbolic model we use a volatility parameter derived from the variance given in (9). The implicit hyperbolic volatility $\sigma_{\text{Imp.Hyp}}$ is computed in the same way as in the Black-Scholes case. The empirically observed $\sigma_{\text{Imp.Hyp}}$ is shown in figure 7 (top left) for

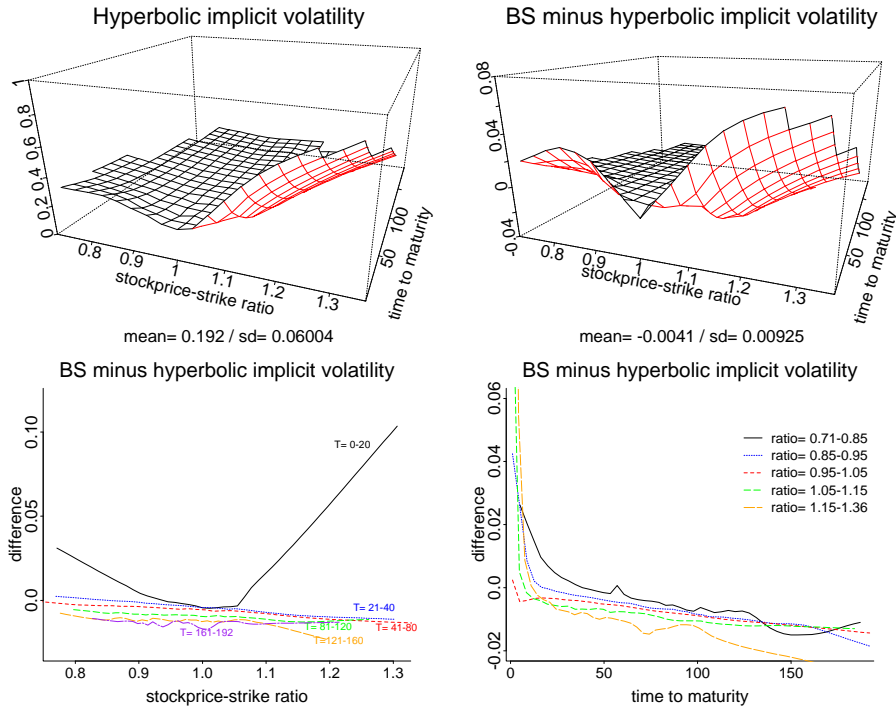


Figure 7: *Implicit hyperbolic volatility and comparison of the implicit volatilities of the Black-Scholes and the hyperbolic model (Deutsche Bank calls from July 1992 to August 1994, $n=68803$).*

Deutsche Bank call options. The implicit Black-Scholes volatility for the same data set is given in figure 6 (top left). At first sight the implicit hyperbolic volatility produces a smile effect similar to the one arising in the Black-Scholes setting. Plotting the difference $\sigma_{\text{Imp.BS}} - \sigma_{\text{Imp.Hyp}}$ of implicit volatilities of the two models (top right) shows that in the hyperbolic case the smile effect is reduced according to the W-shape. The two remaining plots in figure 7 give this difference as a function of moneyness and as a function of time to maturity.

Another way to analyse the smiles in both models is to fit the following linear model for the implicit volatilities

$$\sigma_{\text{Imp},i} = b_0 + b_1 T_i + b_2 (\rho_i - 1)^2 / T_i + e_i, \quad (11)$$

where e_i is the random error term and i the number of the trade in the option data set. The cross-term $(\rho - 1)^2 / T$ reflects the degression of the smile effect with increasing time to maturity T . The regression function was chosen as parsimonious as possible. Results of this regression are given in table I.

Table I: *Fitted coefficients for call options from July 1992 to August 1994.*

		b_0	b_1	b_2	R^2
Bayer	Black-Scholes	0.20675	-0.000487	37.318	0.5545
	Hyperbolic	0.20858	-0.000407	33.764	0.5119
Daimler Benz	Black-Scholes	0.23526	-0.000474	34.463	0.6168
	Hyperbolic	0.24091	-0.000050	29.753	0.5379
Deutsche Bank	Black-Scholes	0.19710	-0.000408	48.937	0.5523
	Hyperbolic	0.19958	-0.000340	41.334	0.4733
Siemens	Black-Scholes	0.17484	-0.000131	85.037	0.4744
	Hyperbolic	0.17785	-0.000084	75.072	0.4091
Thyssen	Black-Scholes	0.25491	-0.000256	81.269	0.5091
	Hyperbolic	0.25794	-0.000211	75.531	0.4686

Note, that T is measured in trading days and therefore the time-to-maturity effect is of a relevant order. As the value of the coefficients for $(\rho - 1)^2 / T$ and for T are smaller for the hyperbolic model, we conclude that this model reduces the smile and the time-to-maturity effect.

Given the fact that hyperbolic distributions provide a rather accurate statistical model for daily stock returns, the observations above show that the smile is not just a consequence of poor modelling in the Black-Scholes case. A better model reduces the effect, but the smile is an intrinsic market phenomenon. Options which are not at the money face additional risk such as liquidity and thus they are more expensive.

VII Pricing performance

An alternative approach to test an option pricing model is to compare observed market prices with the model prices. In contrast to volatility comparisons pricing performance analyses price differences, hence the unit of measurement is a currency unit, in this case Deutsche Mark. Note, that the same difference in volatility has a greater effect on the price if the time to maturity is longer.

