



Non-Linear Value-at-Risk ^{*}

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Abstract. Value-at-risk methods which employ a linear (“delta only”) approximation to the relation between instrument values and the underlying risk factors are unlikely to be robust when applied to portfolios containing non-linear contracts such as options. The most widely used alternative to the delta-only approach involves revaluing each contract for a large number of simulated values of the underlying factors. In this paper we explore an alternative approach which uses a quadratic approximation to the relation between asset values and the risk factors. This method (i) is likely to be better adapted than the linear method to the problem of assessing risk in portfolios containing non-linear assets, (ii) is less computationally intensive than simulation using full-revaluation and (iii) in common with the delta-only method, operates at the level of portfolio characteristics (deltas and gammas) rather than individual instruments.

1. Introduction

Consider a portfolio consisting of quantities $\mathbf{x} = (x_1, x_2, \dots, x_n)'$ of assets 1, 2, ..., n with time t values $\mathbf{v} = (v_1, v_2, \dots, v_n)'$.¹ Then the change in the price of the portfolio, V , over the next interval Δt is given by:

$$\Delta V = \sum_{i=1}^n x_i \Delta v_i, \quad (1.1)$$

where Δv_i (ΔV) denotes the change in the value of asset i (the portfolio) over the interval t to $t + \Delta t$. The value-at-risk of the portfolio \mathbf{x} for some defined probability level α , is defined as the level of loss, $\Delta V^*(\alpha)$, such that the probability that $\Delta V \leq \Delta V^*$ is equal to α . When the joint distribution of the change in asset values can be taken as multivariate normal with a known mean and variance, the calculation of ΔV^* is straightforward. In many cases, however, and particularly when some of the assets are options, the assumption of multivariate normality will be inappropriate, even when appropriate for the underlying rates and prices.

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¹ All the analysis in this paper takes place over the interval t to $t + \Delta t$. We therefore dispense with a time subscript except where necessary.

In this case one of two approaches is typically employed. In the first, the so-called “delta-only” method, the non-linear relation between asset values and the underlying rates and prices is replaced by a linear approximation based on each asset’s “delta”. Assume that the value of each asset i depends on time and K “factors”, $\mathbf{f} = \{f_1, f_2, \dots, f_K\}$. Then $\Delta^\delta V$, the first order approximation to ΔV , is given by:

$$\Delta^\delta V = \sum_{i=1}^n x_i \frac{\partial v_i(\mathbf{f}, t)}{\partial t} \Delta t + \sum_{i=1}^n x_i \sum_{k=1}^K \frac{\partial v_i(\mathbf{f}, t)}{\partial f_k} \Delta f_k \quad (1.2)$$

$$\equiv \mu_p + \sum_{k=1}^K \delta_k \Delta f_k, \quad (1.3)$$

where μ_p is the change in portfolio value resulting from the passage of time:

$$\mu_p = \sum_{i=1}^n x_i \frac{\partial v_i(f)}{\partial t} \Delta t, \quad (1.4)$$

and δ_k , the aggregate delta, is given by:

$$\delta_k = \sum_{i=1}^n x_i \frac{\partial v_i(f)}{\partial f_k}. \quad (1.5)$$

Assuming \mathbf{f} has a multivariate normal distribution with known parameters, $\Delta^\delta V$ has a univariate normal distribution and the mean and variance of $\Delta^\delta V$, and from this the value at risk, are easily computed.

The most widely used alternative to the delta-only approach involves revaluing each contract for a large number of simulated values of the underlying factors. In a portfolio with many instruments, this procedure may be computationally expensive or even infeasible.

In this paper we propose an alternative approach in which the change in value of an asset is approximated, not as a *linear* function of the underlying factors but as a *linear-quadratic* function. In many cases this provides a better approximation to the true distribution than the delta-only approach while being substantially less resource intensive than full valuation. Including a quadratic term in the approximation to ΔV implies using the *gamma* of an option as well as the delta. This approach is also discussed in Wilson (1994), Fallon (1996), Rouvinez (1997) and Jahel, Perraudin and Sellin (1997) as well as in previous work by the authors (Britten-Jones and Schaefer, 1995). Rouvinez uses Imhof’s (1961) numerical technique to invert the characteristic function of the quadratic approximation and so recover the exact distribution. Jahel et. al. also use characteristic functions to compute the moments of the approximation and then fit a parametric distribution to the moments. Fallon’s approach, like ours, uses an approximation to the distribution derived from the

moments. An extensive review of alternative approaches to the calculation of VaR is given in Duffie and Pan (1997).

Wilson (1994) also uses a linear-quadratic approximation but the statistic he derives – “capital-at-risk” [CAR] – differs significantly from the standard definition of VaR. In fact, Wilson’s CAR is *always conservative* with respect to VaR. The relation between CAR and VaR is discussed in the Appendix.

The paper is organised as follows. In Section 2 we deal with the univariate case and show how, by completing the square, we may derive the exact distribution of a quadratic approximation to the relation between asset value and the underlying rates and prices. Section 2 also includes some examples comparing VaR calculated under linear and linear-quadratic approximations to VaR under full revaluation. Section 3 deals with a special case where the relation between VaR’s under delta-only and delta-gamma approximations can be characterised in terms of a single parameter involving the delta and gamma of the position and the volatility of the underlying factor. Section 4 discusses the multivariate case and shows how the distribution of a quadratic approximation to portfolio value may be itself approximated. Section 5 gives an example which uses the approximation described in Section 4 to compute VaR in the multivariate case. Section 6 concludes.

2. The Univariate Case

Consider the portfolio $\mathbf{x} = (x_1, x_2, \dots, x_n)'$ defined earlier and assume that the number of underlying prices or rates on which the values of the assets depend is unity ($K = 1$). We now introduce, $\Delta^\gamma v_i$, a quadratic approximation to Δv_i , the change in value of the i th asset:

$$\Delta^\gamma v_i = \frac{\partial v_i}{\partial t} \Delta t + \frac{\partial v_i}{\partial f} \Delta f + \frac{1}{2} \frac{\partial^2 v_i}{\partial f^2} (\Delta f)^2 \quad (2.6)$$

$$\equiv \mu_i + \delta_i \Delta f + \frac{1}{2} \gamma_i (\Delta f)^2, \quad (2.7)$$

where δ_i and γ_i are, respectively, the delta and gamma of asset i with respect to the factor f . As before, the term μ_i captures the first order effect of the change in value due to the passage of time:

$$\mu_i = \frac{\partial v_i}{\partial t} \Delta t.$$

The corresponding delta-gamma approximation² to the change in value of the portfolio is given by:

² The approximation is quadratic in the factors, but only linear in time, as we only include the first order effect of time. This is justifiable, since in most cases, the passage of time, by itself, does not result in large value changes.

$$\Delta^\gamma V = \sum_{i=1}^n x_i \mu_i + \left(\sum_{i=1}^n x_i \delta_i \right) \Delta f + \frac{1}{2} \left(\sum_{i=1}^n x_i \gamma_i \right) (\Delta f)^2 \quad (2.8)$$

$$\equiv \mu_p + \delta \Delta f + \frac{1}{2} \gamma (\Delta f)^2, \quad (2.9)$$

where $\mu_p = \sum_{i=1}^n x_i \mu_i$, $\delta = \sum_{i=1}^n x_i \delta_i$, and $\gamma = \sum_{i=1}^n x_i \gamma_i$. This is to be contrasted with the standard (delta-only) linear approximation:

$$\Delta^\delta V = \mu_p + \delta \Delta f. \quad (2.10)$$

We assume throughout that the underlying factors are normally distributed; thus here we assume:

$$\Delta f \sim N(\mu_f, \sigma_f^2),$$

where μ_f and σ_f are, respectively, the mean and standard deviation of Δf . Thus, in Equation (2.8) the term which is linear in Δf is normally distributed and the quadratic term is distributed as a non-central chi-squared. The problem of characterising the distribution of $\Delta^\gamma V$, the delta-gamma approximation to ΔV , is therefore that of characterising the distribution of a sum of a normal variate and a non-central chi-squared variate. Fortunately, this problem is easily solved if we simply complete the square in Equation (2.8). Thus:

$$\Delta^\gamma V = \mu_p + \delta \Delta f + \frac{1}{2} \gamma (\Delta f)^2 = \mu_p^* + \frac{1}{2} \gamma (e + \Delta f)^2, \quad (2.11)$$

where:

$$e = \frac{\delta}{\gamma} \text{ and } \mu_p^* = \mu_p - \frac{1}{2} \frac{\delta^2}{\gamma}. \quad (2.12)$$

Since Δf is normally distributed, $(e + \Delta f)$ is also normal with mean $e + \mu_f$ and variance σ_f^2 . It follows that:

$$\frac{\Delta^\gamma V - \mu_p^*}{\gamma \sigma_f^2 / 2} = \left(\frac{e + \Delta f}{\sigma_f} \right)^2 \equiv w \sim \text{non-central } \chi^2 : \nu, d \quad (2.13)$$

where

$$\nu = 1 \text{ and } d = \left(\frac{e + \mu_f}{\sigma_f} \right)^2.$$

Thus, under the delta-gamma approximation, the value-at-risk of the portfolio may be computed directly from the cdf. of the non-central chi-squared distribution

defined in Equation (2.13). For a given probability level, say α , let $w^*(\alpha)$ be such that:

$$\Pr(w \leq w^*(\alpha)) = \alpha.$$

Then the value at risk, for probability level α is $\Delta^\gamma V^*(\alpha)$, where:

$$\Delta^\gamma V^*(\alpha) = \mu_p^* + \frac{1}{2}\gamma\sigma_f^2 w^*(\alpha).$$

Including the gamma term in the approximation for the change in portfolio value, changes the form of the distribution from normal to a translated non-central chi-squared. The materiality of the difference in value-at-risk computed under these two approximations depends on the relation between the portfolio delta and gamma. If delta is large and gamma small then almost all the risk of the portfolio will be associated with directional, or delta, risk in the underlying. In this case the delta-only approximation will provide almost the same results as the delta-gamma. If delta is small and gamma large then little of the risk will be associated with directional changes in the underlying and the delta-only approximation will be poor.

