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Journal of Empirical Finance 7 (2000) 531–554

Journal of EMPIRICAL FINANCE

www.elsevier.com/locate/econbase

Value-at-Risk: a multivariate switching regime approach

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Accepted 5 October 2000

Abstract

This paper analyses the application of a switching volatility model to forecast the distribution of returns and to estimate the Value-at-Risk (VaR) of both single assets and portfolios. We calculate the VaR value for 10 Italian stocks and a number of portfolios based on these stocks. The calculated VaR values are also compared with the variance–covariance approach used by JP Morgan in RiskMetrics™ and GARCH(1,1) models. Under backtesting, the VaR values calculated using the switching regime beta model are preferred to both other methods. The Proportion of Failure and Time Until First Failure tests [The Journal of Derivatives (1995) 73–84] confirm this result. © 2000 Elsevier Science B.V. All rights reserved.

JEL classification: C22; C52; G28
Keywords: Value-at-Risk; Switching regime models; Multivariate approach; Factors models

1. Introduction

Value at Risk (VaR) is a risk-management technique that has been widely used to assess market risk. VaR for a portfolio is simply an estimate of a specified percentile of the probability distribution of the portfolio's value change over a

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PII: S0927-5398(00)00022-0

given holding period. The specified percentile is usually in the lower tail of the distribution, e.g., the 95th percentile or the 99th percentile.

Calculation of portfolio VaR is often based on the variance–covariance approach and makes the assumption, among others, that returns follow a conditional normal distribution. We show that this assumption is at odds with reality and often results in misleading estimates of VaR.

There is substantial empirical evidence (Hsieh, 1988; Meese, 1986) that the distribution of returns on equities and other assets is typically leptokurtic, that is, the unconditional return distribution shows high peaks and fat tails. This feature can arise from a number of different reasons, in particular: jumps, correlation between shocks and changes in volatility, and time series volatility fluctuations usually characterized by persistence.

The literature commonly describes persistence in time series volatility using ARCH or GARCH models that give rise to unconditional symmetric and leptokurtic distributions. Here leptokurtosis follows from persistence in the conditional variance, which produces the clusters of "low volatility" and "high volatility" returns.

In RiskMetrics™, volatility is estimated using the exponentially weighted moving average (EWMA) approach, which places more emphasis on more recent history in estimating volatility. As Phelan (1995) demonstrates, this approach is a restrictive case of the GARCH model.

However, these models do not account for jumps in stock returns. Nevertheless, as risk measurement focuses in particular on the "tails" distribution, jumps deserve careful study.

For this reason, we suggest a new and relatively simple method for estimating VaR: the "switching regime approach". This approach is able to (i) consider the conditional non-normality of returns, (ii) take into account time-varying volatility characterized by persistence, and (iii) deal with events that are relatively infrequent (e.g., some changes in the level of volatility).

The solutions proposed in the VaR literature to the last problem have been the use of: (i) the ex-post historical simulation approach, (ii) the (ex-ante) Student's *t*-distributions, and (iii) a mixture of two normal distribution, as proposed by RiskMetrics (Longerstay, 1996).

Each of these solutions is only partially able to deal with the problems of skewness and kurtosis in the return distribution as they do not entirely correct the under-estimation of risk.

In the financial literature, the non-normality of asset returns has attracted particular attention both as a problem in its own, and because of its implications for the evaluation of contingent claims, in particular options. A number of different time series models have been employed to capture these distributional feature: stationary fat-tailed distributions such as Student's *t* (Rogalski and Vinso, 1978) and the jump diffusion process (Akgiray and Booth, 1988); Gaussian ARCH or GARCH models (Bollerslev et al., 1992); chaotic models; non-standard classes of stochastic processes such as stable processes (see Mandelbrot, 1963), and subordinated stochastic processes (Clark, 1973; Geman and Ané, 1996; Müller et al., 1993).

In our paper we model this phenomenon using the "switching regime" approach that gives rise to a non-normal return distribution in a simple and intuitive way. The improved forecast of return distribution obtained with this approach is important since VaR methodology is indeed based on forecasting the distribution of future values of a portfolio.

The approach is similar to the mixture of distributions (proposed by JP Morgan to embed skewness and kurtosis in a VaR measure), but with the difference that the unobserved random variable characterizing the regime is the outcome of an unobserved k-state Markov chain instead of a Bernoulli variable. The advantage of using a Markov chain as opposed to a Bernoulli specification is that the former allows conditional information to be reflected in the forecast and it captures the well known fact that high volatility is usually followed by high volatility.

The purpose of this paper is to describe the application of this approach to the estimation of VaR and so allow for a more realistic model of the tail distribution of financial returns. We focus on the measurement of market risk in equity portfolios and illustrate our method using data on 10 Italian stocks and the MIB30 Italian Index.

The plan of the paper is as follows. Section 2 provides a description of the evaluation framework for VaR estimates. Section 3 describes the different switching regime models used to estimate VaR. Section 4 shows the results of the empirical investigation of these models on the Italian equity market and compares the results with (i) RiskMetrics and (ii) GARCH(1,1) approaches. Section 5 concludes.

2. Evaluation of VaR estimates

2.1. VaR definition

VAR is a measure of market risk for a portfolio of financial assets[1] and measures the level of loss that a portfolio could lose, with a given degree of confidence a, over a given time horizon h. Analytically it can be formulated as follows:

$$Pr\left[W_{t+h} - W_t < -\mathrm{VaR}_W(h)\right] = a \qquad (1)$$

where W_t is the portfolio value at time t and $\mathrm{VaR}_W(h)$ is the VaR value of the portfolio W with a time horizon h.

[1] There is an extensive recent literature on VaR. Nevertheless, for an introduction to VaR see Linsmeier and Pearson (1996).

The confidence level $(1-a)$ is typically chosen to be at least 95% and often as high as 99% or more (a equal to 5% or 1%). The time horizon h varies with the use made of VaR by management and asset liquidity.

It is possible to express the VaR measure in terms of return of the portfolio instead of portfolio value. Analytically it can be formulated as follows:

$$Pr\left[R_{W_{t+h}} < -\mathrm{VaR}_R(h)\right] = a \tag{2}$$

where $R_{W_{t+h}} = \ln(W_{t+h}/W_t)$ is the portfolio return at time $t+h$ and $\mathrm{VaR}_R(h)$ is the VaR value of portfolio returns R_W with a time horizon h.

Clearly, VaR is simply a specific quantile of a portfolio's potential loss distribution over a given holding period.

Assuming $R_{W_t} \sim f_t$, where f_t is a general return distribution, the VaR for time $t+h$, estimated using a model indexed by m, conditional on the information available at time t and denoted $\mathrm{VaR}_m(h,a)$, is the point in f_{mt+h} model m's estimated return distribution that corresponds to its lower a percent tail. That is $\mathrm{VaR}_m(h,a)$ is the solution to:

$$\int_{-\infty}^{\mathrm{VaR}_m(h,a)} f_{mt+h}(x)\,\mathrm{d}x = a \tag{3}$$

Different models can be used to forecast the return distribution and so to calculate VaR. Given the widespread use of VaR by banks and regulators, it is important to determine the accuracy of the different models used to estimate VaR.