However, one remaining problem is to choose the volatility. We estimate volatility parameters following different approaches. Firstly, we compute historical volatilities for time windows of 30 and 60 calendar days before the trading day of the option using the classical variance estimator. These are named Hist30 and Hist60 in the sequel. Secondly, we apply implicit volatilities observed before each trade. For the estimator `Imp.median n` we took the running median of the implicit volatilities of the last n quoted options. Because of its robustness the running median proved to be a better estimator than means or trimmed means. Note, that this means that we follow out-of-sample approaches both for the historical and the implicit volatility. Cox, Rubinstein (1985) describe the option pricing service of Fisher Black, who used historical and implicit volatilities and some other market parameters for computing a volatility parameter. Thus, both procedures are used in practice.

In figure 8 we provide a typical plot of the pricing performance for the Black-Scholes and for the hyperbolic model. The difference of model price minus market price of the call options increases with time to maturity. In the case of the Bayer calls considered above we observe a reduction of the mispricing in the hyperbolic model using an estimator based on historical volatilities.

A comparison of the pricing performance of both models within a single plot is given in figure 9. We compute for each quote the difference of the absolute pricing errors of the two models: absolute pricing error using Black-Scholes minus absolute pricing error using the hyperbolic model. The plot reveals a distinct correction of the mispricing by the hyperbolic model for call options with longer maturities.

Looking at the smile plots in section V the deficiencies of the Black-Scholes model are stronger for options near to expiration. The pricing error measured in Deutsche Mark is bigger for options with longer maturities. Consequently, we have to analyse both, smile and pricing performance, to get a complete picture.

Finally, we choose a global approach to compare the two models. In table II we give the mean error (and the standard deviation) of the pricing

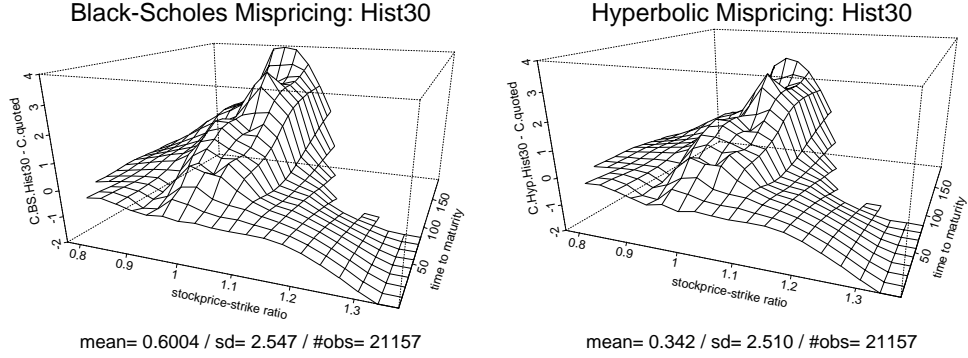


Figure 8: *Mispricing for Bayer calls (July 1, 1992 to August 19, 1994) with volatility estimator Hist30: Black-Scholes model (left) and hyperbolic model (right).*

errors for Bayer calls, i.e.

$$\frac{1}{N} \sum_{i=1}^N (C_{\text{model},i} - \hat{C}_i) \quad (12)$$

where \hat{C}_i is the quoted option price of trade $i = 1, \dots, N$. It is obvious that prices based on implicit volatility estimators are closer to the market. The hyperbolic model leads to significantly smaller errors for both, historical and implicit volatility estimators.

Another interesting aspect is to take trading volume into consideration. We compute the weighted mean of the difference of the absolute errors.

$$\sum_{i=1}^N (|C_{\text{BS},i} - \hat{C}_i| - |C_{\text{Hyp},i} - \hat{C}_i|) \cdot \text{Vol}_i \Big/ \sum_{i=1}^N \text{Vol}_i, \quad (13)$$

where Vol_i is the volume of trade $i = 1, \dots, N$ with quoted option price \hat{C}_i . The model prices are named $C_{\text{Hyp},i}$ and $C_{\text{BS},i}$. We also compute the (unweighted) median and standard deviation of the difference of the absolute errors. Positive values for the weighted mean and the median are obtained when the hyperbolic model produces a smaller pricing error. Table III shows again that mispricing is reduced by modelling asset prices using the hyperbolic Lévy motion.

Table II: *Mispricing of Bayer calls (July 1, 1992 to August 19, 1994) with different volatility estimators: mean and standard deviation of the difference of model price and quoted prices.*

estimator	Black-Scholes		Hyperbolic	
	mean	st. dev.	mean	st. dev.
Hist30	0.6	2.548	0.342	2.51
Hist60	0.616	2.427	0.388	3.532
Imp.median10	0.379	2.145	0.149	2.105
Imp.median20	0.361	2.097	0.134	2.064
Imp.median30	0.364	2.108	0.137	2.073

Table III: *Comparison of the mispricing for Bayer calls: Black-Scholes absolute errors minus absolute errors in the hyperbolic model.*

estimator	difference of the absolute errors		
	weighted mean	median	st. dev.
Hist30	0.0264	0.0056	0.3961
Hist60	-0.0260	0.0037	2.6366
Imp.median10	0.0077	0.0022	0.3572
Imp.median20	0.0117	0.0036	0.3489
Imp.median30	0.0162	0.0044	0.3492

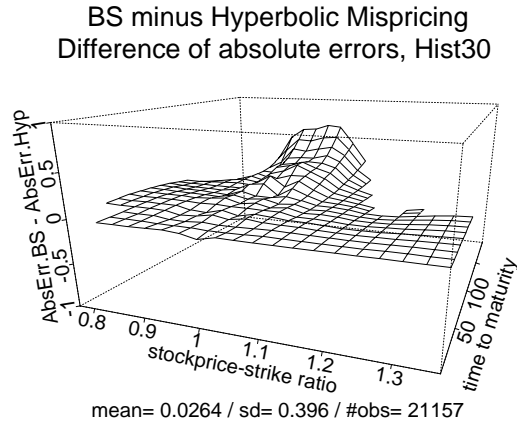


Figure 9: *Comparison of the mispricing with volatility estimator Hist30 (Bayer call options, July 1, 1992 to August 19, 1994).*

