An Integrated Market and Credit Risk Portfolio Model

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We present a multi-step model to measure portfolio credit risk that integrates exposure simulation and portfolio credit risk techniques. Thus, it overcomes the major limitation currently shared by portfolio models with derivatives. Specifically, the model is an improvement over current portfolio credit risk models in three main aspects. First, it defines explicitly the joint evolution of market factors and credit drivers over time. Second, it models directly stochastic exposures through simulation, as in counterparty credit exposure models. Finally, it extends the Merton model of default to multiple steps. The model is computationally efficient because it combines a Mark-to-Future framework of counterparty exposures and a conditional default probability framework.

Credit risk modeling is one of the most important topics in risk management and finance today. The last decade has seen the development of models for pricing credit risky instruments and derivatives, for assessing the credit worthiness of obligors, for managing exposures of derivatives and for computing portfolio credit losses for bonds and loan portfolios. In light of these financial innovations and modeling advances the Basle Committee on Banking Supervision has taken the first steps to amend current regulation and is reviewing the applicability of internal credit risk models for regulatory capital (Basle Committee on Banking Supervision, 1999a, 1999b).

However, common practice still treats market and credit risk separately. When measuring market risk, credit risk is commonly not taken into account; when measuring portfolio credit risk, the market is assumed to be constant. The two risks are then "added" in *ad hoc* ways, resulting in an incomplete picture of risk.

There are two categories of credit risk measurement models: Counterparty Credit Exposure models and Portfolio Credit Risk models.

Derivative desks traditionally manage credit risk by monitoring and placing limits on counterparty credit exposures. **Counterparty exposure** is the economic loss that will be incurred on all outstanding transactions if a counterparty defaults, unadjusted by possible future recoveries. **Counterparty exposure models** measure and aggregate the exposures of all transactions with a given counterparty. In the BIS regulatory model, potential exposures are given by an add-on factor multiplying the notional of each transaction (Basle Committee on Banking Supervision, 1988). Although simple to implement, the model has been widely criticized because it does not accurately account for future exposures. Since exposures of derivatives such as swaps depend on the level of the market when default occurs, models must capture not only the actual exposure to a counterparty at the time of the analysis but also its potential future changes. Recently, more advanced methods based on Monte Carlo simulation (Aziz and Charupat 1998) have been implemented by financial institutions. By simulating counterparty portfolios through time over a wide range of scenarios, these models explicitly capture the contingency of the market on derivative portfolios and credit risk. Furthermore, they can accurately model natural offsets, netting, collateral and various mitigation techniques used in practice.

Since their main focus is on risk at the counterparty level, counterparty credit risk models do not generally attempt to capture portfolio effects such as the correlation between counterparty defaults. In contrast, **Portfolio Credit Risk (PCR) models** measure credit capital and are specifically designed to capture portfolio effects, specifically obligor correlations. They include CreditMetrics (JP Morgan 1997), CreditRisk$^+$ (Credit Suisse Financial Products 1997), Credit Portfolio View (Wilson 1997a and 1997b) and KMV's Portfolio Manager (Kealhofer 1996). Although superficially they appear quite different—the models differ in their distributional assumptions, restrictions, calibration and solution—Gordy (1998) and Koyluoglu and Hickman (1998) show an underlying mathematical equivalence among these models. However, empirical work shows generally that all PCR models yield similar results if the input data is consistent (Crouhy and Mark 1998; Gordy 1998).

A major limitation of all current PCR models is the assumption that market risk factors, such as interest rates, are deterministic. Hence, they do not account for stochastic exposures. While this assumption has less consequence for portfolios of loans or floating rate instruments, it has great impact on derivatives such as swaps and options. Ultimately, a comprehensive framework requires the full integration of market and credit risk.

In this paper, we present a multi-step, stochastic model to measure portfolio credit risk that integrates exposure simulation and portfolio credit risk methods. Through the explicit modeling of stochastic exposures, the model overcomes the major limitation currently shared by portfolio models in accounting for the exposure caused by instruments with embedded derivatives. By combining a Mark-to-Future framework of counterparty exposures (see Aziz and Charupat 1998) and a conditional default probability framework (see Gordy 1998; Koyluoglu and Hickman 1998; Finger 1999), we minimize the number of scenarios where expensive portfolio valuations are calculated, and can apply advanced Monte Carlo or analytical techniques that take advantage of the problem structure.

