TAKING VAR TO PIECES

Portfolio value-at-risk is not usually the sum of its parts, unless those parts are chosen carefully. Mark Garman **explains how**

Current methods of calculating value-at-risk prescribe either of two basic calculations: (a) the total, diversified VAR for a portfolio; or (b) the undiversified VAR for some subset of a portfolio. The portfolio subset might comprise all trades of a certain type or involving a certain asset, the individual trades themselves or even the solitary (mapped) cashflows. Except in rare and exceptional cases, however, the undiversified VAR numbers of the components of a portfolio almost never add up to the diversified VAR of the portfolio.

Neither does the undiversified VAR provide any hint as to whether the corresponding components act to "hedge" the remainder of the portfolio or serve only to increase its risk. This leads us to the search for a useful definition of "component VAR" (CVAR). A good definition would have at least three properties:

- ☐ if the components partition the portfolio (ie, are disjoint and exhaustive), then the CVARs should add up to the (diversified) portfolio VAR;
- ☐ if the component were to be deleted from the portfolio, the CVAR should tell us, at least approximately, how the portfolio VAR will change; and, therefore
- ☐ CVAR will be negative for components that act to hedge the rest of the portfolio.

In this paper, we show that such a definition of CVAR may be based upon the VARdelta (or DelVAR) concept (Garman, 1996[1]).

VARdelta is a portfolio metric appropriate to the analytic ("variance-covariance") methodology of VAR. The relationship of the VARdelta to the VAR is analogous to the relationship between the option delta and the option price. In this case, however, it measures the sensitivity of VAR to the injection of a unit of cashflow in each dimension (or "vertex", as per JP Morgan's RiskMetrics[2]) of the cashflow space. Garman (1996) also shows how to analyse a new, candidate trade's effect on portfolio VAR. Perhaps surprisingly, the same technique can be applied to trades already present in a portfolio, to form a useful and meaningful definition of CVAR.

Background

Let a portfolio of trades P be defined and let **m(P)** be a cashflow "mapping" function, ie, a function that determines the amounts of cashflows on a set of defined vertices, given that portfolio. Let the vector p be the amounts of such vertex cashflows, ie, **p** = **m(P),** stated in present-value, numeraire-based terms. Let Q be the horizon- and confidence level-scaled covariance matrix describing the covariance structure of the vertices.

For example, suppose a 5%, one-day VAR is desired. Then the scale factor would be $1.645 \times 1.645 / 262 = 0.0103$, where 1.645 is the number of standard deviations of a 5% cumulative normal distribution and 262 is the approximate number of trading days each year. This number would then multiply the covariante matrix describing the (annualised) vertex returns.

Then the portfolio VAR is given as:

$$\mathrm{VAR}(p) \equiv \sqrt{p'Qp}$$

In Garman (1996), the VARdelta quantity was calculated as:

$$\mathrm{VARdelta}(p) \equiv \nabla\sqrt{p'Qp} = \frac{Qp}{\sqrt{p'Qp}} = \frac{Qp}{\mathrm{VAR}(p)}$$

It was also reported that the incremental VAR of a new, candidate trade A with cashflow map **a = m(A)** is (approximately) given by the inner product of VARdelta and **m(A),** ie:

$$\mathrm{Incr}(a,p) = a' \bullet \mathrm{VARdelta}(p)$$

The paper also pointed out that this formula can be used to analyse the impact of the next trade upon portfolio VAR, possibly in a real-time setting, since this inner product is very quickly calculated. We will now apply similar reasoning to a trade within the portfolio.