Admitting a nonzero μ in the hyperbolic model leads to lower prices for call options when time to maturity increases (see figure 4). Therefore we can expect to correct Black-Scholes overpricing, which is frequently observed for options with longer maturities (see Geske, Torous (1990) for CBOE calls). The preference of the traders enters the hyperbolic model via the parameter μ and may change during the observation period. Hence it may be necessary to adjust μ for shorter time intervals. A different way to estimate this parameter is described in section IX. To illustrate the potential correction we computed prices using some freely chosen parameters μ . In table IV we provide the pricing errors for the volatility estimator Hist30 and Bayer call options. The table shows that it is possible to reduce the mean error for an appropriately chosen μ . Note, that the standard errors do not increase and that only the absolute value and not the sign of μ has an impact on the pricing performance in the hyperbolic model.

VIII Prediction of volatilities

Based on the insight in volatility estimation we got in the preceding sections we developed a short-term volatility estimator using implicit hyperbolic volatilities. The reduction of the smile in the hyperbolic model allows to construct a more robust estimator. It is called delta.dax and is designed

Table IV: *Mispricing in the hyperbolic model for different μ : mean and standard deviation of model price minus quoted price (Bayer calls from July 1, 1992 to August 19, 1994, volatility estimator Hist30, * marks the value for μ estimated from historical data).*

μ	mean error	st. dev.
0.0	0.573	2.538
-0.000311*	0.342	2.51
-0.0005	-0.004	2.524
-0.000622	-0.288	2.579
0.000311	0.331	2.505
0.0005	-0.018	2.505
0.000622	-0.3	2.541

to predict the volatility for the next trade. Thus it will support a market maker to determine his next quote. What he needs to know is the most recent price of the underlying security and some history of prices of the option and the underlying. The points in the following figure 10 (left) show the option prices that were finally reported for a call with strike 300 and expiration in December 1993 during a period from 150 to 100 trading days prior to expiration. The line represents the predicted prices derived from the hyperbolic model. The dotted line shows predicted hyperbolic prices using a classical volatility estimator based on historical volatility. Note, that at $T = 115$ the underlying overshoots the strike leading to an overestimation of the current volatility by the historical estimator.

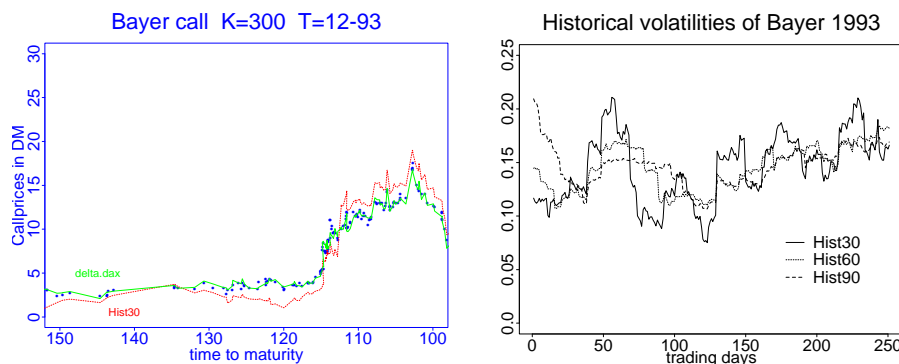


Figure 10: *Market and model prices with different predictors for a call option and historical volatility estimation with different time windows.*

In comparison to an estimator based on historical data implicit volatilities yield volatility parameters which are robust against single outliers in the returns of the underlying asset. The non-robustness of historical estimators is clearly visible in figure 10 (right) where the estimates are plotted for different time windows (30, 60, 90 calendar days). Using implicit volatilities from models with a reduced smile allows also to reduce the bias in the implicit parameter estimation which comes from options out of or in the money. Next to estimators based on implicit volatilities, GARCH-type models are widely discussed (see Bollerslev (1986) for the introduction of the GARCH model and see Bollerslev, Chou, Kroner (1992) for an extensive survey of ARCH modelling in finance). We also include this type of model in the following comparison of volatility forecasts.

Out-of-sample predictive power analysis (see Day, Lewis (1992) and Pagan, Schwert (1990)) is the right tool to check the forecast quality of the various estimators for the implicit volatility. Here we estimate regressions of the form

$$\sigma_t^2 = b_0 + b_1 \hat{\sigma}_t^2 + e_t, \quad (14)$$

where σ_t denotes the implicit volatility at time t , $\hat{\sigma}_t$ is the one-step-ahead forecast of the alternative estimators and e_t is the forecast error. We compare the forecasts of the estimator `delta.dax` with those of historical variance estimators with various time windows and with those of rolling GARCH(1,1) models. For the latter we follow Day, Lewis (1992) to estimate the parameters α and β of the GARCH model with a constant sample size of $n = 500, 600, 700$ observations of past daily returns to estimate a conditional volatility forecast. Note, that we also observe the often reported nearly persistence of volatility ($\alpha + \beta \approx 1$). For each step we shift the sample window by one day.

However, here we want to compare the estimates for the implicit volatility of a single call option. This is in contrast to the above mentioned study, where a whole sample of option contracts were used to estimate the implicit market volatility. The results are shown in the following table.