The effect of the relative size of delta and gamma on the relation between VaR under the delta-only and delta-gamma approximations is discussed further in Section 3.

2.1. AN EXAMPLE

Table 2.1 illustrates the effect on VaR of gamma and the time interval over which the distribution is assessed. The Table summarises the positions of a number of hypothetical portfolios and their corresponding deltas and gammas.

Each portfolio combines a put and a call and thus has a lower delta and a higher gamma than a similar position in either one type of option or the other.

For portfolio 1, VaR is assessed over a short – one day – interval; here the gamma effect is relatively unimportant. Portfolio 2 has a lower delta and a higher gamma (both in absolute terms) and VaR is assessed over a longer period (one week). Thus we expect that a delta-only approach will perform relatively less well for portfolio 2 than portfolio 1. Finally, portfolio 3 is identical to portfolio 1 but here VaR is assessed over a 10-day interval, as the Basle Committee have recently proposed.³ Even though gamma and delta in this case are the same as those for portfolio 1, the effect of gamma will be greater as a result of the longer interval over which VaR is assessed.

The calculations described below show that, particularly for portfolios 2 and 3, VaR calculated using a delta-gamma approach is significantly more accurate than a delta-only approach. But first it is useful to compare the linear and quadratic approximations (Equations (1.2) and (2.8) respectively) to the “true” (Black-Scholes) relation between portfolio value and the underlying.

³ See Basle Committee on Banking Supervision (1996).

Table I. Portfolio positions used in Example

Price of the underlying asset is 100 with assumed volatility level of 30% p.a. The (annually compounded) riskless interest rate is 10% and the strike price for each option is 101. We assume a zero dividend yield

Portfolio	1	2	3
Maturity of options (days)	60	42	60
Quantity of calls	-1.0	-1.0	-1.0
Quantity of puts	-0.5	-0.6	-0.5
Time horizon	1 day	1 week	10 days
Aggregate delta	-0.314	-0.239	-0.314
Aggregate gamma	-0.049	-0.063	-0.049

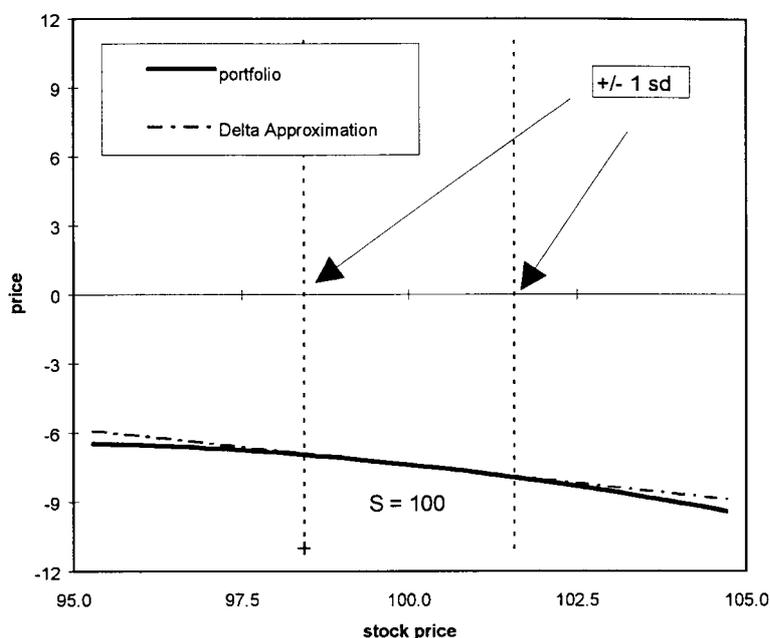


Figure 1. Relation between value of portfolio 1 and price of underlying stock using (i) “true” (Black-Scholes) value and (ii) delta-approximation.

Figure 1 shows the “true” value of portfolio 1 at the end of one day as a function of the underlying asset price and also the value under the linear approximation. The vertical dotted lines show ± 1 standard deviation limits for the underlying asset price, S . We see that over the majority of the range of likely values for S the linear approximation is quite close to the true value. In this case, therefore, we would not expect to find that a VaR measure based on a quadratic approximation, performs much better than a linear approximation. In Figure 2 we see that this is not the

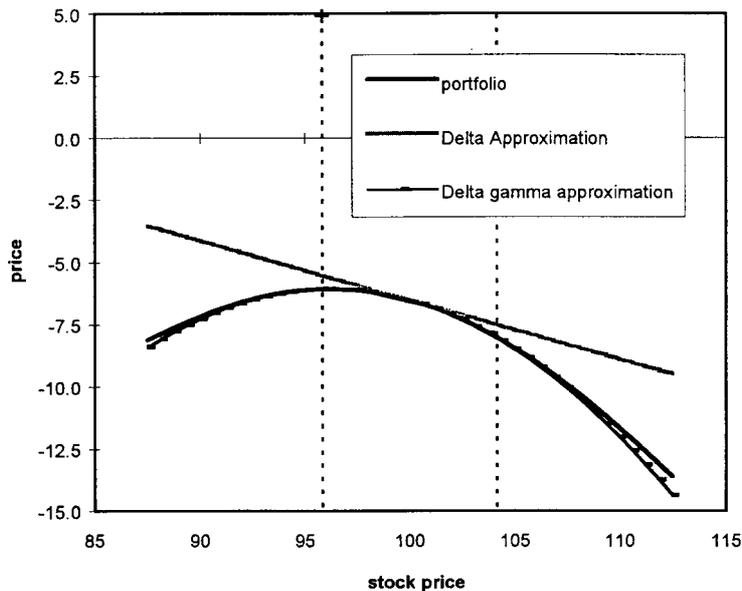


Figure 2. Relation between value of portfolio 2 and price of underlying stock using (i) “true” (Black-Scholes) value, (ii) delta-approximation and (iii) delta-gamma approximation.

case for portfolio 2, which has a higher gamma and a lower delta. Here, for a one week horizon, a quadratic relation provides a much better approximation to the true value. Thus we expect to find that, for portfolio 2, VaR based on a quadratic approximation performs better than a linear approximation.

Figure 3 shows the cumulative distribution of change in value for portfolio 1 using (a) complete revaluation under Black Scholes, (b) a linear (delta-only) approximation and (c) a quadratic (delta-gamma) approximation. In this case, for the reasons observed earlier, the difference between the alternative approaches is small. Figure 4 shows the corresponding results for portfolio 2. Once again, the results correspond closely to those expected from Figure 2: the distribution of the quadratic approximation is much closer to that of the true value than the delta-only approximation. Figure 5 shows the results for example 3 which is identical to example 1 except that the distribution of value is assessed over 10 days rather than one. Here we find, as in example 2, that the quadratic approximation performs much better than the linear approximation.

These examples illustrate three main points. First, that when gamma is large relative to delta, the linear approximation may perform quite poorly. Second, in these cases the VaR based on a quadratic approximation may provide a much better estimate of the true value, and third, that even when a linear approximation works quite well over short periods, it may perform poorly over longer periods.