2.2. Alternative evaluation methods

As discussed by Kupiec (1995) a variety of methods are available to test the null hypothesis that the observed probability of occurrence over a reporting period equals a. In our work, two methods are used to evaluate the accuracy of the VaR model: the Proportion of Failure (PF) test (Kupiec, 1995) and the time until first failure (TUFF) test (Kupiec, 1995).

The first test is based on the probability under the binomial distribution of observing x exceptions in the sample size T.[2] In particular:

$$Pr(x;a,T) = \binom{T}{x} a^x (1-a)^{T-x} \tag{4}$$

VaR estimates must exhibit that their unconditional coverage a, measured by $\hat{a} = x/T$, equals the desired coverage level a_0 (usually equal to 1% or 5%). Thus,

[2] x exceptions means the number of times the observed value $R_{W_{t+h}}$ is lower than $\mathrm{VaR}_R(h)$.

the null hypothesis is H_0: $a = a_0$, and the corresponding Likelihood ratio statistic is:

$$LR_{PF} = 2\left[\ln\left(\hat{a}^x (1-\hat{a})^{T-x}\right) - \ln\left(a_0^x (1-a_0)^{T-x}\right)\right] \tag{5}$$

which is asymptotically distributed as $\chi^2(1)$.

The TUFF test is based on the number of observations before the first exception. The relevant null hypothesis is, once again, H_0: $a = a_0$ and the Likelihood ratio statistic is:

$$LR_{TUFF}(\tilde{T},\hat{a}) = -2\ln\left[\hat{a}(1-\hat{a})^{\tilde{T}-1}\right] + 2\ln\left[\left(1/\tilde{T}\right)\left(1-1/\tilde{T}\right)^{\tilde{T}-1}\right] \tag{6}$$

where $\tilde{T}$ denotes the number of observations before the first exception. The LR_{TUFF} test statistic is also asymptotically distributed as $\chi^2(1)$.

Unfortunately, as Kupiec observed, these tests have a limited power to distinguish among alternative hypotheses. However, this approach has been adopted by regulators in the analysis of internal models to define the zones (green, yellow and red) into which the different models are categorized in backtesting. In particular, for a backtest with 250 observations, regulators place a model in the green zone if x (the exception number) is lower than 4; from 5 to 9 these models are allocated to the yellow zone and the required capital is increased by an incremental factor that ranges from 0.4 to 0.85. If x is greater than 9, the incremental factor is 1.

3. Switching regime models

The risk profile of a firm or of the economy as a whole does not remain constant over time. A variety of systematic and unsystematic events may change the business and financial risk of firms significantly. It is argued here that this might derive from the presence of discontinuous shifts in return volatility.

The change in regime should not be regarded as predictable but as a random event. The effect of these risk shifts should be taken into account by risk analysts in the forecasting process, by risk managers in the assessment of market risk and capital allocation, and by regulators, in the definition of capital requirements.

3.1. Simple switching regime models

A Simple Switching Regime Model (SSRM) can be written as:

$$R_t = \mu(s_t) + \sigma(s_t)\,\varepsilon_t \tag{7}$$

where $R_t = \ln(P_t/P_{t-1})$, $\varepsilon_t \sim \text{IIN}(0,1)$, P_t is the stock price or the index price, s_t is a Markov chain with k states and transition probability matrix $\mathbf{\Pi}$. In particular if $k=2$, we have:

$$R_t = \begin{cases} \mu_0 + \sigma_0 \varepsilon_t \text{ if } s_t = 0 \\ \mu_1 + \sigma_1 \varepsilon_t \text{ if } s_t = 1 \end{cases}$$

and the transition matrix $\mathbf{\Pi}$ is:

$$\mathbf{\Pi} = \begin{bmatrix} p & 1-p \\ 1-q & q \end{bmatrix} \tag{8}$$

where the parameters p and q are probabilities that volatility remains in the same regime. In the model, the variance and mean of returns change only as a result of periodic, discrete events.[3]

Switching regime models have been applied by Rockinger (1994) and van Norden and Schaller (1993) to stock market returns, assuming that returns are characterized by a mixture of distributions. This gives rise to a fat-tailed distribution, a feature of the return data which has been extensively documented since the early work by Mandelbrot (1963).

This approach is different to the mixture of two normal distributions proposed by JP Morgan as a new methodology for measuring VaR (Longerstay, 1996). In the JP Morgan approach, the discontinuous shift random variable is a Bernoulli variable, that is, s_t assumes the values 0 and 1 with respective probabilities of π and $(1-\pi)$. The future value of this variable (s_{t+1}) is independent of the value s_t, that is, future values of the state variable are independent on the current state. In the JP Morgan approach the distribution of future returns depends only on the unconditional probabilities of the Markov chain:

$$\pi = \begin{bmatrix} (1-q)/(2-p-q) \\ (1-p)/(2-p-q) \end{bmatrix} \tag{9}$$

instead of the conditional probabilities p and q. The two approaches are the same only if p and q are equal to 0.5.

[3] Switching regime models is a methodology that has encountered great success in macroeconomics applications. In the path-breaking works by Quandt (1958), as well as Goldfeld and Quandt (1973, 1975) it was used to describe markets in disequilibrium. Hamilton (1989, 1994) has brought about a Renaissance of this methodology by modelling business cycles. In Engel and Hamilton (1990), the switching approach is successfully applied to exchange rates. Firstly, applications to finance have been scarce, noteworthy exceptions being Pagan and Schwert (1990), Turner et al. (1989), as well as van Norden and Schaller (1993), Rockinger (1994) and Hamilton and Susmel (1994). Now there is high interest for this type of models: see for example, Billio and Pelizzon (1997), Ang and Bekaert (1999), Campbell and Li (1999), Khabie-Zeitoun et al. (1999), Jeanne and Masson (1999).

The advantage of using a Markov chain as opposed to a Bernoulli specification for the random discontinuous shift is that the former allows to conditional information to be used in the forecasting process. This allows us to: (i) fit and explain the time series, (ii) capture the well known cluster effect, under which high volatility is usually followed by high volatility (in presence of persistent regimes), (iii) generate better forecasts compared to the mixture of distributions model, since switching regime models generate a time conditional forecast distribution rather than an unconditional forecasted distribution.

To calculate the VaR, under the SSRM process, it is necessary to determine the critical value of the conditional distribution for which the cumulative density is a. Assuming $k=2$, the critical value (and so the VaR) is defined as:

$$a = \sum_{s_{t+h}=0,1} Pr(s_{t+h}|I_t) \int_{-\infty}^{\mathrm{VaR}} N(x, \mu(s_{t+h}), \sigma^2(s_{t+h})|I_t) \mathrm{d}x \tag{10}$$

where N is the normal distribution, I_t is the available information at date t, $Pr(s_{t+h}|I_t)$ is obtained by the Hamilton's filter (see Hamilton, 1994), $\mu(s_{t+h})$ and $\sigma^2(s_{t+h})$ are respectively the mean and the variance with $\mu(0)=\mu_0$, $\mu(1)=\mu_1$, $\sigma^2(0)=\sigma_0^2$ and $\sigma^2(1)=\sigma_1^2$.

The SSRM we present above is a special case of other, more general, switching regime models that we present below.