According to Pagan, Schwert (1990) the estimates of b_0 and b_1 will be approximately 0 and 1 respectively, if the forecast of conditional volatility is unbiased. Hence only `delta.dax` comes close to an unbiased estimation with a multiple R^2 of 0.246. The R^2 s of the other approaches are very low, however, they are comparable to those found by Day, Lewis (1992) and Pagan, Schwert (1990). Note, that the accuracy of the GARCH-fit is decreasing with increasing sample size. For n smaller than 500 the model could not be fitted appropriately. The coefficient of the predicted variance is sometimes even negative. This can be interpreted as follows: Comparing the estimates for b_0 using `Hist30` and `GARCH(n=700)` one can deduce that b_0 represents the mean variance and the forecast term has no influence ($b_1 \approx 0$). In those cases where $b_1 < 0$ the mean variance is overestimated by b_0 , hence

Table V: *Out-of-sample predictive power for Daimler Benz call K=700, T=11-93.*

estimator	b_0 (st. dev.)	b_1 (st. dev.)	R^2
delta.dax	0.05 (0.017)	0.775 (0.078)	0.246
Hist30	0.202 (0.011)	0.091 (0.055)	0.008
Hist60	0.423 (0.035)	-0.933 (0.16)	0.094
Hist90	0.366 (0.064)	-0.673 (0.294)	0.016
GARCH(n=500) $\hat{\alpha}=0.06 \hat{\beta}=0.93$	0.227 (0.003)	-0.109 (0.021)	0.077
GARCH(n=600) $\hat{\alpha}=0.05 \hat{\beta}=0.88$	0.229 (0.003)	-0.105 (0.021)	0.071
GARCH(n=700) $\hat{\alpha}=0.06 \hat{\beta}=0.89$	0.21 (0.003)	0.077 (0.021)	0.039

it is reduced by a negative coefficient b_1 . This behaviour repeats for other call options.

Thus if one zooms into the market down to a particular option, only estimators based on implicit volatilities provide an efficient tool to forecast tomorrows behaviour.

IX Statistical martingale measures

In section IV we presented Esscher transforms for option pricing to cope with the variety of possible equivalent martingale measures. Eberlein, Jacod (1997) showed that under all these measures the range of the pricing operator covers the whole possible no-arbitrage interval. Thus we have to single out one appropriate measure for pricing. This measure has to reflect the riskprofile of the market, i.e. the risk-sensitivity of the traders participating in the market. One has to pay a certain price for a certain risk profile (the so-called risk premium). A particular equivalent martingale measure reflects this risk by reweighting certain events, such as large price changes for example. With all these considerations in mind it seems to be the best to let the market (data) itself choose the pricing measure. We rely on a method

which is often used in interest rate modelling. Because of the intricate forms of these models, a change of measure is often a very difficult problem leading to the idea of martingale modelling. In the latter case the model is already formulated in the martingale setting, however parameter estimation is then often the resulting problem.

In our case we may apply an implicit estimation approach. The idea is to use equation $r = \log M(1, 1; \theta)$ (see (36)) which fixes the martingale setting, but allows for many solutions in θ , i.e. for many martingale measures. Now we let the market data itself choose the appropriate solution. Denote by $\hat{C}_i = \hat{C}(S_i, K_i, T_i, r_i)$ the quoted price of a European call option with strike K_i and maturity T_i , whereas the actual price of the underlying stock is given by S_i and the current interest rate is r_i . Denote by $C_i(\theta)$ the price of the same option under a hyperbolic martingale measure, i.e. a martingale measure which is constructed from the hyperbolic distribution with parameter vector θ . The price of an European call option is in this case given by

$$C_i(\theta) = e^{-r_i T_i} \int (S_i e^x - K_i)_+ f_{T_i}(x; \theta) dx \quad (15)$$

Then we solve the following optimization problem for a given sample of option values $\{\hat{C}_i \mid i \leq N\}$ observed at the secondary market

$$\min_{\theta} \sum_{i=1}^N (C_i(\theta) - \hat{C}_i)^2 \quad \text{over all } \theta \in \Theta, \quad (16)$$

where θ has to satisfy the restriction $r = \log M_{\theta}(1)$ and the usual parameter restrictions. In our case the parameter space Θ is a subset of \mathbb{R}^4 given by

$$\Theta = \left\{ \theta = (\alpha, \beta, \delta, \mu) \mid \alpha > 0, |\beta| < \alpha, \delta > 0, \right. \\ \left. \mu = r - \left(\log \frac{K_1 (\delta \sqrt{\alpha^2 - (\beta + 1)^2})}{K_1 (\delta \sqrt{\alpha^2 - \beta^2})} - \frac{1}{2} \log \frac{\alpha^2 - (\beta + 1)^2}{\alpha^2 - \beta^2} \right) \right\}. \quad (17)$$

The solution of the optimization problem (16) provides us with a martingale measure which is optimal in the sense that it minimizes the (Euclidian) distance of the prices derived from it to the actual prices observed at the market over a given period of time. Hence we construct an option value process from a statistical point of view. The prices obtained by this approach are free of arbitrage opportunities, because the model is build up already in the risk-neutral setting.

Problem (16) is computationally very demanding, because usually we have a large number of quoted option contracts at hand. For example for Daimler Benz intraday quotes of option prices from July 1, 1992 to August 10, 1994 we obtain 63015 observations, for Bayer we have 21157 quotes in the same period. We give a graphical comparison between the underlying

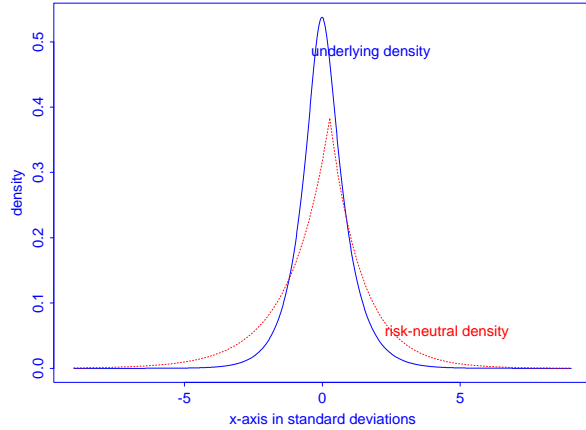


Figure 11: *Fitted densities for Bayer.*

(Maximum-Likelihood) and the risk-neutral density of X_1 . These densities define the measures P and Q .