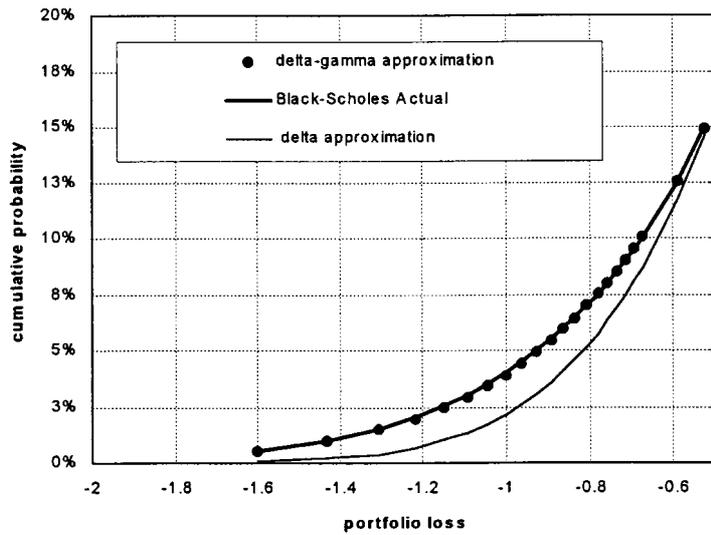


Figure 3. The distribution of value change for portfolio 1 (1 day horizon) under (i) “true” (Black-Scholes) value, (ii) delta-approximation and (iii) delta-gamma approximation.

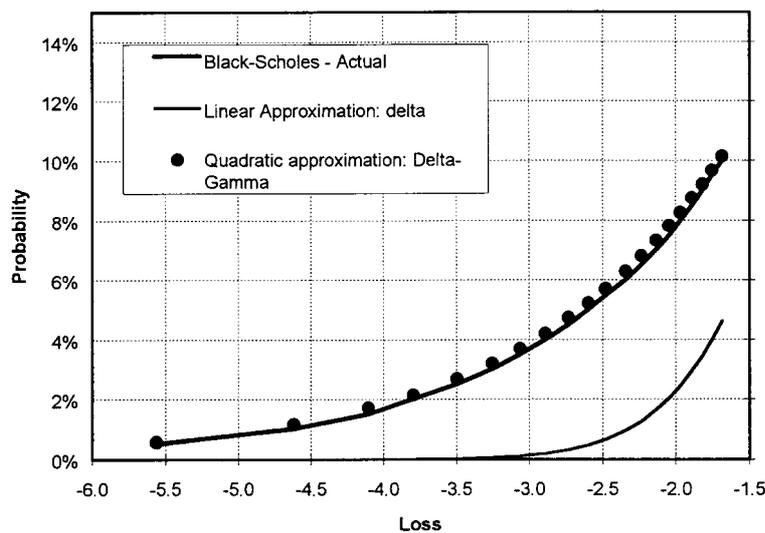


Figure 4. The distribution of value change for portfolio 2 (1 week horizon) under (i) “true” (Black-Scholes) value, (ii) delta-approximation and (iii) delta-gamma approximation.

2.2. POSITIONS WHICH ARE BOTH CONCAVE AND CONVEX IN THE UNDERLYING ASSET

Finally we give an example where the delta-gamma method does not perform well. In the examples discussed earlier the portfolio was convex or concave over the likely range of prices of the underlying. In these cases the portfolio value was

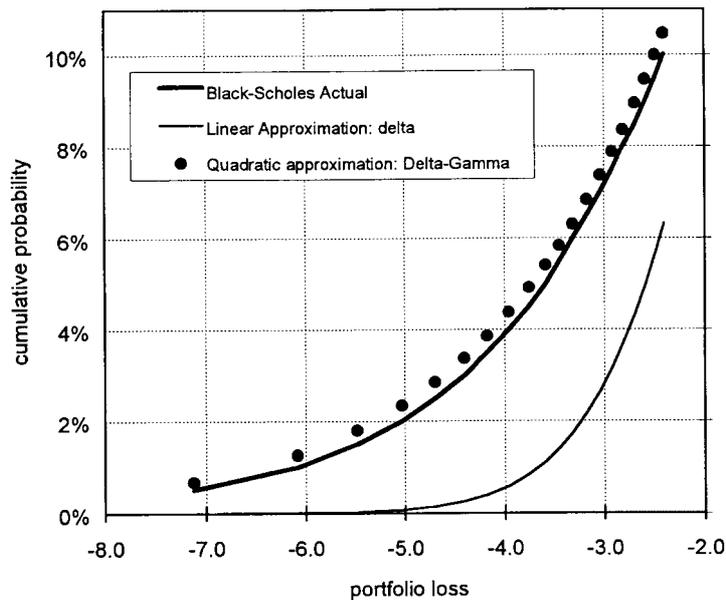


Figure 5. The distribution of value change for portfolio 3 (as portfolio 1 but with 10 day horizon) under (i) “true” (Black-Scholes) value, (ii) delta-approximation and (iii) delta-gamma approximation.

quite well approximated by a quadratic function of the price of the underlying and, consequently, the delta-gamma performed well.

For many option positions, however, the portfolio value may have both convex and concave regions in the range of likely prices of the underlying. Here, a quadratic approximation may not provide a good fit and, in this event, the delta-gamma method is likely to be unreliable.

Table 2.2 shows a position in which a put and a call are sold with a strike price of 95 and a call is bought with a strike of 105. As Figure 6 shows, although the actual position experiences substantial losses when the stock price falls, the delta-gamma approximation is convex and the maximum loss in this case (approximately 0.6) occurs when the stock price rises. As a result, as Figure 7 shows, the distribution of losses under the delta-gamma approach provides a very poor approximation to the distribution of actual portfolio value. Indeed, in this particular case, the delta approach performs much better.

3. The Relation between VaR under Delta-Only and Delta-Gamma Approximations: A Special Case

In the special case where: (i) the expected change in the underlying factor, μ_f , is zero and (ii) VaR is measured relative to the expected change in value of the portfolio, the ratio between VaR computed under the delta and delta-gamma ap-

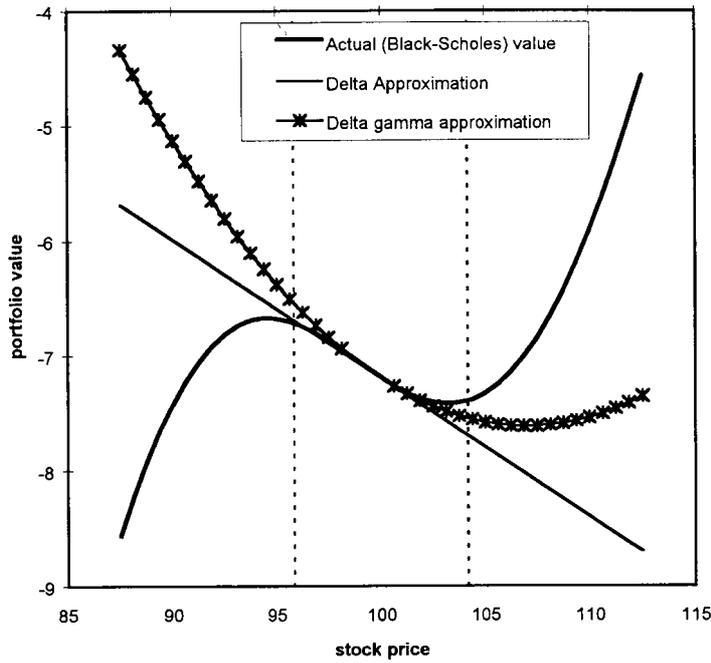


Figure 6. Portfolio value in the case where the relation between the portfolio value and the price of the underlying stock case is neither convex nor concave.

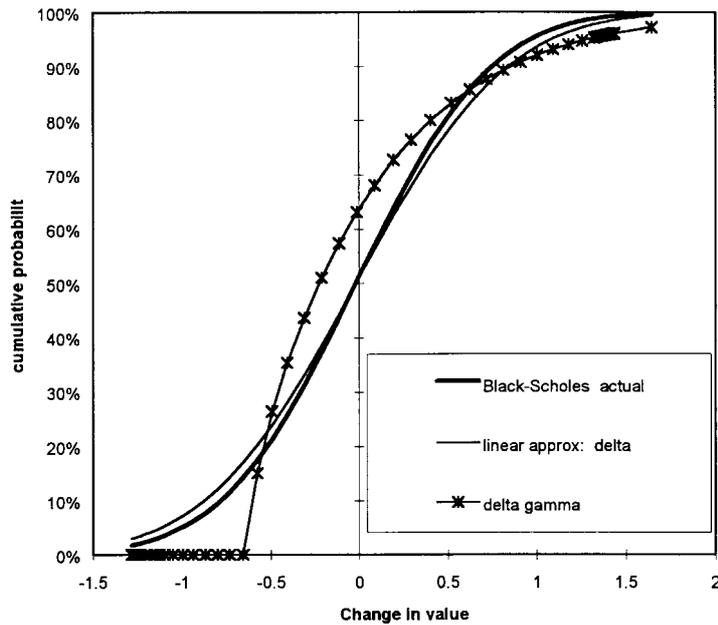


Figure 7. Distribution of portfolio value under (i) full revaluation, (ii) delta only approximation and (iii) delta-gamma approximation when the portfolio is neither convex nor concave in the value of the underlying.