We restrict this paper to a "default mode" model; that is, the model measures credit losses arising exclusively from the event of default. However, default mode models cannot account for deals that have direct contingency on migrations (e.g., credit trigger features) without further modifications. Although perhaps computationally intensive, it is not difficult to extend the model to account for migration losses. Note, however, that since credit migrations are actually changes in expectations of future defaults, a multi-step model captures migration losses indirectly.

Specifically, the model presented in this paper is an improvement over current portfolio models in three main aspects:

- First, it defines explicitly the joint evolution of market risk factors and credit drivers. **Market factors** drive the prices of securities and **credit drivers** are non-idiosyncratic factors that drive the credit worthiness of obligors in the portfolio. Factors are general and can be microeconomic, macroeconomic, economic and financial.

- Second, it models directly stochastic exposures through simulation, as do the Counterparty Credit Exposure models. In this sense, it constitutes an integration of counterparty exposure and Portfolio Credit Risk models.

- Finally, it extends the Merton model of default (1974), as used, for example, in CreditMetrics, to multiple steps. It explicitly solves for multi-step thresholds and conditional default probabilities in a general simulation setting.

The rest of the paper is organized as follows. We begin by introducing a general framework for Portfolio Credit Risk Models. The framework is first illustrated through the commonly known single-step model with deterministic exposures. Next, we present the multi-step, stochastic model in two stages. First, we extend the single–step model with deterministic exposures to account for stochastic exposures, and second, we extend that model to a multi-step version. The paper closes with some concluding remarks and outline of future work.

Framework for Portfolio Credit Risk models

Current portfolio models fit within a generalized underlying modeling framework. Gordy (1998) and Koyluoglu and Hickman (1998) first introduced the framework to facilitate the comparison between the various models. Finger (1999) further points out that formulating the models in this framework permits the use of powerful numerical tools known in probability that can improve computational performance by dramatically reducing the number of scenarios required. The main idea behind the framework is that conditional on a scenario all defaults and rating changes are independent. A state-of-the-world is a complete specification at a point in time of the relevant economic and financial credit drivers and market factors (macroeconomic, microeconomic, financial, industrial, etc.) that drive the model. A **scenario** is defined by a set of states-of-the-world over time. In a single-period model there is a direct correspondence between a state-of-the-world and a scenario; in a multi-period model a scenario corresponds to a path of states-of-the-world over time.

In this section, we introduce the basic components of the framework, which we subsequently use to present various models. We make several steps explicit in the framework, which were previously implicit in the original presentations. This further specification permits us to present the models in a manner that better explains the assumptions made and allows us to address the generalizations of the model.

The framework consists of five parts:

- **Part 1: Risk factors and scenarios**. This is a model of the evolution of the relevant systemic risk factors over the analysis period. These factors may include both credit drivers and market factors.

- **Part 2: Joint default model**. Default and migration probabilities vary as a result of changing economic conditions. An obligor's probabilities are conditioned on the scenario at each point in time. The relationship between its conditional probabilities and the scenario is obtained through an intermediate variable, called the obligor's credit worthiness index. Correlations among obligors are determined by the joint variation of conditional probabilities across scenarios.

- **Part 3: Obligor exposures, recoveries and losses in a scenario**. The amount that will be lost if a credit event occurs (default or migration) as well as potential recoveries are computed under each scenario. Based on the level of the market factors in a scenario at each point in time, Mark-to-Future (MtF) exposures for each counterparty are obtained accounting for netting, mitigation and collateral. Similarly, recovery rates in the event of default can be state dependent.

- **Part 4: Conditional portfolio loss distribution in a scenario**. Conditional upon a scenario, obligor defaults are independent. Various techniques based on the property of independence of obligor defaults can be applied to obtain the conditional portfolio loss distribution.

- **Part 5: Aggregation of losses in all scenarios**. Finally, the unconditional distribution of portfolio credit losses is obtained by averaging the conditional loss distributions over all possible scenarios.

We illustrate the framework with a single time step Portfolio Credit Risk model with deterministic exposures, PCR_SD. Common notation and key concepts are also introduced.