We compare now skewness and kurtosis of the daily returns X_1 for the estimates corresponding to the different methods. Skewness is defined as $\gamma_1 = m_3 m_2^{-3/2}$ and kurtosis is defined as $\gamma_2 = m_4 m_2^{-2} - 3$, where m_k denotes the k -th central moment of the corresponding distribution. With the normal distribution one would obtain $\gamma_1 = 0$ and $\gamma_2 = 0$. In Eberlein, Keller (1995) the empirical estimates of γ_1 and γ_2 were found significantly different from zero. In particular the kurtosis values were much higher than those for the normal model. We provide the values of the two variables for the hyperbolic distribution given the two sets of parameter estimates in table VI.

Table VI: *Kurtosis and skewness.*

	$\gamma_2^{\text{underlying}}$	$\gamma_2^{\text{risk-neutral}}$	$\gamma_1^{\text{underlying}}$	$\gamma_1^{\text{risk-neutral}}$
Bayer	1.62	3.22	-0.06	-0.82
Daimler Benz	2.2	3	0.27	-0.11
Deutsche Bank	2.06	3.03	-0.48	0.29
Siemens	0.94	3.27	-0.45	0.9
Thyssen	1.85	3.14	-0.12	0.65

Under the optimal risk-neutral measure determined by the procedure described above the values of kurtosis increase, i.e. there is more mass in

the tails of the risk-neutral probability distribution. The values of skewness under the risk-neutral measure are all except one of sign opposite to the one under the underlying distribution. This can be explained in the following way. Under any martingale measure the discounted price process S^* is required to be a martingale. Hence in our case the return process under measure Q is skewed in the other direction than under the underlying measure P to compensate the intrinsic drift that arises from the exponential form of the price process. We suggest the following way to compare the models with reality. We compute the distance

$$d_k = \frac{1}{N} \sum_{i=1}^N |C_i(\text{model}_k) - \hat{C}_i|, \quad (18)$$

where \hat{C}_i again denotes the quoted price of an option and $C_i(\text{model}_k)$ the corresponding price under model $_k$. The number d_k represents the average price difference of model $_k$ to reality. Hyp.risk-neutral denotes the hyperbolic model where the parameters are estimated from option data as described above, Hyp.Esscher denotes the hyperbolic model introduced in Eberlein, Keller (1995) and BS denotes the Black-Scholes model where the volatility is estimated by the historical standard deviation of the returns of the stocks over the whole period.

Table VII: *Comparison of different models with reality.*

	$d_{\text{Hyp.risk-neutral}}$	$d_{\text{Hyp.Esscher}}$	d_{BS}
Bayer	0.319	0.519	0.755
Daimler Benz	0.643	0.909	1.602
Deutsche Bank	0.609	0.955	1.443
Siemens	0.421	0.588	1.196
Thyssen	0.318	0.405	0.638

Evidently the hyperbolic model fit to the quoted prices is significantly better than the fit by the standard Black-Scholes model. The decrease in distance is substantial from Black-Scholes to the hyperbolic model using Esscher transforms, and the quoted price can be fitted even better using the statistical martingale approach. Thus using the flexibility of the hyperbolic model allows for a more accurate description of market reality.

X Value at risk

Value at risk (VaR) is now a major concern of many economic practitioners involved in risk management. It has become the standard measure for the market risk linked to holding a portfolio of various assets. Value at risk is defined as the potential loss given a level of probability because of market movements. Hence the notion of VaR is essentially that of quantile. We cite here the recommendation of the Group of Thirty given in 1993 for risk assessment:

Market risk is best measured as “Value at Risk” using probability analysis based upon a common confidence interval (e.g. two standard deviations) and time horizon (e.g a one-day exposure).²

In the simple case of a linear position, e.g. holding a portfolio consisting only of stock, we give the following comparison of the hyperbolic model to the normal model. For a holding period of one day figure 12 shows the loss functions of the two models together with the one observed for the NYSE industrial index with respect to level of probability. For example the 99% VaR for the hyperbolic model is given by \$0.0184 close to the observed market value of \$0.0193. The value derived from the normal model is \$0.0153. Note, that the loss functions intersect, which is consistent with figure 1. The location of the intersection is 0.05 which is often used as the level of probability for VaR. However, it is misleading to argue that for this particular level of probability the normal model is adequate, because the point of intersection is random.

Now we look at non-linear positions. Value at risk for derivatives is the challenging feature of risk management. We compare the hyperbolic model with the Black-Scholes model. We do not apply any delta-gamma approximation for the option risk. Instead we apply full valuation of possible portfolio losses. Let us consider a portfolio consisting of one call option sold short. Then VaR is defined as

$$VaR = C(S_h, K, T - h, r) - C(S_0, K, T, r), \quad (19)$$

where S_h is determined by the 95%-quantile and h is the exposure period. Note that the option is sold short, hence the value of the portfolio declines when the value of the option rises. First, we use the Esscher approach to evaluate VaR. Second, we follow Madan, Chang (1995) and use the risk-neutral density to compute VaR, because it captures market option prices better as pointed out above.