3.2. Switching regime beta models

The SSRM does not provide an explicit link between the return on the stock and the return on the market index. The Switching Regime Beta Model (SRBM) is a sort of market model or better a single factor model in the APT framework where the return of a single stock i is characterized by the regime switching of the market index and the regime switching of the specific risk of the asset. The SRBM can be written as:

$$\begin{cases} R_{mt} = \mu_m(s_t) + \sigma_m(s_t)\varepsilon_t, & \varepsilon_t \sim \mathrm{IIN}(0,1) \\ R_{it} = \mu_i(s_{it}) + \beta_i(s_t, s_{it}) R_{mt} + \sigma_i(s_{it})\varepsilon_{it}, & \varepsilon_{it} \sim \mathrm{IIN}(0,1) \end{cases} \tag{11}$$

where s_t and s_{it} are two independent Markov chains and ε_{it} and ε_t are independently distributed.

In such a framework the conditional mean of the risky asset is given by the parameter $\mu_i(s_{it})$ that is specific to the asset plus the factor loading ($\beta_i(s_t, s_{i,t})$) on the conditional mean of the factor. The factor loading compensates for the risk of the asset, which depends on the factor: higher covariances demand higher risk premium. The variance is the sum of variance of the index market weighted by the factor loading and the variance of the idiosyncratic risk.

To calculate VaR, we use the approach as before. Assuming that $k=2$ for both the Markov chains we have:

$$a = \sum_{s_{t+h}=0,1} \sum_{s_{i,t+h}=0,1} Pr(s_{t+h}, s_{i,t+h} | I_t) \int_{-\infty}^{\mathrm{Var}} N(x, \mu(s_{t+h}, s_{i,t+h}),$$

$$\sigma^2(s_{t+h}, s_{i,t+h}) | I_t) \mathrm{d}x \tag{12}$$

where N is the normal distribution with $\mu(s_{t+h}, s_{i,t+h}) = \mu_i(s_{i,t+h}) + \beta_i(s_{t+h}, s_{i,t+h})\ \mu_m(s_{t+h})$ and $\sigma^2(s_{t+h}, s_{i,t+h}) = \beta_i^2(s_{t+h}, s_{i,t+h})\ \sigma_m^2(s_{t+h}) + \sigma_i^2(s_{i,t+h})$, I_t is the available information at date t and $Pr(s_{t+h}, s_{i,t+h} | I_t)$ is obtained, as before, by the Hamilton filter.

The SRBM considers a single asset only, but can be generalized to calculate the VaR for a portfolio of assets taking into account the correlation between different assets.

3.3. Multivariate switching regime model

The generalized version of the SRBM, considering N risky assets, that we call the Multivariate Switching Regime Model (MSRM), can be written as:

$$\begin{cases} R_{mt} = \mu_m(s_t) + \sigma_m(s_t)\varepsilon_t, & \varepsilon_t \sim \mathrm{IIN}(0,1) \\ R_{1t} = \mu_1(s_{1t}) + \beta_1(s_t, s_{1t}) R_{mt} + \sigma_1(s_{1t})\varepsilon_{1t}, & \varepsilon_{1t} \sim \mathrm{IIN}(0,1) \\ R_{2t} = \mu_2(s_{2t}) + \beta_2(s_t, s_{2t}) R_{mt} + \sigma_2(s_{2t})\varepsilon_{2t}, & \varepsilon_{2t} \sim \mathrm{IIN}(0,1) \\ \vdots & \\ R_{Nt} = \mu_N(s_{Nt}) + \beta_N(s_t, s_{Nt}) R_{mt} + \sigma_N(s_{Nt})\varepsilon_{Nt}, & \varepsilon_{Nt} \sim \mathrm{IIN}(0,1) \end{cases} \tag{13}$$

where s_t and s_{jt}, $j=1,\dots,N$ are independent Markov chains, ε_t and ε_{jt}, $j=1,\dots,N$, are independently distributed.

Using this approach we are able to take into account the correlation between different assets. In fact, if we consider $k=2$, two assets, and, for example, $s_t = s_{1t} = 0$ and $s_{2t} = 1$, the variance–covariance matrix between the two assets is:

$$\Sigma(0,0,1) = \begin{bmatrix} \beta_1^2(0,0)\sigma_m^2(0) + \sigma_1^2(0) & \beta_1(0,0)\beta_2(0,1)\sigma_m^2(0) \\ \beta_2(0,1)\beta_1(0,0)\sigma_m^2(0) & \beta_2^2(0,1)\sigma_m^2(0) + \sigma_2^2(1) \end{bmatrix} \tag{14}$$

then the correlation between different assets is given by β's parameters and market variance.

In this model, as in the market model, the covariance between asset 1 and asset 2 depends on the extent to which each asset is linked, through the factor loading β, to the market index.

To calculate VaR for a portfolio based on N assets it is enough to use the approach presented above. In particular, considering two assets and assuming that $k=2$ for all the three Markov chains we have:

$$a = \sum_{s_{t+h}=0,1}\sum_{s_{1,t+h}=0,1}\sum_{s_{2,t+h}=0,1} Pr(s_{t+h}, s_{1,t+h}, s_{2,t+h} | I_t)$$

$$\times \int_{-\infty}^{\mathrm{VaR}} N(x, \boldsymbol{w}'\boldsymbol{\mu}(s_{t+h}, s_{1,t+h}, s_{2,t+h}),$$

$$\boldsymbol{w}'\Sigma(s_{t+h}, s_{1,t+h}, s_{2,t+h})\boldsymbol{w} | I_t)\mathrm{d}x \tag{15}$$

where $\boldsymbol{w}$ is the vector of the percentage of wealth invested in the two assets and $\boldsymbol{\mu}(s_{t+h}, s_{1,t+h}, s_{2,t+h})$ is the vector of risky asset mean returns based on the single asset mean returns already described in the SRBM. For example with $s_t = s_{1t} = 0$ and $s_{2t} = 1$, we have:

$$\boldsymbol{\mu}(0,0,1) = \begin{cases} \mu_1(0) + \beta_1(0,0)\mu_m(0) \\ \mu_2(1) + \beta_2(0,1)\mu_m(0) \end{cases} \tag{16}$$

However, MSRM requires the estimation of a number of parameters that grows exponentially with the number of assets. In fact, the number of possible regimes generates by this model is 2^{N+1}.

3.4. The factor switching regime model

One possible solution to the problem that affects the MSRM is to consider the specific risk distributed as $\mathrm{IIN}(0,\sigma_i^2)$ (without a specific Markov chain dependency) and characterize the systematic risk with more than one source of risk. This approach (that we call Factor Switching Regime Model (FSRM)) is in line with the Arbitrage Pricing Theory Model where the risky factors are characterized by switching regime processes. Formally, we can write this model as:

$$\begin{cases} F_{jt} = \alpha_j(s_{jt}) + \theta_j(s_{jt})\varepsilon_{jt}, & \varepsilon_{jt} \sim \mathrm{IIN}(0,1) \\ R_{1t} = \mu_1 + \sum_{j=1}^{g} \beta_{1j}(s_{jt})F_{jt} + \sigma_1\varepsilon_{1t}, & \varepsilon_{1t} \sim \mathrm{IIN}(0,1) \\ R_{2t} = \mu_2 + \sum_{j=1}^{g} \beta_{2j}(s_{jt})F_{jt} + \sigma_2\varepsilon_{2t}, & \varepsilon_{2t} \sim \mathrm{IIN}(0,1) \\ \vdots \\ R_{Nt} = \mu_N + \sum_{j=1}^{g} \beta_{Nj}(s_{jt})F_{jt} + \sigma_N\varepsilon_{Nt}, & \varepsilon_{Nt} \sim \mathrm{IIN}(0,1) \end{cases} \tag{17}$$

where F_{jt} is the value of factor j at time t $(j=1,2,\dots,g)$, $\beta_i(s_{jt})$ is the factor loading of asset i on factor j and s_{jt} is the Markov chain that characterizes factor j. Further, $s_{jt},\ j=1,\dots,g$ are independent Markov chains, $\varepsilon_{jt},\ j=1,\dots,g$, and $\varepsilon_{it},\ i=1,\dots,N$, are independently distributed.