Development

The foundation of CVAR rests upon a simple yet rather surprising theorem, ie, that, for any cashflow map **p**, we have:

$$\mathbf{p' \bullet VARdelta(p)} = \mathbf{p'} \bullet \frac{Qp}{\sqrt{p'Qp}} = \frac{p'Qp}{\sqrt{p'Qp}} = \mathbf{VAR\ (p)} \qquad (1)$$

in other words, the inner product of any cashflow map and its corresponding VARdelta is equal to its VAR. This unusual theoretical result forms the basis of our definition of CVAR.[3]

Suppose now that the cashflow map **p** derives from the arbitrary addition of component cashflow maps, ie:

$$p = \sum_{i=1}^{N} p_i$$

where N is the number of components. Then it follows from equation (1) that:

$$\mathrm{VAR}(p) = \mathbf{p' \bullet VARdelta(p)} = \sum_{i=1}^{N}\left\{p_i' \bullet \mathbf{VARdelta(p)}\right\}$$

Defining CVAR as the terms on the right-hand side of the last equation, we see that the first of the three criteria mentioned at the be-

A possible component VAR risk report

Trade number	Trade description	Mark-to-market	CVAR	Percentage CVAR
547	Currency swap	\$17,563	\$14,845	0.61
582	Currency option	\$214,432	\$141,861	5.82
236	Bond option	\$237,066	(\$121,490)	–4.99
...	...	...	...	...
393	Knock-out option	\$34,834	\$15,765	0.65
Total		**\$3,452,764**	**\$2,436,238**	**100.00**

1 "Improving on VAR", Risk May 1996, pages 61-63

2 See RiskMetrics – Technical Document, fourth edition, December 18, 1996, JP Morgan/Reuters. Further information and publications are available on the JP Morgan World Wide Web site, at: http://www.jpmorgan.com/RiskManagement/RiskMetrics/RiskMetrics.html

3 This result is analogous to the option-pricing theorem that states that the option value is equal to the sum of its value-denominated deltas, and apparently flows from analogous homogeneity properties, namely the first-degree homogeneity of VAR(p) in p

4 Note that the story does not end at this point. A "component" as used here means a "cashflow component" and this is not quite the same as a "portfolio component". To ensure thot these are indeed the same, it may be required that the cashflow mapping function **m(.)** be linear. This means that, for a suitably defined portfolio aggregation operator **"+"**, **m(A+B) = m(A)+m(B)** for all portfolios **A** and **B**. Both the mapping methodology itself and such institutional features as netting agreements may give rise to non-linear mapping functions

ginning of this article – ie, summation of the CVAR to the total portfolio VAR – is indeed satisfied. By Taylor series expansion (as per Garman, 1996), it can also be verified that the two other criteria are also satisfied.

Thus the term:

$$p_i' \bullet \text{VARdelta}(p)$$

is our choice for CVAR.[4] It divides total VAR in an additive fashion, regardless of how the subset cashflow maps are selected. In other words, the cashflow maps can be partitioned by trades, maturities, assets, instrument types or any other criterion and our VARdelta-based definition of CVAR remains valid.

We might go on to use this definition of CVAR in a trade-based risk report (see table). Note that trade 236 has a negative CVAR. This means that it serves as a "hedge" to the rest of the portfolio. If we deleted trade 236, the portfolio VAR would rise, by approximately $121,490. Similarly, trade 547 adds approximately $14,845 to portfolio VAR; were this trade deleted, portfolio VAR would fall by approximately this amount.

This approximation will be most nearly exact when the CVAR is "small". Thus, the CVAR for trade 547 is likely to be a better approximation than the CVAR for trade 236. Just as option deltas capture an option price's sensitivity to small changes in the underlying price, VARdelta - upon which CVAR is based – is exact only for small changes in the cashflow map.

The final column in the table shows CVAR as a percentage of total VAR. This shows how portfolio risk is concentrated in the individual trades, expressed as a fraction of overall risk.

Of course, partitioning the total portfolio by trades, in line with the foregoing, is not the only possibility. Similar component-risk reports might be analogously performed for various branch offices, for short-term versus long-term cashflows, or for a host of other criteria. The CVAR report permits us to "slice and dice" total risk by any category, simply by taking a single inner product for each category. ■

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