For the Black-Scholes model we obtain the following. Denote by q_a the a -quantile of the standard normal distribution, then we get

$$S_h = S_0 e^{\sigma \sqrt{h} q_a + (r - \frac{\sigma^2}{2}) h}. \quad (20)$$

²This citation is taken from Gamrowski, Rachev (1996).

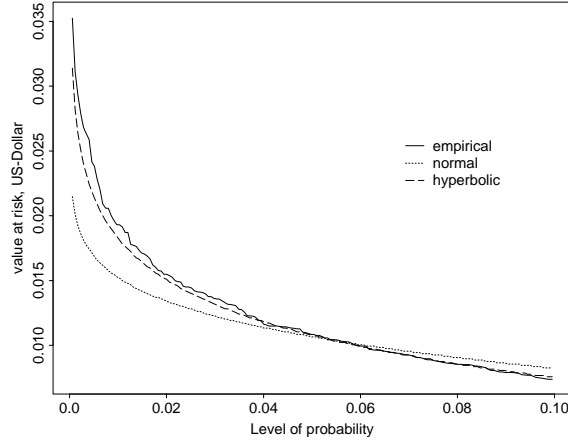


Figure 12: *Loss functions for the NYSE industrial index from 1-1990 to 11-1996.*

For the hyperbolic model $S_t = S_0 e^{X_t}$ we obtain simply

$$S_h = S_0 e^{qhyp_h(a)}, \quad (21)$$

where $qhyp_h(a)$ is the a -quantile of the distribution of X_h which arises from the estimated hyperbolic distribution by convolution.

Following the Group of Thirty recommendation we choose the exposure period $h = 1day$ and assume $a = 5\%$. Further assume that $S_0 = K = 100$ whereas the time to maturity is assumed to be $T = \frac{1}{2}year$ and the annual discount rate is given by $r = 0.08$.

Table VIII: *Value at risk.*

	VaR _{BS}	VaR _{Hyp.Esscher}	VaR _{Hyp.risk-neutral}
Bayer	1.48	1.53	2.58
Daimler Benz	1.75	1.80	2.52
Deutsche Bank	1.31	1.29	2.24
Siemens	1.21	1.21	1.89
Thyssen	1.99	2.05	3.14

Table VIII provides the following estimates of VaR for an option-portfolio risk. First, we compute VaR according to the Black-Scholes model where the volatility is estimated by the standard historical estimator over the whole

period. Second, we apply the hyperbolic model using Esscher transforms and finally we use the statistical martingale measure approach discussed above.

The Black-Scholes model yields low estimates. The values of VaR for the hyperbolic model using Esscher transforms are comparable to those of the Black-Scholes model but indicate the underestimation. Looking at the last column we conclude that the market model provides a much higher value. Portfolios with a large exposure to option risk carry a substantial amount of model risk by misspecification of the underlying model. Using a statistical setting provided by hyperbolic distributions for the option pricing model itself we find a significant underestimation of value at risk according to the standard Black-Scholes model.

XI Summary and outlook

The paper presents an extensive empirical survey of the implications of the hyperbolic model. This model provides an excellent fit for return data of financial markets and allows for a closed option pricing formula. We compare the model with the classical Black-Scholes model. Differences of option prices show the typical W-shape reported already by other authors (underpricing of the Black-Scholes model in-the-money and out-of-the-money and overpricing at-the-money).

Based on large datasets from the options exchange we computed implicit volatility surfaces with respect to moneyness and time to maturity, the so-called (3-dimensional) smile. After applying some appropriate smoothing procedures we find characteristic differences between the hyperbolic smile and the Black-Scholes smile. The W-shape of the difference shows how the misspecification of the Black-Scholes distributional assumptions affects the implicit volatility. Moreover, the W-shape hints at possible corrections for volatility prediction. We think that the remaining hyperbolic smile is an intrinsic market phenomenon.

Undoubtedly volatility is the most important parameter for option pricing, however, it is also necessary to investigate the pricing deficiencies by a mispricing analysis. The latter gives the difference between model price and the quoted price. Using various volatility estimators we obtain a reduction of the error by the hyperbolic model. We also illustrate the role of the drift parameter μ . Its choice is delicate. On one side it allows to correct Black-Scholes overpricing for options with longer times to maturity. On the other side estimating μ from an inappropriate data set can lead to an overcorrection and thus increase the pricing error. It may be necessary to readapt the drift parameter from time to time.

Zooming down into the market to single option series we illustrate the problem of finding a suitable volatility forecast. Comparing estimators de-

rived from the GARCH-model with historical estimators and those based on implicit volatilities we conclude that the latter are the best choice.

Exploiting the flexibility of the hyperbolic model even further, an implicit option value process could be extracted from the quoted prices of option transactions. This results in a substantial improvement of modelling observed option prices as compared with the standard Black-Scholes model. With this approach the market itself chooses the martingale measure leading to absence of arbitrage which is the *conditio sine qua non* for option pricing. We also obtained some insight into the implicit risk-neutral distribution of the underlying asset.

Finally, we provide some new value at risk calculations both for linear and non-linear positions. In the first case we compare the implications of the hyperbolic model to the normal model. The loss function derived from the hyperbolic model is found to be in accordance with the empirically observed one. For non-linear risk the hyperbolic model indicates a substantial underestimation of VaR by the Black-Scholes model. This implies that model risk cannot be neglected.