The FSRM is more parsimonious, in fact the introduction of an extra asset means that only $g+2$ parameters need to be estimated.

This approach is valid when the number of assets in the portfolio is high and the specific risk is easily eliminated by diversification.

Using this approach, the variance–covariance matrix is simply:

$$\Sigma(s_t)=\begin{bmatrix}\beta_1^2(s_t)\theta^2(s_t)+\sigma_1^2 & \beta_1(s_t)\beta_2(s_t)\theta^2(s_t) & \cdots & \beta_1(s_t)\beta_N(s_t)\theta^2(s_t)\\ \beta_2(s_t)\beta_1(s_t)\theta^2(s_t) & \beta_2^2(s_t)\theta^2(s_t)+\sigma_2^2 & \cdots & \cdots\\ \cdots & \cdots & \cdots & \cdots\\ \beta_N(s_t)\beta_1(s_t)\theta^2(s_t) & \cdots & \cdots & \beta_N^2(s_t)\theta^2(s_t)+\sigma_N^2\end{bmatrix}\tag{18}$$

The VaR for a portfolio based on N assets, one factor and $k=2$ is defined as:

$$a=\sum_{s_{t+h}=0,1}Pr\left(s_{t+h}|I_t\right)\int_{-\infty}^{\mathrm{VaR}}N\left(x,w'\mu\left(s_{t+h}\right),w'\Sigma\left(s_{t+h}\right)w|I_t\right)\mathrm{d}\,x\tag{19}$$

where $\boldsymbol{\mu}(s_{t+h})$ is the vector of risky assets mean returns, that is:

$$\boldsymbol{\mu}\left(s_{t+h}\right)=\begin{bmatrix}\mu_1+\beta_1\left(s_{t+h}\right)\alpha\left(s_{t+h}\right)\\ \mu_2+\beta_2\left(s_{t+h}\right)\alpha\left(s_{t+h}\right)\\ \cdots\\ \mu_N+\beta_N\left(s_{t+h}\right)\alpha\left(s_{t+h}\right)\end{bmatrix}\tag{20}$$

4. VaR estimation

4.1. VaR estimation of a single asset

The data used in the empirical analysis are daily returns based on closing price of 10 Italian Stocks[4] and the MIB30 Italian market index. The data cover the period from November 29, 1995 to September 30, 1998: a total of 714 daily observations. Table 1 gives summary statistics and a normality test: all the series fail to pass the Jarque–Bera normality test.

[4] Comit, Credit, Fiat, Imi, Ina, Mediobanca, Ras, Saipem, Telecom, Tim.

Table 1
Data description (Jarque–Bera test critical value at 5% is 10.597)

	Mean	Standard error	Asymmetry	Kurtosis	Jarque–Bera test
Comit	0.0017	0.0239	−0.0679	5.1638	136.8291
Credit	0.0020	0.0233	0.2916	4.8368	108.1382
Fiat	−0.0002	0.0211	0.0713	5.3558	162.2487
Imi	0.0013	0.0229	0.0362	5.2777	151.2175
Ina	0.0011	0.0194	0.4588	8.2976	845.6066
Mediobanca	0.0006	0.0245	0.0768	5.5915	196.4285
Ras	0.0001	0.0188	−0.3125	6.1520	301.4666
Saipem	0.0010	0.0196	−0.0548	5.2827	152.0853
Telecom	0.0013	0.0203	−0.1053	4.3811	56.5893
Tim	0.0018	0.0201	0.0601	3.6938	14.2079
MIB30	0.0010	0.0156	−0.2469	4.8238	103.8987

The first step in the empirical analysis is the estimation of all the models presented above using all the observations in the data set. We assume that all the Markov Chains have two states: [0,1] and we estimate the parameters[5] using Maximum Likelihood and Hamilton's filter.[6]

In order to determine the future distribution, we need to know the actual regime. The probabilistic inference of being in one of the two regimes can be calculated for each date t of the sample using the Hamilton's filter and smoother algorithm (Hamilton, 1994). For an illustrative purpose, the resulting series for the MIB30 market index and 2 stocks are shown in Figs. 1–4. It is easy to observe how rapidly the probability of switching from one regime to the other changes during time. This demonstrates the ability of the model to capture the effect of potential changes in the volatility of returns. It is interesting to observe that even if the estimation is carried out one stock at a time (that is the MIB30 market index and each single stock), the market Markov chain behavior is almost the same. This means that the identification of the market index parameters is possible from each single equation.

The second step is the estimation of VaR for different single stocks. We split the sample of 714 observations in two sets: the first 250 observations and the remaining 464. We estimated the daily VaR at 1%, 2.5% and 5% level of significance for the second subset (the last 464 observations in the data set). The

[5] Parameters $\mu_i(0)$ and $\mu_i(1)$ are not statistically different and we improved the estimation by considering $\mu_i(s_{it})$ as a constant term μ_i. The estimated betas are quite different in each regime: we performed a test on the null hypothesis that they are all equals and in all cases this hypothesis is rejected at the 5% level, even if, in some cases, two by two they are not statistically different. The parameter estimation will be provided by request.

[6] See Hamilton (1994) for an analytic description of it.

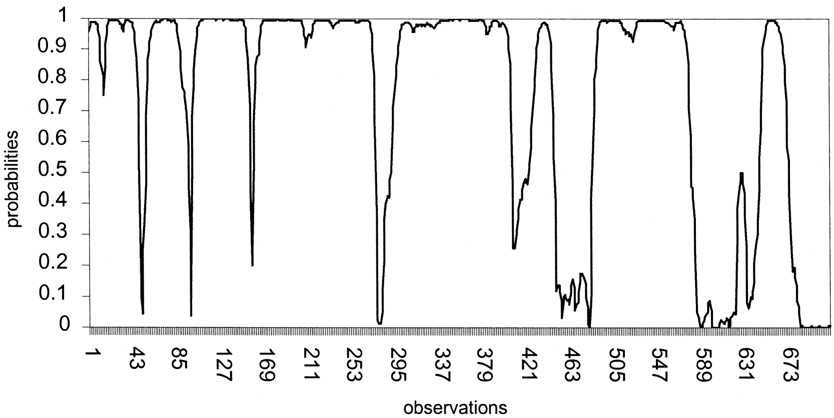

Fig. 1. SRBM: smoothed probabilities of regime 0 for the market chain when estimated together with Comit.

VaR estimations performed are based on the first 250 observations of the data set and the parameters are re-estimated increasing the sample every 50 observations. Between the different estimations, VaRs are determined using the same parameters and the forecasted probability of switching day by day determined by the Hamilton filter augmenting the data set each day with one observation.

In order to analyse the results of our different models we performed a backtesting analysis, that is, we analysed the number of exceptions observed in the 464 daily VaR values estimated. The goodness-of-fit test is applied to the null hypothesis that the frequency of such exceptions is respectively equal to 1%, 2.5% and 5% with respect to the different VaR estimations.