Table II. Portfolio which is both convex and concave in the underlying

Type	Put	Call	Call
Strike	95	95	105
Time to maturity (weeks)	4	4	4
Quantity	-1.0	-1.5	+2.5

proximations depends on a single parameter, θ , which is a function of the portfolio delta and gamma and the volatility of the underlying asset.⁴

The delta-only approximation to value is given by Equation (1.2):

$$\Delta^\delta V = \mu_\delta + \delta \Delta f,$$

and, under assumptions (i) and (ii) above, the corresponding delta-only VaR at probability level α , $\text{VaR}_\delta(\alpha)$ is given by:

$$\text{VaR}_\delta(\alpha) = \phi^{-1}(\alpha) \delta \sigma_f \quad (3.14)$$

where $\phi^{-1}(\alpha)$ is the α^{th} quantile of the normal distribution and σ_f is the standard deviation of Δf over the interval t to $t + \Delta t$.

The change in value under the corresponding delta-gamma approximation is given by Equation (2.11) which, using the values for e and $\mu_{p\gamma}^*$ in Equation (2.12), becomes:

$$\Delta^\gamma V = \mu_\gamma + \delta \Delta f + \frac{1}{2} \gamma (\Delta f)^2 = \mu_\gamma - \frac{1}{2} \frac{\delta^2}{\gamma} + \frac{1}{2} \gamma \sigma^2 \left(\frac{\delta}{\sigma \gamma} + z \right)^2, \quad (3.15)$$

where:

$$z \equiv \frac{\Delta f}{\sigma} \sim N(0, 1).$$

Setting $E(\Delta^\gamma V) = E(\Delta^\delta V)$ in Equation (3.15) we have that:

$$\mu_\gamma = \mu_\delta - \frac{1}{2} \gamma \sigma^2,$$

and (3.15) therefore becomes:

$$\Delta^\gamma V = \mu_\delta - \frac{1}{2} \gamma \sigma^2 - \frac{1}{2} \frac{\delta^2}{\gamma} + \frac{1}{2} \gamma \sigma^2 \left(\frac{\delta}{\sigma \gamma} + z \right)^2,$$

⁴ In this section we suppress, for clarity, the subscript p denoting “portfolio”.

We now compute the deviation of $\Delta^\gamma V$ from its mean, μ_δ , in “units” of the delta-only VaR. Defining:

$$\Delta V_{\delta-g}^* \equiv \frac{\Delta^g V - \mu_\delta}{\text{VaR}_\delta(\alpha)},$$

and using Equation (3.14) we have:

$$\Delta V_{\delta-\gamma}^* = -\frac{1}{2} \frac{\gamma\sigma}{\phi^{-1}(\alpha)\delta} - \frac{1}{2} \frac{\delta}{\gamma\phi^{-1}(\alpha)\sigma} + \frac{1}{2} \frac{\gamma\sigma}{\phi^{-1}(\alpha)\delta} \left(\frac{\delta}{\sigma\gamma} + z\right)^2 \quad (3.16)$$

$$= \frac{1}{2\phi^{-1}(\alpha)} \left[-\frac{1}{\theta} - \theta + \frac{1}{\theta}(\theta + z)^2 \right]. \quad (3.17)$$

where:

$$\theta = \frac{\delta}{\sigma\gamma}. \quad (3.18)$$

From Equation (3.17) we have that:

$$2\phi^{-1}(\alpha)\Delta V_{\delta-\gamma}^*\theta + (1 + \theta^2) = (\theta + z)^2 \equiv \tilde{w} \sim \chi_{nc}^2(1, d),$$

where:

$$d = \theta^2.$$

Let $\bar{w}(\alpha)$ be such that:

$$\Pr(\tilde{w} \leq \bar{w}(\alpha)) = \alpha,$$

then:

$$\Pr(2\phi^{-1}(\alpha)\Delta V_{\delta-\gamma}^*\theta + (1 + \theta^2) \leq \bar{w}(\alpha)) = \alpha,$$

or:

$$\Pr\left(\Delta V_{\delta-\gamma}^* \leq \frac{\bar{w}(\alpha) - (1 + \theta^2)}{2\phi^{-1}(\alpha)\theta}\right) = \alpha. \quad (3.19)$$

The right hand side of the inequality in 3.16 is the ratio of the VaR under a delta-gamma approximation to the delta VaR and, as Equation (3.20) shows, for a given probability level, α is a function of $\theta = \delta/\sigma\gamma$:

$$\frac{\Delta^\gamma V^*(\alpha)}{\Delta^\delta V(\alpha)} = \frac{\bar{w}(\alpha) - (1 + \theta^2)}{2\phi^{-1}(\alpha)\theta}. \quad (3.20)$$

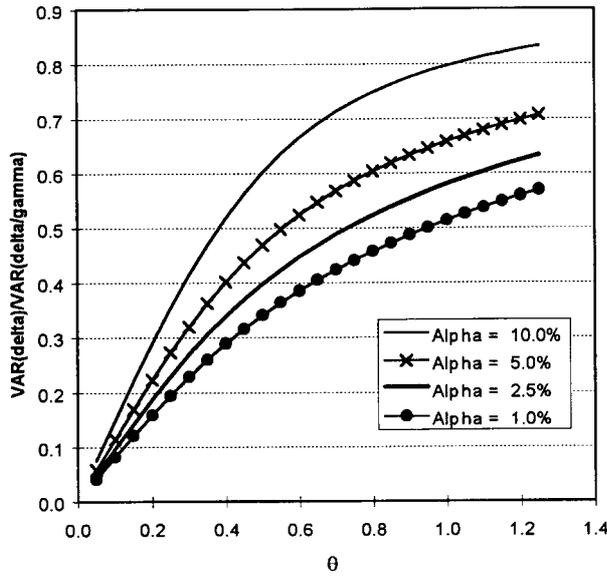


Figure 8. The figure shows the ratio of VAR under the delta-only approximation to VAR under the delta-gamma approximation as a function of θ , defined as $\theta = \frac{\delta}{\sigma\gamma}$.

Figure 8 shows the ratio of the delta VaR to the delta-gamma VaR, the *inverse* of the expression in 3.20, for values of α of 1%, 2.5%, 5% and 10% and for a range of values of θ . As θ tends to zero, delta risk constitutes a diminishing fraction of total portfolio risk and $\Delta^\delta V(\alpha)/\Delta^\gamma V^*(\alpha)$ tends to zero. As θ increases $\Delta^\delta V(\alpha)/\Delta^\gamma V^*(\alpha)$ increases although, at the 99th percentile, its value is less than 0.5 for values of θ less than 0.95.⁵

3.1. THE “GAMMA ADJUSTED” DELTA METHOD

One approach incorporating the effect of gamma risk is to use the delta method but to adjust the delta (or the standard deviation of the underlying factor) to reflect the increase in volatility created by exposure to gamma risk.⁶ Using Equations (2.9) and (3.18) we may show that the standard deviation of the delta-gamma approximation in the univariate case is:

$$\sigma(\Delta^\gamma V) = \delta\sigma_f \left(1 + \frac{1}{2\theta^2}\right)^{1/2} \equiv \delta'\sigma_f,$$

where δ' , the “adjusted delta” is given by:

$$\delta' \equiv \delta \left(1 + \frac{1}{2\theta^2}\right)^{1/2}.$$

⁵ The values of θ for the three portfolios analysed earlier are, respectively, 3.42, 0.91 and 1.08.

⁶ See, e.g., Duffie and Pan (1997).

Using this approach the “gamma adjusted” VaR for probability level α is therefore given by:

$$k(\alpha)\delta\sigma_f \left(1 + \frac{1}{2\theta^2}\right)^{1/2},$$

where $k(\alpha)$ is as defined earlier. Now, following the analysis of the previous section we may now show that, $\text{VaR}_\gamma(\alpha)/\text{VaR}_\delta(\alpha)$, the VaR under the delta-gamma approximation *measured in units of the “gamma adjusted” delta VaR* is given by:

$$\frac{\text{VaR}_\gamma(\alpha)}{\text{VaR}_\delta(\alpha)} = \frac{\bar{w}(\alpha) - (1 + \theta^2)}{\sqrt{2}k(\alpha) (1 + 2\theta^2)^{1/2}}.$$

The inverse of this ratio is plotted against θ in Figure 7. Although the ratio is much closer to one for low values of θ than in the case of the delta-only approximation the deviations from unity remain significant. As in the case of the delta-only approximation, a major problem with the gamma adjusted delta approach is that the understatement of VaR (overstatement in some case) varies significantly with the probability level. Thus, for $\theta = 1$, for example, the ratio varies from 0.63 for $\alpha = 1\%$ to 0.98 for $\alpha = 10\%$. Thus it is not possible to find an “adjustment factor” which would correct the bias in the gamma adjusted VaR for different probability levels.