There is plenty of room for generalizations and refinements of this new basic model. In particular introducing stochastic volatility and stochastic interest rates will lead to further improvements. It is clear that as in the Black-Scholes setting a stochastic volatility process would have a greater effect than stochastic interest rates. Another aspect which we investigated is consistency of the model with respect to various time scales. As was pointed out in Eberlein, Keller (1995), weekly or monthly stock returns are much closer to the normal distribution than daily returns. Therefore for long term historic studies the classical geometric Brownian motion model is to a certain degree appropriate. Given the increased trading frequency of modern financial markets and the sensitivity of derivative prices, the other direction is more interesting. The shift from the floor to electronic exchanges makes high frequency data sets available. Thus, one can easily analyse intraday returns such as hourly or thirty minutes returns. Comparison of these empirical distributions with the corresponding distributions generated by our model which is based on daily data, shows that the model is highly consistent. These results as well as results on modelling term structures will appear elsewhere.

Acknowledgement

We would like to thank Deutsche Börse AG, Frankfurt, for a number of data sets concerning stock and option prices. We also thank an anonymous referee for some very useful remarks.

Appendix

A Moment generating and characteristic functions

Lemma 1 *The moment generating function of the hyperbolic distribution is given by*

$$M(u) = e^{\mu u} \frac{\sqrt{\alpha^2 - \beta^2}}{K_1(\delta\sqrt{\alpha^2 - \beta^2})} \frac{K_1(\delta\sqrt{\alpha^2 - (\beta + u)^2})}{\sqrt{\alpha^2 - (\beta + u)^2}} \quad |\beta + u| < \alpha \quad (22)$$

Proof. Because μ is a location parameter it is enough to consider the case where $\mu = 0$. Define

$$a(\alpha, \beta, \delta) = \frac{\sqrt{\alpha^2 - \beta^2}}{2\alpha\delta K_1(\delta\sqrt{\alpha^2 - \beta^2})} \quad (23)$$

then we get for $|\beta + u| < \alpha$

$$\begin{aligned} M(u) &= \int e^{ux} f_{(\alpha, \beta, \delta, 0)}(x) dx \\ &= a(\alpha, \beta, \delta) \int \exp\left(-\alpha\sqrt{\delta^2 + x^2} + (\beta + u)x\right) dx \\ &= \frac{a(\alpha, \beta, \delta)}{a(\alpha, \beta + u, \delta)} \\ &= \frac{\sqrt{\alpha^2 - \beta^2}}{K_1(\delta\sqrt{\alpha^2 - \beta^2})} \frac{K_1(\delta\sqrt{\alpha^2 - (\beta + u)^2})}{\sqrt{\alpha^2 - (\beta + u)^2}} \end{aligned} \quad (24)$$

■

Hence all moments of the hyperbolic distribution exist and we can calculate the moments of the hyperbolic Lévy motion X_t by computing the derivatives of the moment generating function. At first by stationarity and independence of the increments we obtain linearity of the moments of X_t in t

$$\begin{aligned} EX_t^k &= \frac{\partial^k}{\partial u^k} M_{X_t}(u) \Big|_{u=0} = \frac{\partial^k}{\partial u^k} (M(u))^t \Big|_{u=0} \\ &= tM^{(k)}(0) = tEX_1^k, \end{aligned}$$

because $M(0) = 1$. By computing the derivatives of M we get

$$E[X_1] = \mu + \frac{\delta\beta}{\sqrt{\alpha^2 - \beta^2}} \frac{K_2(\zeta)}{K_1(\zeta)} \quad (25)$$

and for the variance

$$Var[X_1] = \delta^2 \left(\frac{K_2(\zeta)}{\zeta K_1(\zeta)} + \frac{\beta^2}{\alpha^2 - \beta^2} \left[\frac{K_3(\zeta)}{K_1(\zeta)} - \left(\frac{K_2(\zeta)}{K_1(\zeta)} \right)^2 \right] \right), \quad (26)$$

where $\zeta = \delta\sqrt{\alpha^2 - \beta^2}$. Moreover, we obtain the following result concerning the characteristic function of the hyperbolic distribution.

Lemma 2 *The characteristic function of the hyperbolic distribution is given by*

$$\phi(u) = e^{i\mu u} \frac{\sqrt{\alpha^2 - \beta^2}}{K_1(\delta\sqrt{\alpha^2 - \beta^2})} \frac{K_1(\delta\sqrt{\alpha^2 - (\beta + iu)^2})}{\sqrt{\alpha^2 - (\beta + iu)^2}}, \quad (27)$$

Proof. According to Lemma 1 the radius of convergence of the moment generating function of the hyperbolic distribution with parameters $(\alpha, \beta, \delta, \mu)$ is given by $\alpha - \beta$. Hence with an analytic extension argument we obtain for the characteristic function $\phi(u) = Ee^{iuX_1} = M(iu)$ the desired formula. This representation is valid for all real u . ■

Thus in the symmetric centred case where $\beta = \mu = 0$ (and consequently the shape parameter $\zeta = \alpha\delta$) we get

$$\phi(u; \alpha, \delta) = \frac{\alpha}{K_1(\delta\alpha)} \frac{K_1(\delta\sqrt{\alpha^2 + u^2})}{\sqrt{\alpha^2 + u^2}}, \quad (28)$$

which is of course a real-valued function because of the symmetry. With (28) it is clear that the distribution of X_t is not closed under convolution, i.e. it is for $t \neq 1$ a law of the convolution semigroup generated by the hyperbolic distribution, but not itself a hyperbolic distribution. Hence this process is computationally demanding and we often have to rely on numerical methods. For example in the symmetric centred case the density of $\mathcal{L}(X_t)$ is given by the Fourier inversion formula

$$f_t^{\alpha, \delta}(x) = \frac{1}{\pi} \int_0^\infty \cos(ux) (\phi(u; \alpha, \delta))^t du. \quad (29)$$

Using the fast Fourier method (FFT) the integral can be computed rather efficiently in real-time. An example of such a computation is shown in figure 13, where the shape and the scale parameters were chosen as $\zeta = \delta = 1$.