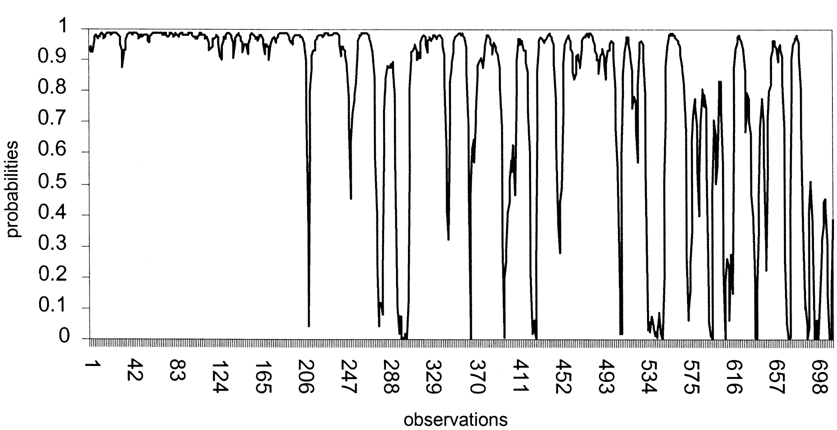

Fig. 2. SRBM: smoothed probabilities of regime 1 for the Comit-specific chain.

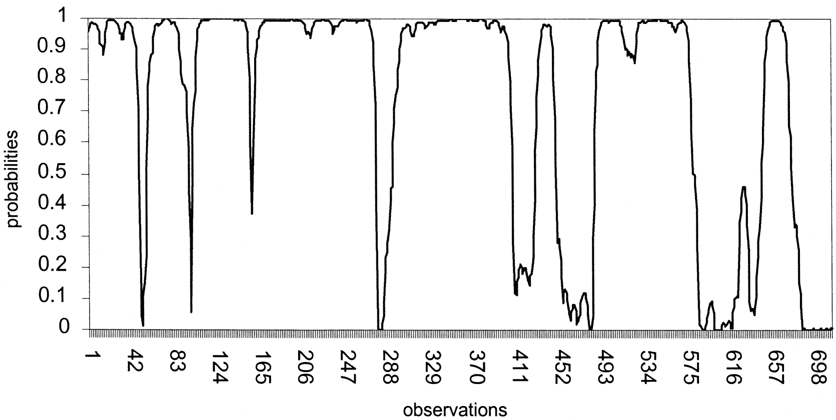

Fig. 3. SRBM: smoothed probabilities of regime 0 for the market chain when estimated together with Tim.

We also compared these results with the performance of other approaches suggested in the VaR literature: the JP Morgan RiskMetrics (RiskMetrics, 1995) variance–covariance approach and the GARCH approach.

The RiskMetrics approach assumes that returns conditionally follow a joint normal distribution with mean zero and time varying variance–covariance matrix estimated using an exponentially weighted moving average approach. In particular at time t the estimated covariance between asset i and asset j is determined as:

$$\sigma_{i,j} = (1-\lambda)\sum_{h=0}^{H} \lambda^h R_{i,t-h} R_{j,t-h} \tag{21}$$

where λ is the decay factor and in the RiskMetrics approach is chosen to be 0.94.

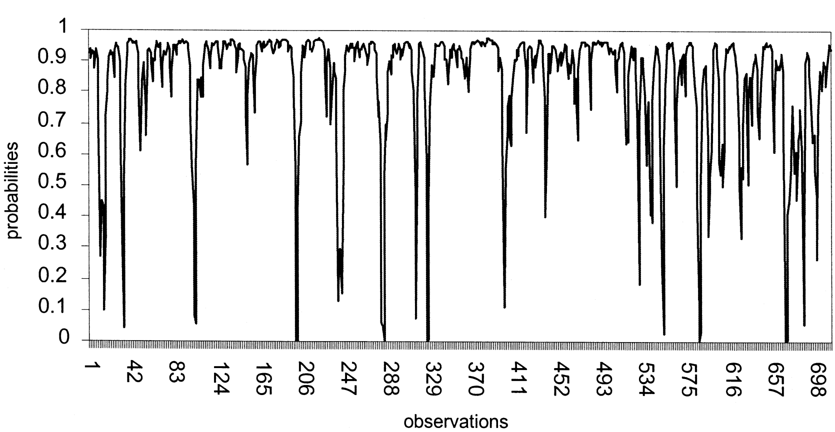

Fig. 4. SRBM: smoothed probabilities of regime 1 for the Tim-specific chain.

For the RiskMetrics variance–covariance model we considered both the approach suggested by JP Morgan where the VaR for each asset is based on its volatility (we call this RM) and the approach for equities: where the calculation is based on the risk of the Market index and the beta of the stocks (we call this RMB). Specifically, in the RM model we have:

$$\mathrm{VaR}_i = \Phi(a)\sqrt{\sigma_{i,i}} \tag{22}$$

where: VaR_i is the VaR of the generic stock i, $\Phi(a)$ is the cumulate standard normal distribution coefficient, $\sigma_{i,i}$ is the asset i variance.

In the RMB model we calculate VaR as:

$$\mathrm{VaR}_i = \beta_{i,m}\mathrm{VaR}_m \tag{23}$$

where: VaR_m is the VaR of the Market index, $\beta_{i,m} =$ is the beta of the beta of asset i with respect to the market index.

For VaR calculation based on RiskMetrics approach we used a moving window of 250 observations.

Regards the GARCH approach we consider four models. The first is a GARCH model with conditional normal distribution and zero mean (in line with the RiskMetrics approach). In particular, we assume that returns follows a GARCH(1,1), that is:

$$R_t = u_t = \eta_t \varepsilon_t \tag{24}$$

where: $\varepsilon_t \sim N(0,1)$, $E(u_t^2 | I_{t-1}) \equiv \eta_t^2 = a + bu_{t-1}^2 + \gamma\eta_{t-1}^2$, I_t is the information set.

With this model we have:

$$\mathrm{VaR}_i = \Phi(a)\eta_{t+1} \tag{25}$$

where: η_{t+1} is the GARCH asset i variance.

The second is a beta-GARCH model (or one factor model), that we call GARCHB, with the following return specification:

$$\begin{cases} R_{mt} = \mu_m + u_{mt}, \\ R_{it} = \alpha_i + \beta_i R_{mt} + u_{it}, \end{cases} \tag{26}$$

where: $u_{it} = \eta_{it}\varepsilon_{it}$, $\varepsilon_{it} \sim N(0,1)$, $E(u_{it}^2 \mid I_{t-1}) \equiv \eta_{it}^2 = a + bu_{i,t-1}^2 + \gamma\eta_{i,t-1}^2$, $u_{m,t} = \sigma_{m,t}\varepsilon_{mt}$, $\varepsilon_{mt} \sim N(0,1)$, $E(u_{mt}^2 \mid I_{t-1}) \equiv \sigma_{m,t}^2 = a_m + b_m u_{m,t-1}^2 + \gamma_m \sigma_{m,t-1}^2$, $\mathrm{Cov}(u_{it}, R_{m,t} | I_{t-1}) = 0$.