4. The Multivariate Case

The multivariate case is parallel to the univariate case considered earlier except that the distribution of the delta-gamma approximation now involves the sum of non-central chi-square variates. We assume that the vector of changes in the underlying factors follows a multivariate normal with mean $\boldsymbol{\mu}_f$ and covariance matrix $\boldsymbol{\Sigma}$:

$$\Delta\mathbf{f} \sim N_n(\boldsymbol{\mu}_f, \boldsymbol{\Sigma}). \quad (4.21)$$

The quadratic approximation to ΔV is obtained, as before, as a second order Taylor series expansion of the portfolio value:

$$\Delta V^\gamma \equiv \mu + \frac{\partial V'}{\partial f} \Delta\mathbf{f} + \frac{1}{2} \Delta\mathbf{f}' \frac{\partial^2 V}{\partial f \partial f'} \Delta\mathbf{f}. \quad (4.22)$$

where μ is the change in value resulting from the passage of time. Now we assume that the first and second derivatives of portfolio value with respect to stock prices have been calculated from the deltas and gammas of the individual instruments. We thus assume that aggregate delta:

$$\boldsymbol{\delta} = \frac{\partial V}{\partial f} = \begin{pmatrix} \frac{\partial V}{\partial f_1} \\ \vdots \\ \frac{\partial V}{\partial f_K} \end{pmatrix} \quad (4.23)$$

and aggregate gamma,

$$\mathbf{\Gamma} = \frac{\partial^2 V}{\partial f \partial f'} = \begin{pmatrix} \frac{\partial^2 V}{\partial f_1 \partial f_1} & \cdots & \frac{\partial^2 V}{\partial f_1 \partial f_K} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 V}{\partial f_K \partial f_1} & \cdots & \frac{\partial^2 V}{\partial f_K \partial f_K} \end{pmatrix} \quad (4.24)$$

can be calculated.

Portfolio value is thus approximated by a quadratic function of normally distributed variables. In matrix notation we write this as

$$\Delta V^\gamma \equiv \mu_\gamma + \boldsymbol{\delta}' \Delta \mathbf{f} + \frac{1}{2} \Delta \mathbf{f}' \mathbf{\Gamma} \Delta \mathbf{f}.$$

Before examining the complete probability distribution of this quadratic function, we first calculate its volatility.

4.1. PORTFOLIO VOLATILITY

Portfolio value V has a variance approximately equal to the sum of the variances of the delta and gamma terms plus the covariance between the delta and gamma terms. In this section we assume that the factor changes have mean zero, i.e. $\mathbf{f} \sim N_n(\mathbf{0}, \boldsymbol{\Sigma})$. The variance of portfolio value is given by

$$\begin{aligned} \text{Var}[\Delta V^\gamma] &= \text{Var} \left[\boldsymbol{\delta}' \Delta \mathbf{f} + \frac{1}{2} \Delta \mathbf{f}' \mathbf{\Gamma} \Delta \mathbf{f} \right] \\ &= \text{Var}[\boldsymbol{\delta}' \Delta \mathbf{f}] + \frac{1}{4} \text{Var}[\Delta \mathbf{f}' \mathbf{\Gamma} \Delta \mathbf{f}] + \\ &\quad \text{Cov} \left[\boldsymbol{\delta}' \Delta \mathbf{f}, \frac{1}{2} \Delta \mathbf{f}' \mathbf{\Gamma} \Delta \mathbf{f} \right]. \end{aligned} \quad (4.25)$$

Now the covariance term in (4.25) equals zero from Stein's (1981) lemma,⁷ thus the variance of portfolio value is comprised of two parts, a delta or linear term and a gamma or quadratic term. Consider the gamma term $\Delta \mathbf{f}' \mathbf{\Gamma} \Delta \mathbf{f}$. We first orthogonalize the factors and define a new random vector $\Delta \mathbf{y}$:

$$\Delta \mathbf{y} = \boldsymbol{\Sigma}^{-1/2} \Delta \mathbf{f}, \quad (4.26)$$

⁷ Stein's lemma states that the covariance between a normally distributed variable x , and a smooth function $f(y)$ of a normally distributed variable equals the covariance between x and y , multiplied by the expected value of the first derivative of the function:

$$\text{cov}(x, f(y)) = E[f'(y)] \times \text{cov}(x, y).$$

where $\Sigma^{-1/2}$ is a symmetric matrix satisfying $\Sigma^{-1/2}\Sigma^{-1/2} = \Sigma^{-1}$.⁸ By construction $\Delta\mathbf{y}$ is multivariate normal, with a covariance matrix equal to the identity matrix:

$$\Delta\mathbf{y} \sim N_n(\mathbf{0}, \mathbf{I}_n). \quad (4.27)$$

The gamma term can thus be written,

$$\frac{1}{2}\Delta\mathbf{f}'\Gamma\Delta\mathbf{f} = \frac{1}{2}\Delta\mathbf{y}'\Sigma^{1/2}\Gamma\Sigma^{1/2}\Delta\mathbf{y} = \frac{1}{2}\Delta\mathbf{y}'\mathbf{A}\Delta\mathbf{y} \quad (4.28)$$

where $\mathbf{A} = \Sigma^{1/2}\Gamma\Sigma^{1/2}$. Now decompose the square matrix \mathbf{A} using a spectral decomposition:

$$\mathbf{A} = \mathbf{C}\mathbf{\Lambda}\mathbf{C}', \quad (4.29)$$

where $\mathbf{\Lambda}$ is a diagonal matrix containing the eigenvalues $(\lambda_1, \dots, \lambda_n)$ of \mathbf{A} and \mathbf{C} is an orthogonal matrix containing the eigenvectors of \mathbf{A} . We see that the gamma term is a linear combination of independent χ_1^2 variables:

$$\frac{1}{2}\Delta\mathbf{f}'\Gamma\Delta\mathbf{f} = \frac{1}{2}\Delta\mathbf{y}'\mathbf{C}\mathbf{\Lambda}\mathbf{C}'\Delta\mathbf{y} = \frac{1}{2}\mathbf{x}'\mathbf{\Lambda}\mathbf{x} = \frac{1}{2}\sum_{i=1}^n \lambda_i x_i^2, \quad (4.30)$$

where:

$$\mathbf{x} \sim N_n(\mathbf{0}, \mathbf{I}_n). \quad (4.31)$$

Since the variance of x_i^2 is 2, the variance of the gamma term is simply given by half the sum of the squared eigenvalues of \mathbf{A} :

$$\text{Var}\left(\frac{1}{2}\Delta\mathbf{f}'\Gamma\Delta\mathbf{f}\right) = \left(\frac{1}{2}\right)^2 \sum_{i=1}^n \lambda_i^2 \times 2 \quad (4.32)$$

$$= \frac{1}{2} \sum_{i=1}^n \lambda_i^2. \quad (4.33)$$

As the square of a matrix has eigenvalues that are squares of the eigenvalues of the original matrix, and because the sum of eigenvalues equals the trace of the matrix, the above expression simplifies to,

$$\text{Var}\left(\frac{1}{2}\Delta\mathbf{f}'\Gamma\Delta\mathbf{f}\right) = \frac{1}{2}\text{tr}\mathbf{A}^2 = \frac{1}{2}\text{tr}[(\Sigma\Gamma)^2]. \quad (4.34)$$

We can now write total portfolio variance as:

$$\text{Var}(\Delta V^\gamma) = \boldsymbol{\delta}'\Sigma\boldsymbol{\delta} + \frac{1}{2}\text{tr}[(\Sigma\Gamma)^2]. \quad (4.35)$$

⁸ The symmetric decomposition of a non-negative definite matrix can be formed from its spectral decomposition. See Mardia, Kent, and Bibby (1994, p. 475) for details.

Thus portfolio volatility can easily be calculated from the portfolio's aggregate deltas and gammas and from the covariance matrix of factor innovations.

4.2. SKEWNESS AND OTHER MOMENTS

Of course portfolio volatility by itself is less informative when gamma is significant as the resultant distribution of portfolio value is not normal. Higher order moments such as skewness are helpful in this case for two reasons. First it is interesting to know how the presence of derivatives may impart skewness to a portfolio's distribution; second, higher order moments can be used to construct approximations to the complete probability distribution.