Next, the model is extended to allow for stochastic exposures in a single-step setting, PCR_SS. Finally, we present a third model, PCR_MS, which allows for multiple time steps and stochastic exposures.

A set of four tables (Appendix 1) summarize the features of the models and highlight the similarities and differences of the models presented here. Table A1 presents a summary of the features of the three PCR models. Table A2 summarizes definitions of the risk factors and scenarios in Part 1 of the framework. Table A3 summarizes the components of the joint default model of Part 2. Table A4 summarizes the calculations for conditional obligor losses, conditional portfolio losses and unconditional losses of Parts 3 to 5 of the framework.

PCR_SD - Single-step with deterministic exposures

The first model, PCR_SD, measures single-step portfolio credit losses with deterministic obligor exposures and recovery rates. This is a two-state form of the CreditMetrics model. We consider a default mode model, where default is driven by a Merton model.

Consider a portfolio with N obligors or accounts. Each obligor belongs to one of $N_s < N$ sectors. We assume that obligors in a sector are statistically identical. The grouping of obligors into sectors facilitates the estimation and solution of the problem.

Part 1. Risk factors and scenarios

Consider the single period $[t_0, t]$ where, generally, $t = 1$ year. In this single period model a scenario corresponds to a state-of-the-world. At the end of the horizon, t, the scenario is defined by q^c systemic factors, the credit drivers, which influence the credit worthiness of the obligors in the portfolio.

Denote by $\boldsymbol{x}(t)$ the vector of factor returns at time t; i.e., $\boldsymbol{x}(t)$ has components $x_k(t) = ln\{r_k(t)/r_k(t_0)\}$, where $r_k(t)$ is the value of the k-th factor at time t. Assume that at the horizon the returns are normally distributed: $\boldsymbol{x}(t) \sim N(\mu, \boldsymbol{Q})$, where μ is a vector of mean returns and $\boldsymbol{Q}$ is a covariance matrix. Denote by $\boldsymbol{Z}(t)$, the vector of normalized factor returns; i.e., $Z_k(t) = (x_k(t) - \mu_k) / \sigma_k$. For ease of exposition, and without loss of generality, assume that the factor returns are independent; independent factors can always be obtained, for example, by applying Principal Component Analysis to the original economic factors.

Part 2. Joint default model

The joint default model consists of three components. First, the definition of unconditional default probabilities. Second, the definition of a credit worthiness index for each obligor and the estimation of a multi-factor model that links the index to the credit drivers. Finally, a model of obligor default, which links the credit worthiness index to the probabilities of default, is used to obtain conditional default probabilities. Below, we explain these components in more detail.

Denote by τ_j the time of default of obligor j, and by $p_j(t)$ its **unconditional probability of default,** the probability of default of an obligor in sector j by time t:

$$p_j(t) = Pr\{\tau_j \le t\} \tag{1}$$

Note that all obligors in sector j have the same unconditional probability of default. We assume that unconditional probabilities for each sector are available from an internal model or from an external agency.

The **credit worthiness index**, Y_j, of obligor j determines the credit worthiness or financial health of that obligor at time t. Whether an obligor is in default can be determined by considering the value of its index. We assume that Y_j, a continuous variable, is related to the credit drivers through a linear, multi-factor model:

$$Y_j(t) = \sum_{k=1}^{q^c} \beta_{jk} Z_k(t) + \sigma_j \varepsilon_j \tag{2}$$

where

$$\sigma_j = \sqrt{1 - \sum_{k=1}^{q^c} \beta_{jk}^2}$$

is the volatility of the idiosyncratic component associated with sector j, β_{jk} is the sensitivity of the index of obligor j to the k-th factor and ε_j, $j = 1, 2, \ldots, N$, are independent and identically distributed standard normal variables. Thus, the first term on the right side of Equation 2 is the systemic component of the index while the second term is the specific, or idiosyncratic, component. Note that the distribution of the index is standard normal; it has zero mean and unit variance.

Since all obligors in a sector are statistically identical, obligors in a given sector share the same multi-factor model. However, while all obligors in a sector share the same β_{jk} and σ_j, each has its own idiosyncratic, uncorrelated component, ε_j.