With this model we have[7]:

$$\mathrm{VaR}_i = \mu_{i,t+1} + \Phi(a)\sigma_{i,t+1} \tag{27}$$

where: $\mu_{i,t+1} = \alpha_i + \beta_i E(R_{m,t+1} | I_t) = \alpha_i + \beta_i \mu_{m,t}$, $\sigma i,t+1 = \sqrt{\beta_I^2 \sigma_{m,t+1}^2 + \eta_{i,t+1}^2}$.

[7] For a derivation of conditional moments in a factor model see Gouriéroux (1992, p. 218).

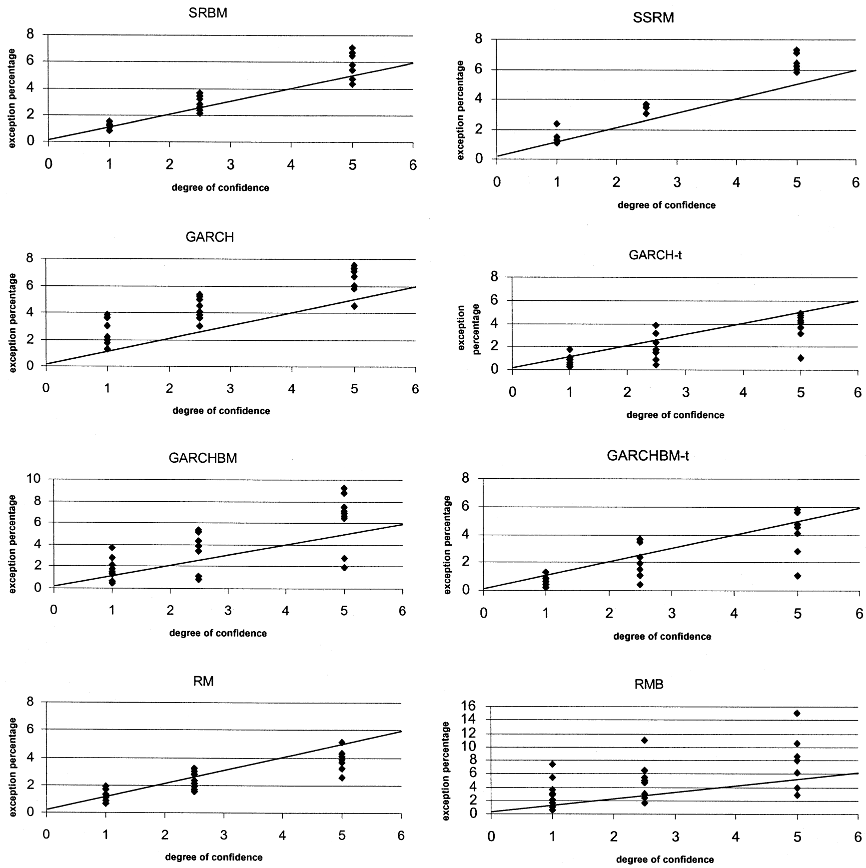

Fig. 5. Backtesting over 464 observations. Each point represents one stock for different degree of confidence.

The third and the fourth models are only an extension of the previous ones. We also consider in fact Student's *t* innovation and we generate GARCH-*t*(1,1) and GARCHB-*t*(1,1). In these cases, we estimate the degrees of freedom for each stock.

Fig. 5 shows the backtesting analysis results of SSRM, SRBM, RM, RMB and the two versions of GARCH(1,1) and beta GARCH models (GARCH, GARCHB, GARCH-t and GARCHB-t) for the individual stocks.[8] Table 2 reports an analysis

[8] Regards GARCH-*t* model, the degrees of freedom estimated are: Comit 9, Credit 8, Fiat 8, Imi 10, Ina 5, Ras 6, Saipem 6, Telecom 8, Tim 16. The estimation of Mediobanca degree of freedom does not converge and then we consider only the Gaussian version of the GARCH(1,1) model.

Table 2
Mean absolute error of α-values over 10 single asset portfolios

α^* (%)	SRBM	SSRM	GARCH	GARCH-*t*	GARCHB	GARCHB-*t*	RM	RMB
5	0.009	0.014	0.017	0.012	0.025	0.013	0.011	0.035
2.5	0.004	0.009	0.019	0.010	0.017	0.009	0.005	0.026
1	0.002	0.004	0.014	0.004	0.011	0.004	0.005	0.020

of the mean absolute difference between the observed and theoretical confidence level.

Table 3
p-Values of the Proportion of Failure test (PF) for the different models (given the poor performance of SSRM and RMB models we avoid to present their test value)

PF test	Degree of confidence (%)	SRBM *p*-value (%)	GARCH *p*-value (%)	GARCHB *p*-value (%)	GARCH-*t* *p*-value (%)	GARCHB-*t* *p*-value (%)	RM *p*-value (%)
Comit	5.0	42.96	96.60	16.50	35.63	55.81	24.97
	2.5	7.02	7.79	2.33	48.92	13.28	68.29
	1.0	1.32	0.04	0.00	86.82	54.38	7.16
Credit	5.0	42.96	3.09	4.90	3.50	1.82	0.88
	2.5	62.62	1.19	21.55	0.91	2.71	25.72
	1.0	86.82	15.52	30.57	41.33	16.48	41.33
Fiat	5.0	13.28	4.90	7.55	79.66	63.41	35.63
	2.5	86.82	13.28	2.33	14.02	85.71	14.02
	1.0	66.86	54.38	15.52	75.97	41.33	75.97
Imi	5.0	3.09	3.09	0.06	16.64	79.66	48.55
	2.5	85.72	21.56	2.33	25.72	42.12	85.72
	1.0	30.58	0.35	54.38	86.82	75.97	7.16
Ina	5.0	16.50	3.09	1.90	35.63	35.63	86.54
	2.5	21.56	0.05	0.05	14.02	14.02	48.92
	1.0	54.38	0.00	0.00	3.95	16.49	0.35
Mediobanca	5.0	16.50	4.36	0.06			16.64
	2.5	21.56	3.02	2.71			62.62
	1.0	86.82	0.33	41.33			75.97
Ras	5.0	37.45	0.27	1.82	0.00	0.00	24.97
	2.5	27.35	0.04	0.91	0.04	0.00	62.62
	1.0	41.82	0.02	16.49	16.49	16.49	54.38
Saipem	5.0	23.36	4.36	0.06	16.64	35.63	33.31
	2.5	13.28	7.16	0.12	25.72	85.72	7.16
	1.0	31.64	0.35	0.14	41.33	75.98	1.32
Telecom	5.0	64.96	4.36	64.96	36.60	79.66	48.55
	2.5	16.50	3.02	16.50	25.72	42.12	85.72
	1.0	31.64	0.35	31.64	41.33	41.33	54.38
Tim	5.0	70.48	11.32	11.32	48.55	42.96	6.25
	2.5	68.29	3.02	7.79	85.72	21.56	86.82
	1.0	30.58	1.32	3.02	41.33	75.97	82.92

From Fig. 5 and Table 2 it is evident that the SRBM performs quite well for almost every percentile and every stock. In fact the results are close to the theoretical values and it does not seem that the model persistently either under or over estimates any of the confidence level. In particular, it is interesting to observe that the SRBM performs always better than the SSRM, which suggests that the link with the market is fundamental to risk estimation.