The 'completion-of-the-square' approach used in the univariate case can also be used in the multivariate case so long as the gamma matrix is non-singular (i.e. when Γ^{-1} exists). The technique is as follows:

$$\begin{aligned}\Delta V^\gamma &\equiv \mu_\gamma + \delta' \Delta \mathbf{f} + \frac{1}{2} \Delta \mathbf{f}' \Gamma \Delta \mathbf{f} \\ &= \mu_c + \frac{1}{2} (\Delta \mathbf{f} + \Gamma^{-1} \delta)' \Gamma (\Delta \mathbf{f} + \Gamma^{-1} \delta),\end{aligned}$$

where $\mu_c = \mu_\gamma - \frac{1}{2} \delta' \Gamma^{-1} \delta$. Now $\Delta \mathbf{f} + \Gamma^{-1} \delta$ is normally distributed with a mean $\boldsymbol{\mu} + \Gamma^{-1} \delta$ and covariance $\boldsymbol{\Sigma}$. Redefining \mathbf{y} as:

$$\mathbf{y} = \boldsymbol{\Sigma}^{-1/2} (\Delta \mathbf{f} + \Gamma^{-1} \delta),$$

where \mathbf{A} , as before, is given by:

$$\mathbf{A} = \boldsymbol{\Sigma}^{1/2} \Gamma \boldsymbol{\Sigma}^{1/2},$$

we can express the change in portfolio value as

$$\Delta V^\gamma = \mu_c + \frac{1}{2} \mathbf{y}' \mathbf{A} \mathbf{y},$$

where \mathbf{y} is normally distributed:

$$\mathbf{y} \sim N(\boldsymbol{\mu}_f + \Gamma^{-1} \delta, \mathbf{I}).$$

As before we can decompose the square matrix A using a spectral decomposition:

$$\mathbf{A} = \mathbf{C} \boldsymbol{\Lambda} \mathbf{C}', \quad (4.36)$$

where $\boldsymbol{\Lambda}$ is a diagonal vector containing the eigenvalues $(\lambda_1, \dots, \lambda_n)$ of \mathbf{A} , and \mathbf{C} is an orthogonal matrix containing the eigenvectors of A . We can then express the change in portfolio value as a linear combination of independent χ_1^2 variables:

$$\Delta V^\gamma \approx \mu_c + \frac{1}{2} \Delta \mathbf{y}' \mathbf{C} \boldsymbol{\Lambda} \mathbf{C}' \Delta \mathbf{y} = \mu_c + \frac{1}{2} \mathbf{z}' \boldsymbol{\Lambda} \mathbf{z} = \mu_c + \frac{1}{2} \sum_{i=1}^n \lambda_i z_i^2, \quad (4.37)$$

where

$$\mathbf{z} \sim N_n(\mathbf{C}'(\boldsymbol{\mu} + \boldsymbol{\Gamma}^{-1}\boldsymbol{\delta}), \mathbf{I}_n). \quad (4.38)$$

This representation expresses the change in portfolio value as the sum of a set of non-central chi-square variables. Mathai and Provost (1991, pp. 53–54) derive the integer moments for this distribution. In our notation the r^{th} moment of the change in portfolio value $E(\Delta V^\gamma)^r$ is given by

$$E(\Delta V^\gamma)^r = \left\{ \sum_{r_i=0}^{r-1} \binom{r-1}{r_i} m(r-1-r_i) \sum_{r_j=0}^{r_i-1} \binom{r_i-1}{r_j} m(r_i-1-r_j) \dots \right\}$$

where the function m is defined by,

$$\begin{aligned} m(0) &= \mu_\gamma + \boldsymbol{\delta}'\boldsymbol{\mu}_f + \frac{1}{2}\boldsymbol{\mu}_f'\boldsymbol{\Gamma}\boldsymbol{\mu}_f + \frac{1}{2}\text{tr}\boldsymbol{\Gamma}\boldsymbol{\Sigma} \\ m(1) &= \frac{1}{2}\text{tr}(\boldsymbol{\Gamma}\boldsymbol{\Sigma})^2 + \boldsymbol{\delta}'\boldsymbol{\Sigma}\boldsymbol{\delta} \\ m(2) &= \text{tr}(\boldsymbol{\Gamma}\boldsymbol{\Sigma})^3 + 3\boldsymbol{\delta}'\boldsymbol{\Sigma}\boldsymbol{\Gamma}\boldsymbol{\Sigma}\boldsymbol{\delta} \\ m(k) &= \frac{k!}{2}\text{tr}(\boldsymbol{\Gamma}\boldsymbol{\Sigma})^{k+1} + \frac{(k+1)!}{2}\boldsymbol{\delta}'(\boldsymbol{\Sigma}\boldsymbol{\Gamma})^{k-1}\boldsymbol{\Sigma}\boldsymbol{\delta}, \quad k \geq 1, \end{aligned}$$

and an empty product is assumed to have a value of unity. From this we can calculate any desired integer moments of the distribution of portfolio value. Expressions for the first three moments are shown below.

$$\begin{aligned} E(\Delta V^\gamma) &= m(0) \\ &= \mu_\gamma + \boldsymbol{\delta}'\boldsymbol{\mu}_f + \frac{1}{2}\boldsymbol{\mu}_f'\boldsymbol{\Gamma}\boldsymbol{\mu}_f + \frac{1}{2}\text{tr}\boldsymbol{\Gamma}\boldsymbol{\Sigma} \end{aligned} \quad (4.39)$$

$$\begin{aligned} E(\Delta V^\gamma)^2 &= m(1) + m(0)^2 \\ &= E[\Delta V^\gamma]^2 + \frac{1}{2}\text{tr}(\boldsymbol{\Gamma}\boldsymbol{\Sigma})^2 + \boldsymbol{\delta}'\boldsymbol{\Sigma}\boldsymbol{\delta} \end{aligned} \quad (4.40)$$

$$\begin{aligned} E(\Delta V^\gamma)^3 &= m(2) + 3m(1)m(0) + m(0)^3 \\ &= E[\Delta V^\gamma]^3 + \text{tr}(\boldsymbol{\Gamma}\boldsymbol{\Sigma})^3 + 3\boldsymbol{\delta}'\boldsymbol{\Sigma}\boldsymbol{\Gamma}\boldsymbol{\Sigma}\boldsymbol{\delta} \\ &\quad + 3\left(\frac{1}{2}\text{tr}(\boldsymbol{\Gamma}\boldsymbol{\Sigma})^2 + \boldsymbol{\delta}'\boldsymbol{\Sigma}\boldsymbol{\delta}\right) E[\Delta V^\gamma]. \end{aligned} \quad (4.41)$$

Solomon and Stephens (1977)⁹ have suggested that the random number given by $K_1\omega_p^{K_2}$, where ω_p has a χ^2 distribution with p degrees of freedom, and K_1 , and K_2 are constants, can provide a good approximation to the density of a sum of non-central chi-square variates, such as $\Delta V^\gamma - \mu_c$, when K_1 , K_2 and p are chosen to match the first three moments of $\Delta V^\gamma - \mu_c$.¹⁰

⁹ See Mathai and Provost (1992), pp. 172–173.

¹⁰ See also Johnson, Kotz and Balakrishnan (1977), pp. 444–450.

Now the moments ($r = 1, 2, \dots$) of the χ^2 distribution about zero are given by

$$E\omega_p^r = \frac{2^r \Gamma(r + p/2)}{\Gamma(p/2)}, \quad r = 1, 2, \dots$$

where $\Gamma(\cdot)$ is the gamma function. The moments about zero of $K_1\omega_p^{K_2}$, μ'_1 , μ'_2 , μ'_3 , ..., are given by:

$$\mu'_s = (K_1)^s 2^{sK_2} \frac{\Gamma(sK_2 + p/2)}{\Gamma(p/2)}, \quad s = 1, 2, 3, \dots \quad (4.42)$$

Let:

$$R_2 = \frac{\mu'_2}{(\mu'_1)^2} = \frac{\Gamma(p/2) \Gamma(2K_2 + p/2)}{(\Gamma(K_2 + p/2))^2}, \quad (4.43)$$

and:

$$R_3 = \frac{\mu'_3}{(\mu'_1)^3} = \frac{(\Gamma(p/2))^2 \Gamma(3K_2 + p/2)}{(\Gamma(K_2 + p/2))^3}. \quad (4.44)$$

Using (4.39), (4.40) and (4.41), Equations (4.43) and (4.44) may be solved for K_2 and p . Finally, K_1 may be obtained using (4.39) and the expression for μ'_1 . Once the parameter values are found the probability of losses of greater than a certain magnitude can be read from a Table of χ^2 .

5. Multivariate Examples

In this section we present two examples of the application of the delta-gamma approximation in the multivariate case. In the first we consider a portfolio consisting of puts and calls on three uncorrelated stocks; Table 5 gives the prices and volatilities of the underlying stocks and the strike prices, maturities and quantities of the options. In this example the position has negative gamma in all three of the underlying stocks; Table 5 gives the aggregate deltas and gammas.

We now proceed according to Sections 4.1 and 4.2 and compute the eigenvalues and non-centrality parameters of the quadratic approximation given in Equation (4.36). From this we compute the three eigenvalues and non-centrality parameters in Equation (4.37) and then compute the first three moments from Equation (4.39), (4.40) and (4.41). Finally, we solve for the parameters K_1 , K_2 , and p as described above. Using these values we compute the cumulative density function (cdf) of the Solomon-Stephens approximation.