The **conditional probability of default** of an obligor in sector j, $p_j(t; \mathbf{Z})$, is the probability that an obligor in sector j defaults at time t, conditional on scenario $\mathbf{Z}$:

$$p_j(t;\mathbf{Z}) = Pr\{\tau_j \le t | \mathbf{Z}(t)\} \tag{3}$$

The estimation of conditional probabilities requires a conditional default model which describes the functional relationship between the credit worthiness index Y_j (and hence the systemic factors) and the default probabilities p_j.

We assume that default is driven by a Merton model (Merton 1974). In the Merton model (Figure 1), default occurs when the assets of the firm fall below a given boundary or threshold, generally given by its liabilities. We consider that an obligor defaults when its credit worthiness index, Y_j, falls below a pre-specified threshold estimated from historical data. In this setting, an obligor's credit worthiness index, Y_j, can be interpreted as the standardized return of its asset levels. Default occurs when this index falls below α_j, the **unconditional default threshold.**

From an econometrics perspective, the Merton model is referred to as a probit model. It is conceptually straightforward to substitute a different default model, such as a logit model, as presented in Wilson (1997a, 1997b).

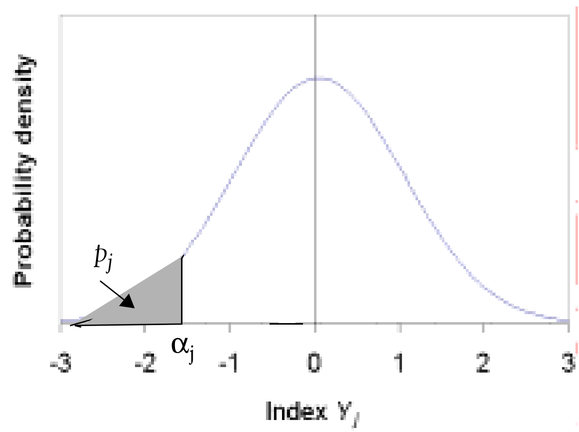

Figure 1: Merton model of default

The first step in the model defines the conditional default threshold, α_j, for each obligor. The second step calculates the conditional default probabilities in each scenario.

The unconditional default probability of obligor j is given by

$$p_j = Pr\{Y_j < \alpha_j\} = \Phi(\alpha_j) \tag{4}$$

where Φ denotes the normal cumulative density function. For simplicity, we have dropped the dependence on time, t, from the notation.Thus, the unconditional threshold, α_j, is obtained by the inverse of Equation 4:

$$\alpha_j = \Phi^{-1}(p_j) \tag{5}$$

The conditional probability of default is then the probability that the credit worthiness index falls below the threshold in a given scenario:

$$p_j(\mathbf{Z}) = Pr\{Y_j < \alpha_j | \mathbf{Z}\} \tag{6}$$

$$= Pr\left\{ \sum_{k=1}^{q^c} \beta_{jk} Z_k + \sigma_j \varepsilon_j < \alpha_j \mid \mathbf{Z} \right\}$$

$$= Pr\left\{ \varepsilon_j < \frac{\alpha_j - \sum_{k=1}^{q^c} \beta_{jk} Z_k}{\sigma_j} \right\}$$

$$= \Phi\left(\frac{\alpha_j - \sum_{k=1}^{q^c} \beta_{jk} Z_k}{\sigma_j} \right)$$

$$= \Phi(\hat{\alpha}_j(\mathbf{Z}))$$

The **conditional threshold**, $\hat{\alpha}_j(\mathbf{Z})$, is the threshold that the idiosyncratic component of obligor j, ε_j, must fall below for default to occur in scenario $\mathbf{Z}$.

Note that obligor credit worthiness index correlations are uniquely determined by the default model and the multi-factor model, which links the index to the credit driver returns. The correlations between obligor defaults are then obtained from the functional relationship between the index and the event of default, as determined by the Merton model. For example, the indices of obligors that belong to the same sector are perfectly correlated if their idiosyncratic component is zero.