Moreover, the SRBM performs better than all GARCH models. In fact the Gaussian GARCH and GARCHB models do not work very well as the number of extreme observations deviates significantly from the theoretical values for almost all the stocks. Generally, the values are higher than the theoretical ones and means

Table 4
p-Values of the Time Until First Failure Test (TUFF) for the different models

TUFF test	Degree of confidence (%)	SRBM *p*-value (%)	GARCH *p*-value (%)	GARCHB *p*-value (%)	GARCH-*t* *p*-value (%)	GARCHB-*t* *p*-value (%)	RM *p*-value (%)
Comit	5.0	1.44	1.44	1.44	1.44	1.44	1.44
	2.5	0.24	0.66	0.66	0.66	0.66	0.66
	1.0	62.15	0.24	0.24	87.13	87.13	0.24
Credit	5.0	22.01	1.44	1.44	1.44	1.44	1.44
	2.5	10.99	22.01	0.66	0.00	22.01	0.66
	1.0	36.33	10.99	7.87	4.93	4.93	7.87
Fiat	5.0	6.68	1.44	1.44	1.44	1.44	95.84
	2.5	59.83	50.05	71.84	6.68	6.68	6.68
	1.0	83.83	59.83	59.83	59.83	59.83	59.83
Imi	5.0	46.53	46.53	46.53	44.30	46.42	6.84
	2.5	97.96	97.96	97.96	97.96	97.96	22.01
	1.0	50.86	44.51	44.51	50.86	50.86	77.00
Ina	5.0	91.53	6.84	91.53	91.53	91.53	6.84
	2.5	47.32	47.32	47.32	1.29	1.29	47.32
	1.0	32.61	8.11	26.63	32.61	32.61	8.11
Mediobanca	5.0	2.35	44.30	15.61			77.88
	2.5	32.61	23.52	23.52			23.52
	1.0	87.88	32.61	94.51			32.61
Ras	5.0	81.24	12.26	92.09	92.09	92.09	81.24
	2.5	12.26	12.26	58.07	70.82	70.82	65.81
	1.0	55.73	55.73	27.76	27.76	27.76	25.50
Saipem	5.0	1.44	1.44	1.44	1.44	1.44	1.44
	2.5	0.66	0.66	0.66	0.66	0.66	0.24
	1.0	0.11	0.11	0.24	32.61	0.24	0.11
Telecom	5.0	46.53	6.84	6.84	46.52	46.52	84.72
	2.5	22.01	22.08	22.01	63.26	63.26	63.26
	1.0	24.37	24.37	24.37	24.37	24.37	24.37
Tim	5.0	46.53	46.53	46.53	46.53	46.53	46.53
	2.5	60.62	22.01	22.01	60.62	60.62	60.62
	1.0	32.61	67.60	67.60	32.61	32.61	67.60

that the models underestimates risk. The Student's t versions perform better but generally overestimate risk.

With regard to the RM and RMB models, the RMB performs poorly, as expected, on single assets. In contrast the RM performs quite well and is preferable to the RMB and GARCH models. However, RM usually overestimates risk at the 5% and the 2.5% confidence levels, while it underestimates risk at the 1% level. This implies that this model is unable to capture the extreme events. Moreover, it overestimates risk given its inability to return quickly to "normal condition" when the market has already returned to normality after a shock. Comparing the results of RM with the SRBM we have that, in most of the cases, the SRBM performs better. In fact, the SRBM is able to account for both the information coming from the market index and those that characterize the single stock. The RM and the RMB models are unable to do this.

An important issue here is testing the goodness of fit for the null hypothesis that the probability of an observation falling in the category of extreme returns is

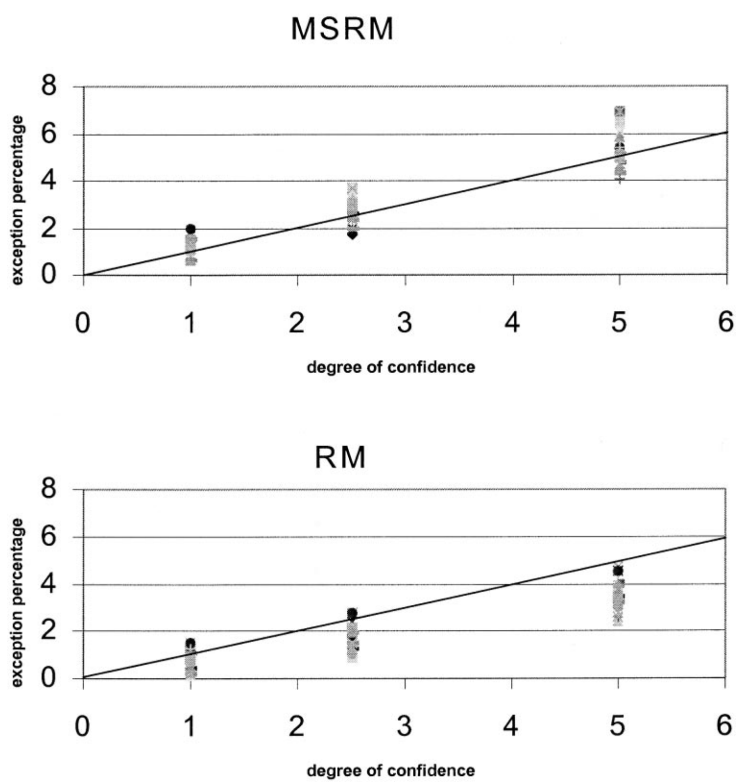

Fig. 6. Two asset portfolios backtesting over 464 observations. Each point represents one portfolio for different degrees of confidence.

Table 5
Mean absolute error of α-values over 21 two asset portfolios

α^* (%)	SRBM	RM
5	0.007594	0.014594
2.5	0.005088	0.007512
1	0.002928	0.004093

equal to 1%, 2,5%, 5%, respectively. As discussed earlier we evaluate the different models using two different tests.

The results of the PF, and TUFF tests are shown in Tables 3 and 4. In almost all the cases, Gaussian GARCH and GARCHB models do not perform well, and in many cases, the null hypothesis is rejected. The Student's t versions (GARCH-t and GARCHB-t) perform better, but they fail to pass tests more frequently than the RM model.

In summary, the SRBM and the RM perform quite well, and, in most of the cases the SRBM results presents a higher p-value. The test results are consistent with those shown in Table 2.

Moreover, for the 1% level using the approach adopted by the regulators, the SRBM always falls in the green zone, while, the RM sometimes falls in the yellow zone.

4.2. VaR estimation of a portfolio of assets

We consider 21 different portfolios, constructed by combining the different stocks previously considered two at a time. Every combination is characterized by three different combinations of weights: 50%–50%, 80%–20%, 20%–80%. To evaluate the performance of our model we use the MSRM presented above and the multivariate RM approach. We do not consider the GARCH(1,1) models and the RMB given their poor performance in the single asset case.

The results are interesting since, in almost all the portfolios, the MSRM performs better than the RM approach as reported in Fig. 6 and Table 5. It is easy to observe that the difference in performance between the MSRM and the RM is greater than in the single asset case. Moreover, the RM model continues to

Table 6
10 Assets equally weighted portfolio backtesting over 464 observations

α^* (%)	One FSRM (%)	RMB (%)
5	5.8	2.6
2.5	3.9	1.08
1	1.7	1.08

Table 7
p-Values of Failure test (PF) and Time Until First Failure Test (TUFF) of the 10 assets equally weighted portfolio

	α^* (%)	One FSRM *p*-value (%)	RMB *p*-value (%)
PF Test	5	42.96	0.9
	2.5	7.81	2.72
	1	15.56	86.82
TUFF Test	5	46.53	46.53
	2.5	22.01	1.29
	1	38.53	32.61

underestimate risk at the 5% and 2.5% confidence level while the MSRM does not.