B Lévy-Khintchine representation

It is well-known that infinitely divisible distributions admit the Lévy-Khintchine representation of their characteristic function given by

$$\phi(u) = \exp\left(i\mu u - \frac{c}{2}u^2 + \int (e^{iux} - 1 - iux1_{|x|\leq 1})F(dx)\right), \quad (30)$$

where μ is the drift term, c is the quadratic variation coefficient and F is a positive measure with $\int (x^2 \wedge 1)F(dx) < \infty$. In the theory of Lévy processes F is called the Lévy measure. It describes the jumps of the process. For a Brownian motion W_t with drift μ the Lévy measure vanishes because of the

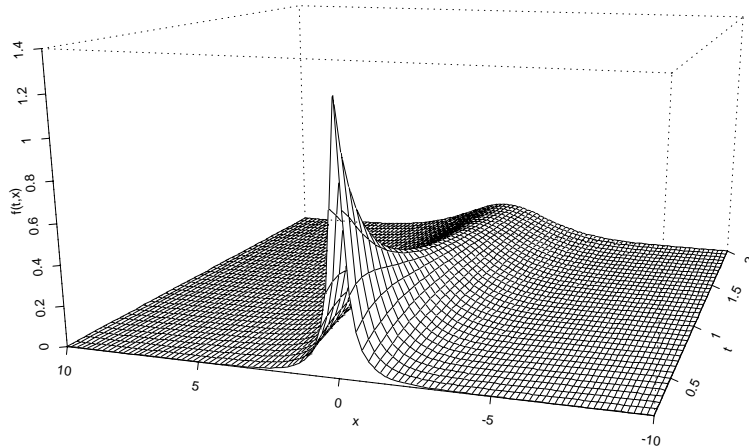


Figure 13: Convolution semigroup densities

continuous paths and the variance is given by ct . For the hyperbolic Lévy motion a lengthy computation given in Eberlein, Keller (1995) yields the following form of the Lévy-Khintchine representation of the characteristic function (28) in the symmetric centred case

$$\phi(u) = \exp \left(\int (e^{iux} - 1 - iux) g(x) dx \right), \quad (31)$$

where the density of the hyperbolic Lévy measure is given by

$$g(x) = \frac{1}{|x|} \left(\int_0^\infty \frac{e^{-\sqrt{2y+\alpha^2}|x|}}{\pi^2 y (J_1^2(\delta\sqrt{2y}) + Y_1^2(\delta\sqrt{2y}))} dy + e^{-\alpha|x|} \right), \quad (32)$$

where J_1 and Y_1 are Bessel functions. From this representation it can be seen that the hyperbolic Lévy motion is a purely discontinuous process because there is no continuous part, i.e. $c = 0$. Using the asymptotics of the various Bessel functions one can deduce that $g(x) \sim \frac{1}{x^2}$ for $x \rightarrow 0$, hence every path of this process has infinitely many jumps in any finite time interval (see Breiman (1968) chapter 14). However, the magnitude of the jumps is such that the process is integrable, which can be deduced from the existence of the moment generating function. This is in contrast to the α -stable Lévy motion, which is also a pure jump process.

C The price measure

For option pricing purposes we need an equivalent martingale measure, i.e. a measure P^θ which is equivalent to the underlying measure P such that the discounted price process $S_t^* = e^{-rt}S_t$ is a martingale. Under our choice of P^θ the return process X_t will again be a hyperbolic process, but now with different parameters. Let us mention that this measure can also be justified by an equilibrium approach.

In the symmetric centred case we proceed as follows. Let f_t be the density of $\mathcal{L}(X_t)$. For some real number θ we can define a new density

$$f_t(x; \theta) = \frac{e^{\theta x} f_t(x)}{M(\theta)^t} \quad (33)$$

Now we choose θ by

$$S_0 = e^{-rt} E^\theta[S_t]. \quad (34)$$

Consequently $S_t^* = e^{-rt}S_t$ is a martingale. Under the corresponding probability P^θ the process is again a Lévy process, which is called the Esscher transform of the original process.

Consider the moment generating function under P^θ

$$M(u, t; \theta) = \int_{-\infty}^{\infty} e^{ux} f_t(x; \theta) dx. \quad (35)$$

Since by stationarity $M(u, t; \theta) = M(u, 1; \theta)^t$ we get from (34)

$$e^r = M(1, 1; \theta) = \frac{M(\theta + 1)}{M(\theta)}. \quad (36)$$

In the symmetric centred case we obtain with $\zeta = \delta\alpha$

$$M(u) = \frac{\zeta}{K_1(\zeta)} \frac{K_1\left(\frac{\sqrt{\zeta^2 - \delta^2 u^2}}{\zeta}\right)}{\sqrt{\zeta^2 - \delta^2 u^2}} \quad \left(|u| < \frac{\zeta}{\delta}\right). \quad (37)$$

Introducing this in (36) we get the value θ^* which defines the martingale measure as the solution of

$$r = \ln \frac{K_1\left(\frac{\sqrt{\zeta^2 - \delta^2(\theta + 1)^2}}{\zeta}\right)}{K_1\left(\frac{\sqrt{\zeta^2 - \delta^2\theta^2}}{\zeta}\right)} - \frac{1}{2} \ln \frac{\zeta^2 - \delta^2(\theta + 1)^2}{\zeta^2 - \delta^2\theta^2}. \quad (38)$$

By numerical methods we find a solution for θ given the (daily) interest rate r and the parameters δ and ζ .

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