Part 3. Obligor exposures and recoveries in a scenario

Define the exposure to an obligor j at time t, V_j, as the amount that will be lost due to outstanding transactions with that obligor if default occurs, unadjusted for future recoveries. An important property of PCR_SD is the assumption that obligor exposure is deterministic, not scenario dependent: $V_j \neq f(\mathbf{Z})$.

The economic loss if obligor j defaults in any scenario is

$$L_j(\mathbf{Z}) = V_j \cdot (1 - \gamma_j) \tag{7}$$

where γ_j is the recovery rate, expressed as a fraction of the obligor exposure. Recovery, in the event of default, is also assumed to be deterministic. (Expressing the recovery amount as a fraction of the exposure value at default does not necessarily imply instantaneous recovery of a fraction of the exposure when default occurs.)

The distribution of conditional losses for each obligor is given by

$$L_j(\mathbf{Z}) = \begin{cases} V_j \cdot (1 - \gamma_j) & \text{with prob.} \quad p_j(\mathbf{Z}) \\ 0 & \text{with prob.} \quad 1 - p_j(\mathbf{Z}) \end{cases} \tag{8}$$

Part 4. Conditional portfolio loss distribution in a scenario

Conditional on a scenario, $\mathbf{Z}$, obligor defaults are independent. This follows from Equation 6 and the assumption that the idiosyncratic components of the indices are independent. To determine whether an obligor default occurs in a scenario, all that remains to be sampled is its idiosyncratic component.

In practice, the computation of conditional losses can be onerous. In the most general case, a Monte Carlo simulation can be applied to determine portfolio conditional losses. However, the observation that obligor defaults are independent permits the application of more effective computational tools. Some of these techniques are described in Credit Suisse (1997), Finger (1999) and Nagpal and Bahar (1999).

For the purpose of exposition only, consider a portfolio with a very large number of obligors, each with a small marginal contribution. In this case, we can use the Law of Large Numbers (LLN) to estimate conditional portfolio losses. As the number of obligors approaches infinity, the conditional loss distribution converges to the mean loss over that scenario; the conditional variance and higher moments become negligible. Hence, the conditional portfolio losses, $L(\mathbf{Z})$, are given by sum of the expected losses of each obligor:

$$L(\mathbf{Z}) = \sum_{j=1}^{N} E\{L_j(\mathbf{Z})\} = \sum_{j=1}^{N} V_j \cdot (1 - \gamma_j) \cdot p_j(\mathbf{Z}) \tag{9}$$

Assuming that the LLN is appropriate simplifies the presentation which permits us to focus this discussion on the differences and similarities among the PCR_SD model and the stochastic and multi-step models that follow. Other methods include the application of the Central Limit Theorem (which assumes the number of obligors is large, but not necessarily as large as that required for the LLN), the application of moment generating functions with numerical integration or the application of probability generating functions with a discretization of exposures.

Part 5. Aggregation of losses in all scenarios

Unconditional portfolio losses are obtained by averaging the conditional losses over all scenarios. The distribution of unconditional portfolio losses is given by

$$Pr\{L_P < \lambda\} = \int_{\mathbf{Z}} Pr\{L(\mathbf{Z}) < \lambda\} dF(\mathbf{Z}) \tag{10}$$

where L_P denotes the unconditional portfolio losses, λ denotes the level of losses and $F(\mathbf{Z})$ is the distribution of $\mathbf{Z}$.

The aggregation of losses is generally obtained by performing a Monte Carlo simulation on the risk factor returns. Alternatively, analytical solutions are available under some restrictions (see Nagpal and Bahar (1999)).

The first column of Table A1 (Appendix 1) summarizes the features of the PCR_SD model. In the first columns of Tables A2 to A4, we summarize the components of the PCR_SD model associated with the framework.

PCR_SS – Single-step model with stochastic exposures

The second model developed in this paper relaxes the assumption of deterministic exposures and recoveries of the previous model. The PCR_SS model measures single-step portfolio credit losses due to default and assumes that obligor exposures and recoveries are stochastic.

Part 1. Risk factors and scenarios

The definition of risk factors and scenarios is similar to that in the PCR_SD model, with the difference that now a set of market factors are introduced which are determinant in the calculations of credit exposures.

Consider the single period $[t_0, t]$. At the end of the horizon, t, the scenario is now defined by q factors of which q^m are market factors and $q^c = q - q^m$ are credit drivers (this separation is for ease of exposition only and in no way restricts the model).