The results are shown in Figure 8 which shows the cdf computed using full evaluation (Black-Scholes), the delta approximation, the *actual* delta-gamma approximation (via simulation) and the Solomon-Stephens approximation to the delta-gamma approximation. While we cannot claim any generality for the results –

Table III. Asset and option characteristics for multivariate example

	Asset 1	Asset 2	Asset 3
Price	100	50	80
Volatility (% p.a.)	0.30	0.40	0.20
Call			
Strike	100	50	80
Maturity (years)	0.1	0.1	0.1
Quantity	-100	-100	-30
Put			
Strike	100	50	80
Maturity (years)	0.1	0.1	0.1
Quantity	-100	-30	-100

Table IV. Portfolio deltas and gammas for multivariate example

hline	Asset 1	Asset 2	Asset 3
Delta	-12.15	-42.35	-25.22
Gamma	-8.313	-8.12	-10.07

the example is just an example – they are encouraging. Although the delta-only approximation performs poorly - as it was bound to do in this example – both the actual delta-gamma approximation and the Solomon-Stephens approximation are very close to the true values.

In the second example we consider a position involving two underlying assets where the portfolio is convex in the price of one underlying and concave in the other. Table 5 summarizes the position and the relation between the portfolio value and the prices of the two underlying stocks is shown in Figure 11. As the figure shows, the position is concave with respect to the price of asset 1 and convex in the price of asset 2. Nonetheless, this relation is well approximated by a function which is quadratic in the two stock prices and, as a result, the distribution of a delta-gamma approximation to portfolio value is close to the distribution using full revaluation.

In this case, however, the Solomon-Stephens approach is not suitable since, as a result of the convexity with respect to one underlying and concavity with respect to the other, the eigenvalues of matrix A (see Equation (4.30)) have different signs. Thus the value of $\Delta V^\gamma - \mu_c$ in this case is unbounded both below and above. When all the eigenvalues are positive $\Delta V^\gamma - \mu_c$ is unbounded above and

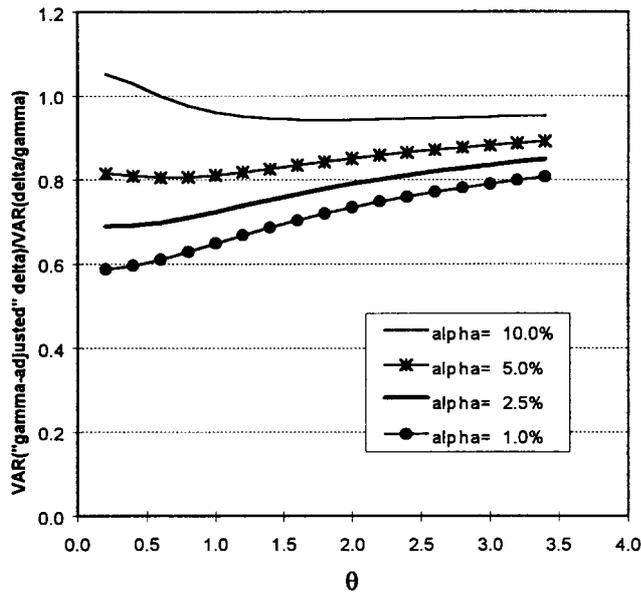


Figure 9. The figure shows the ratio of VAR under the “gamma adjusted” delta approach to VAR under the delta-gamma approximation as a function of θ , defined as $\delta/\sigma\gamma$.

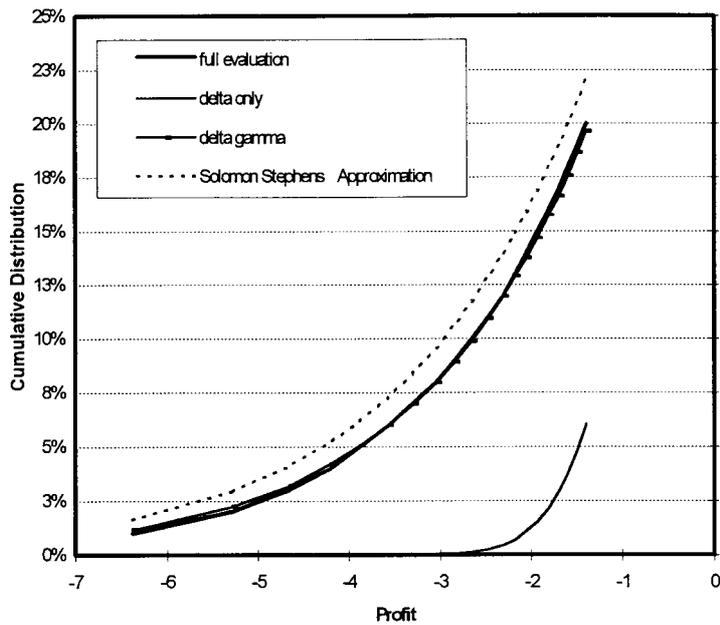


Figure 10. The lower tail of the distribution of portfolio value for the first multivariate example. The graph shows the cdf computed using (i) “true” (Black-Scholes) values, (ii) the delta-only approximation, (iii) the actual values of the delta-gamma approximation and (iv) the Solomon-Stephens approximation to the delta-gamma approximation.

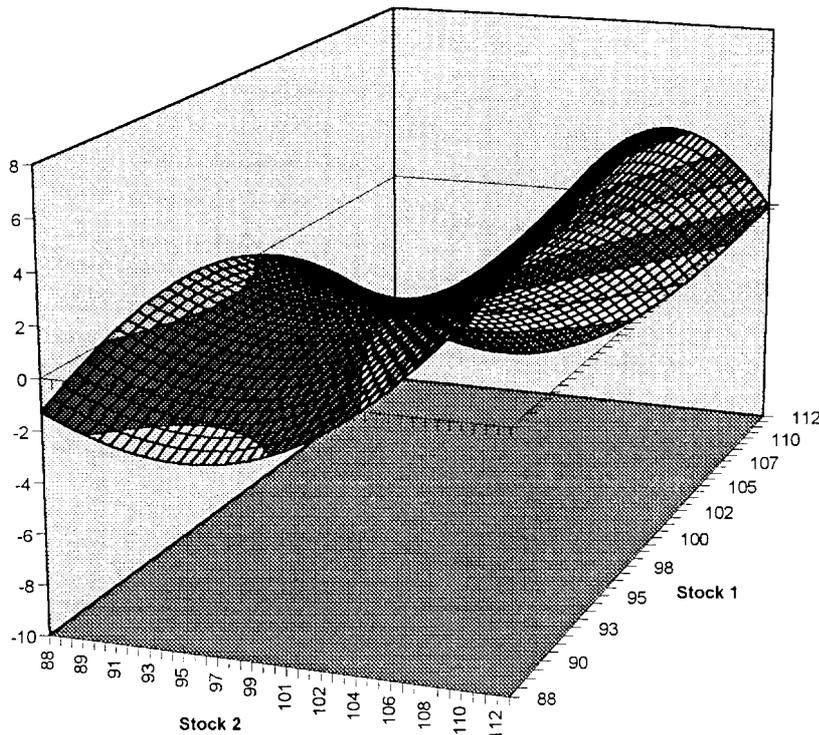


Figure 11. The relation between portfolio value and the prices of the two underlying assets for a portfolio containing options written on two assets whose value is convex with respect to one asset and concave with respect to the other.

bounded below at zero just as the variate proposed by Solomon and Stephens. When the eigenvalues are not all of the same sign, therefore, the Solomon and Stephens approach is unlikely to perform well and, in the example under discussion here, we were unable to find admissible values of K_1 , K_2 , and p which matched the moments of $\Delta V^\gamma - \mu_c$.¹¹

¹¹ In this case it is always possible to proceed by simulation. Alternatively, the expression $\sum_{i=1}^n \lambda_i x_i^2$ could be split into two. Without loss of generality, assume that the first m eigenvalues are positive and the remainder negative. We may therefore write $\sum_{i=1}^n \lambda_i x_i^2$ as the difference between two positively weighted sums of non-central chi-squared variates:

$$\equiv \sum_{i=1}^m \lambda_i^+ x_i^2 - \sum_{i=m+1}^n (-\lambda_i^-) x_i^2.$$

These two expressions may each be approximated using the Solomon-Stephens approach and the two distributions finally combined using Monte-Carlo.

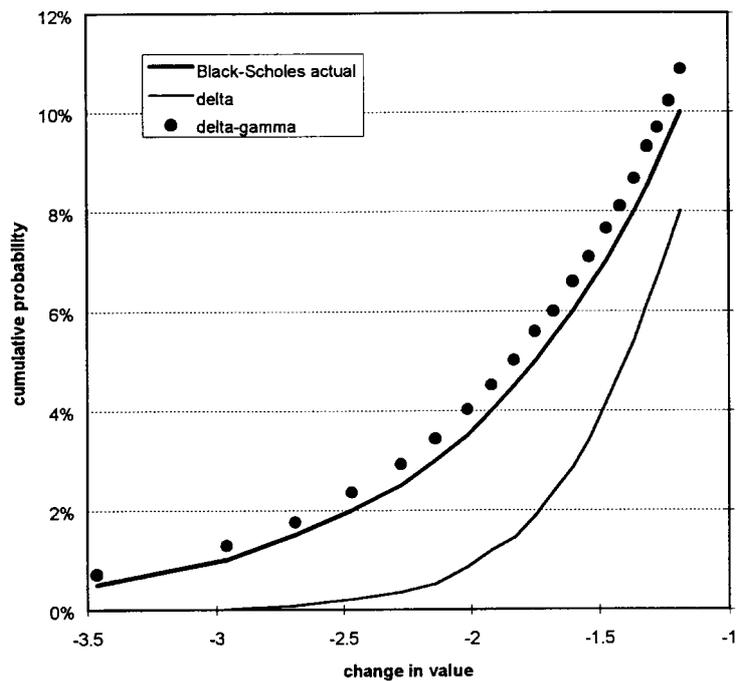


Figure 12. The figure shows the lower tail of the distribution of portfolio value under (i) full revaluation, (ii) delta approximation and (iii) delta-gamma approximation for the example in which the portfolio is concave in the value of one underlying and convex in the other.