When we consider the equally weighted portfolio containing all 10 assets the MSRM model cannot be used because the number of parameters, which it is necessary to estimate is too high. The same reason led JP Morgan to propose the RMB to estimate VaR. The FSRM does not present this problem and, for this reason, we prefer to use, for the equally weighted portfolio VaR estimation, the FSRM with only one factor: the market index (one-FSRM). We compare the result of this model with the RMB model. Tables 6 and 7 give the results.

As the result show, both models perform poorly relative to the previous cases. The one-FSRM underestimates risk for all the confidence level and the RMB overestimates at the 5% and the 2.5% percentile. However, the tests show that the one-FSRM is always accepted at the 1% p-value while the RMB is not (see the 5% percentile of PF test). For the TUFF test, the one FSRM is always better than the RMB. In summary, the one-FSRM is preferred on a statistical basis to the

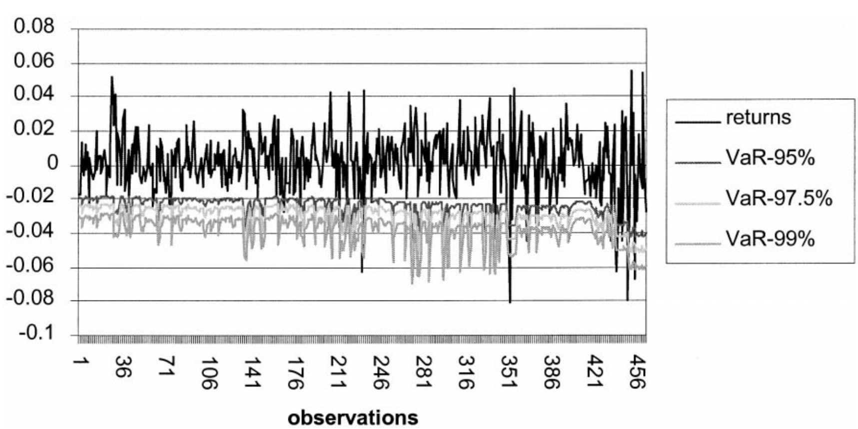

Fig. 7. One FRSM: backtesting VaR values for the equally weighted portfolio at different confidence level.

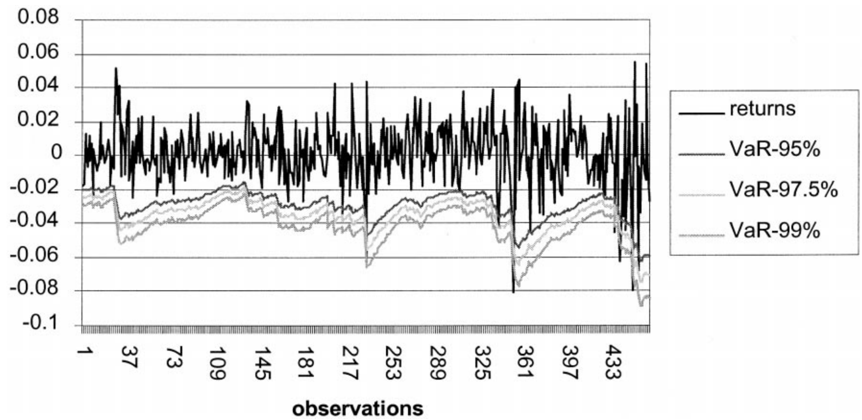

Fig. 8. RMB: backtesting VaR values for the equally weighted portfolio at different confidence level.

RMB. The one-FSRM could be easily improved by increasing the number of factors.

Furthermore, it is interesting to note the implications of the two models for VaR based capital requirements.[9] It is evident from Figs. 7 and 8 that the one-FSRM is better than the RMB to capture quickly the changes in volatility of the returns. This implies that the former model requires a high capital allocation only when the market is highly volatile and avoids high capital when market volatility returns to normality. In contrast, RMB always requires a high capital even when market volatility decreases.

Furthermore, it is interesting to observe that the average VaRs at the different confidence levels are: for the FSRM, 2.46%, 3.05% and 3.86% with standard deviations of 0.62%, 0.77% and 0.89%, respectively; while, for the RMB the VaRs at the different confidence levels are on the average: 2.99%, 3.57% and 4.23% with standard deviations 0.87%, 1.04% and 1.23%, respectively. This means that the FSRM leads to a lower level of capital required and smaller revision to required capital.

5. Conclusion

This paper has analysed the application of switching regime models to measuring VaR in order to account for a non-normal return distribution. Four different

[9] In Jannuary 1996, the Basle Supervisory Commitee issued a Market Risk Ammendment to the 1988 Accord (Basle Commitee on Banking Supervision (1996)), specyfing a minimum capital requirement based on bank internal models for market risks. This approach takes daily VaR calculations from the bank's own risk management system and applies a multiplier to arrive at a required capital set-aside to cover market portfolio risk.

switching models are considered: SSRM, SRBM for one asset, MSRM for two assets portfolio and the FSRM for an equally weighted portfolio containing 10 assets. To illustrate the application of our approach we calculated the VaR for 10 individual Italian stocks and for the MIB30 market index. We compared our results with those obtained from the variance–covariance RiskMetrics approach and two versions of the GARCH(1,1) model.

For portfolios with one and two assets the SRBM and MSRM perform well for almost every percentile and every portfolio. In fact, the results are close to the theoretical values and it does not seem that models persistently under or over estimate each level of confidence. The SRBM model always performs better than the SSRM and this means, as expected, that the link with the market is fundamental for the estimation of equity risk.

The SRBM performs better than GARCH and GARCHB models. In fact the GARCH and GARCHB models do not work well as the number of exceptions deviates significantly from the theoretical values for almost all the stocks. Generally, the values are higher than the theoretical ones for the Gaussian version and lower for the Student's t version. This means that these models always underestimate (Gaussian) or overestimate (Student's t) risk.

With regard to the RiskMetrics approach we considered two models: one (RM) based on the variance estimation of the single asset and the other (RMB) where the risk of a single asset is determined by its beta with respect to the market index. The RM performs better that the RMB for all the portfolios. Moreover, RM and RMB usually overestimate the risk at the 5% and the 2.5% confidence levels, and underestimate the risk at the 1% level. In other words, this approach does not reliably capture the risk of the extreme events. Comparing the RM results with the SRBM for one asset and with the MSRM for the two asset portfolio we find that in most of the cases the SRBM and MSRM performs better than RM and RMB. These results are confirmed by two tests: the PF test and the TUFF test and by the approach adopted by regulators for backtesting.

Finally, we estimate VaR for an equally weighted portfolio containing all 10 stocks. We use the one-FSRM and the RMB in order to estimate VaR. Both models performs poorly. However, the PF test and the TUFF test indicate that the one-FSRM is statistically preferable to the RMB model. This result is important and suggests that, with a higher number of factors, the FSRM has the potential to provide reliance estimates of market risk.

Acknowledgements

We gratefully acknowledge conversations with our supervisors, Alain Monfort and Stephen Schaefer, and help by Tiziana De Antoni. Pelizzon thanks ESRC for financial support.

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