Denote by $\boldsymbol{x}(t)$ the vector of factor returns at time t. In general, we use the superscript m to denote quantities related to market factors and the superscript c to denote those related to credit drivers. Thus, $\boldsymbol{x}^m$ and $\boldsymbol{x}^c$ denote the factor returns of the market factors and credit drivers, respectively. The first q^m components of $\boldsymbol{x}(t)$ correspond to $\boldsymbol{x}^m$ and the following q^c components to $\boldsymbol{x}^c$.

Assume that both credit driver and market factor returns are normally distributed: $\boldsymbol{x}(t) \sim N(\mu, \mathbf{Q})$. Denote by $\mathbf{Z}(t)$ the vector of normalized credit driver returns; i.e., $Z_k(t) = (x_k^c(t) - \mu_k^c(t))/\sigma_k(t)$. As in the PCR_SD model, assume that the components of $\mathbf{Z}$ are independent. Note that normalized, independent returns are required for the credit drivers only; the returns of the market factors can follow more general models.

A scenario is described by an outcome of the returns vector $\boldsymbol{x}$, or equivalently by a joint outcome of the vectors $\boldsymbol{x}^m$ and $\mathbf{Z}$.

Part 2. Joint default model

The joint default model of PCR_SS is identical to that of PCR_SD. Conditional default probabilities are as given by Equation 6. In the PCR_SS model, the vector $\mathbf{Z}$ contains only the standardized returns of the credit drivers, $Z_k(t)$.

Part 3. Obligor exposures and recoveries in a scenario

The main difference between the PCR_SS model and the previous deterministic model is that in this model obligor exposures are stochastic. The exposure to obligor j at time t, $V_j(x^m)$, varies by scenario as a function of the market risk factors; i.e., $V_j = f(x^m)$ (the index t is dropped for simplicity).

Exposure for each obligor is obtained through a single-step MtF simulation of all outstanding transactions, accounting for all netting agreements, mitigation and collateral. We refer to the table of obligor exposures by scenario, $V_j(\boldsymbol{x}^m)$, as the Exposure MtF Table.

We further allow for recoveries in the event of default, γ_j, to be stochastic. The economic loss incurred if obligor j defaults in a given scenario is

$$L_j(\boldsymbol{x}^m, \mathbf{Z}) = V_j(\boldsymbol{x}^m) \cdot (1 - \gamma_j(\boldsymbol{x}^m, \mathbf{Z})) \tag{11}$$

Because it is difficult to estimate the correlations, it is common to assume that recoveries are independent of the risk factors.

The table of obligor conditional losses by scenario, $L_j(\boldsymbol{x}^m, \mathbf{Z})$ is referred to as the Obligor Losses MtF Table.

The loss distribution for each obligor is then

$$L_j(\boldsymbol{x}^m, \mathbf{Z}) = \begin{cases} V_j(\boldsymbol{x}^m) \cdot \left(1 - \gamma_j(\boldsymbol{x}^m, \mathbf{Z})\right) & \text{with prob. } p_j(\mathbf{Z}) \\ 0 & \text{with prob. } 1 - p_j(\mathbf{Z}) \end{cases} \tag{12}$$

Part 4. Conditional portfolio loss distribution in a scenario

As in the PCR_SD model, obligor defaults are independent conditional on a scenario $\mathbf{Z}$. The main difference, of course, is that the exposures and recoveries are now also a function of the scenario. Thus, if the portfolio contains a very large number of obligors, each with a small marginal contribution, the LLN dictates that conditional portfolio losses converge to the sum of the expected losses of each obligor:

$$L(\boldsymbol{x}^m, \mathbf{Z}) = \sum_{j=1}^{N} E\left\{L_j(\boldsymbol{x}^m, \mathbf{Z})\right\} \tag{13}$$

$$= \sum_{j=1}^{N} V_j(\boldsymbol{x}^m) \cdot \left(1 - \gamma_j(\boldsymbol{x}^m, \mathbf{Z})\right) \cdot p_j(\mathbf{Z})$$

Part 5. Aggregation of losses in all scenarios

Unconditional portfolio losses are obtained by averaging the conditional losses over all scenarios:

$$Pr\{L_P < \lambda\} = \int_{(\boldsymbol{x}^m, \mathbf{Z})} Pr\left\{L(\boldsymbol{x}^m, \mathbf{Z}) < \lambda\right\} dF(\boldsymbol{x}^m, \mathbf{Z}) \tag{14}$$

where $F(\boldsymbol{x}^m, \mathbf{Z})$ is the joint distribution of the market risk factors and credit drivers. This integral is generally computed using a Monte Carlo simulation.