Table V. Asset and option characteristics for multivariate example with convexity in one asset and concavity in the other

	Asset 1	Asset 2
Price	100	100
Volatility (p.a.)	30%	30%
Correlation between returns	0.7	
Call		
Strike	101	101
Maturity (weeks)	6	6
Quantity	-1.0	-0.6
Put		
Strike	101	101
Maturity (weeks)	6	6
Quantity	+1.0	+0.5
Delta	-0.2391	+0.2866
Gamma	-0.0625	+0.0586

6. Conclusion

Linear approximations to the relation between derivative values and the underlying price or rate are unlikely to be robust and risk assessment methods based on them will usually fare no better. In this paper we have proposed an approach to VaR which is based on a second order “delta-gamma” approximation and recognises the impact that this will have, not only on variance, but on the form of the distribution.

For the case of portfolios of derivatives on a single underlying asset the paper derived the exact distribution of the delta-gamma approximation and, subject to some further mild assumptions, the relation between the delta-gamma and the delta-only VaR. For a given probability level, the latter depended on a single parameter, θ , which is a function of the portfolio delta and gamma and the volatility of the underlying instrument.

For the multivariate case the paper derived the moments of the delta-gamma approximation and showed that when the matrix of gammas and cross-gammas is invertible the delta-gamma approximation is distributed as the sum of independent non-central chi-square variates. An approximation to this sum, due to Solomon and Stephens (1977) was described and an example of its application to a portfolio of put and call options presented.

Appendix A: Capital-at-Risk vs. Value-at-Risk

One method of implementing the delta-gamma approach that avoids explicit calculation of the probability distribution of a quadratic form is the quadratic programming (QP) approach outlined by Wilson (1994) and Rouvinez (1997). The method is relatively simple and intuitively appealing. The starting point involves a redefinition of value at risk as *the maximum possible loss over a specific time horizon within a given confidence interval*. To the extent that this definition is meaningful, it means the level of loss, $\Delta V^*(\alpha)$, such that the probability that $\Delta V \geq \Delta V^*$ is equal to $1 - \alpha$. In other words, if α is 5% then the VaR under this approach is defined as the level of loss which 95% of the time will be worse than the actual outcome. The quadratic programming approach then seeks to *solve for the market event which maximizes potential losses subject to the constraint that the event and all events generating smaller losses are within a given confidence interval*.

The key step here is that we have gone from a confidence region defined over *portfolio value* changes (one-dimensional) to a confidence region defined over *factor realizations* which are (in general) multidimensional. A $1 - \alpha$ confidence region is a region within which realizations will fall $(1 - \alpha)\%$ of the time. In general, for any random variable or set of random variables, there is an infinite number of different confidence regions corresponding to any particular probability level $1 - \alpha$. To see this consider a random variable z distributed as a standard normal: $z \sim N(0, 1)$. A fifty percent confidence region is $(-0.68, +0.68)$; another is $(0, +\infty)$; still another is $(-\infty, 0)$. The reader can check that z has a 50% probability of falling into each of these regions.

The arbitrariness of confidence regions does not extend however to the calculation of VaR. The region in which all outcomes worse than a particular level fall with a particular probability is uniquely defined by the probability distribution and the chosen probability level α . For z the 5% VaR is simply -1.65, and this is unique since the region so defined has one end fixed implicitly at $+\infty$. The 95% confidence region corresponding to the VaR is $(-1.65, +\infty)$ and z has a 95% chance of falling into this region. Thus VaR has a natural way of uniquely determining the confidence region.

How can we determine a unique confidence region for a set of random variables? The method commonly employed is to focus on regions defined by *iso-density* lines. This is the approach used by Wilson. Viewing a density function as a hill, an iso-density line is a contour line at constant height. In the one dimensional case the iso-density region consists of two points at equal height, one on each side of the distribution. For the multiasset case the confidence region is defined by the region bounded by a multidimensional ellipse. Using the iso-density criterion for the previous univariate case of z results in a 50% confidence region defined by $(-0.68, +0.68)$. The 95% confidence region is $(-1.92, 1.92)$.

An inadequacy is now immediately apparent. The regions described by the VaR definition and the quadratic programming approach are different even in the simple one-dimensional case. The relevant confidence region (over a standard normal variable) for VaR was $(-1.65, +\infty)$. The QP approach uses the region $(-1.92, 1.92)$. Consider the case where there is one factor determining portfolio value, and the change in portfolio value is simply equal to the change in this factor. Assume the factor is standard normal. The QP approach suggests that the 5% VaR is -1.92. The actual VaR at 5% is only -1.65.

In this case, it is fairly clear why the QP approach is not working — the tail areas are counted twice under QP but only on one side under VaR. This is easily fixed, but deeper problems remain.

Consider a more complex non-linear case. To highlight the QP approach, we shall assume that the exact portfolio value is given by a quadratic function of a single factor z which is distributed standard normal. Assume the relation is as follows:

$$\Delta V = -10 + 25000z^2.$$

The maximum loss of -10 occurs at $z = 0$ and, as this is within the 95% confidence region for z , the VaR(5%) under the QP approach is defined to be -10. However losses only occur for z in $(-0.02, 0.02)$. The chance that any loss at all occurs is only 1.6%. So we can see that the true VaR at the 5% level must be a positive number.

The reason for the erroneous result, is that it simply is not possible to use a confidence region (defined over factors) to make inferences over a function of those factors. In fact, if it were possible to do so, much of the work in statistics involving distributions of functions of random variables would be unnecessary!

Simply because a point lies within a 95% confidence region does not mean that it has a 95% chance of occurrence. A point may lay within some 95% region, have a negligible chance of occurring and have a massive loss associated with it. The size of this loss does not give any indication of the true VaR. In short the QP approach is conceptually flawed and will give erroneous results under all but special situations where it will happen to coincide with the correct answer.

It is possible to say something more about the relation between the QP version of VaR and true VaR: the **true VaR is always greater than or equal to the QP VaR**, i.e., the true VaR will be a smaller loss, or a larger profit, than the QP VaR. Thus the QP VaR is conservative in that it predicts that a larger amount of capital is at risk.

To prove this statement, we need to set some notation. Define the true value at risk for a particular probability level α as ΔV^* such that

$$\Pr(\Delta V < \Delta V^*) = \alpha.$$

The QP version of value at risk is defined for a particular confidence region \mathbf{C} . The confidence region \mathbf{C} is defined in the space of factor realizations and must satisfy

$$\Pr(\Delta \mathbf{f} \in \mathbf{C}) = 1 - \alpha.$$

We noted before that confidence regions are not uniquely defined by the probability, but our results apply to **any** confidence region that satisfies the above criterion. The QP value at risk Q^* can now be defined simply as

$$Q^* = \min_{\Delta \mathbf{f} \in \mathbf{C}} \Delta V(\Delta \mathbf{f}).$$

An immediate implication of the QP definition is that the probability of an outcome above (or equalling) Q^* must be at least as great as $1 - \alpha$:

$$\Pr(\Delta V \geq Q^*) \geq 1 - \alpha.$$

This follows immediately since the set of all outcomes above (or equaling) Q^* includes \mathbf{C}

$$\begin{aligned} \Pr(\Delta V \geq Q^*) &= \Pr(\Delta \mathbf{f} \in \mathbf{C}) + \Pr\left(\begin{array}{l} \text{those } \Delta \mathbf{f} \text{ outside of } \mathbf{C} \text{ and} \\ \text{resulting in } V \text{ above } Q^* \end{array}\right) \\ &\geq 1 - \alpha. \end{aligned}$$

Thus the probability of an outcome *worse* than Q^* is lower than (at best equal to) α :

$$\Pr(\Delta V \leq Q^*) \leq \alpha.$$

Since $\Pr(\Delta V < \Delta V^*) = \alpha$, it follows that $Q^* \leq V^*$ provides an overly conservative estimate of value at risk. The extent to which it is conservative requires evaluation of the actual probability distribution, and thus leads us back to the probabilistic approach, such as the approximations discussed above or full simulation.

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