The second column of Table A1 (Appendix 1) summarizes the features of the PCR_SS model. In the second columns of Tables A2 to A4 we summarize the components of the PCR_SS model associated with the framework.

PCR_MS – Multiple-step model with stochastic exposures

The previous single-step, stochastic model, PCR_SS, is now extended to a multi-step setting. Model PCR_MS measures multi-step portfolio credit losses due to default and assumes that obligor exposures and recoveries are stochastic. Default is driven by a multi-step extension of a Merton model.

In this section, the full derivation of the discrete time model is presented. Appendix 2 introduces the problem of determining the credit worthiness process for the continuous time analog. The resolution of this problem will be addressed in future work.

Part 1. Risk factors and scenarios

Assume M multiple discrete time steps during the period $[t_o, T]$: $t_o < t_1 < t_2 < \ldots < t_M = T$. A state-of-the-world at each time t_i is defined by a realization of q factors, out of which q^m are market factors and q^c credit drivers, respectively.

Denote by $\boldsymbol{r}(t_i)$ the vector of risk factor values at time t_i and by $\boldsymbol{x}(t_i)$ the vector of factor returns from time t_0 to t_i; i.e., $\boldsymbol{x}(t_i)$ has components $x_k(t_i) = ln\{r_k(t_i)/r_k(t_0)\}$, where $r_k(t_i)$ is the value of the k-th factor at time t_i. Again, $\boldsymbol{x}^m$ and $\boldsymbol{x}^c$ denote the factor returns of the market factors and credit drivers, respectively.

We assume that the evolution of the vector of risk factor values over time is determined by a set of stochastic differential equations:

$$dr_k(t) = \mu_k(r,t)dt + \sum_{l=1}^{q} \sigma_{kl}(r,t)dw_l\ ,\ k = 1, \ldots, q \qquad (15)$$

where μ_k denotes the instantaneous drift of factor k and w_l, $l = 1,2,\ldots,q$, are uncorrelated Wiener processes. The matrix $\sigma = (\sigma_{kl})$ is such that $\sigma\sigma^T$ forms an instantaneous covariance matrix. In general, the parameters of the stochastic differential equation can be functions of both time and risk factors.

More specifically, assume that the credit driver returns follow an arithmetic Brownian motion with constant coefficients:

$$dx_k^c(t) = \mu_k^c dt + \sum_{l=1}^{q^c} \sigma_{kl}^c dw_l^c\ , \quad k = 1, \ldots, q^c \qquad (16)$$

No additional assumptions are made concerning the process for the market risk factors; the process for the market risk factors is as defined in Equation 15.

A scenario, in the discrete time setting, is then described by an outcome of the return vectors $\boldsymbol{x}(t_i)$, $i = 1,\ldots, M$, (or, equivalently, by a joint outcome of the vectors $\boldsymbol{r}^m$ and $\boldsymbol{r}^c$). Thus, in the multi-step model, a scenario is a path of states-of-the-world over time, i.e., a scenario is defined as $\boldsymbol{x} = \{\boldsymbol{x}(t_i), i = 1,\ldots, M\}$.

Part 2. Joint default model

The joint default model consists of three components: unconditional default probabilities and thresholds, a multi-factor model for the credit worthiness index of each obligor and a default model.

In the discrete time setting, the time of default of obligor j, τ_j, can take values t_i, $i = 1,\ldots, M$. The **unconditional probability of default** at time t_i, $p_j(t_i)$, is the probability that default of obligor j occurs in the i-th time step:

$$p_j(t_i) = Pr\{\tau_j = t_i\} \qquad (17)$$

Denote by $P_j(t_n)$ the **unconditional cumulative probability of default** of an obligor in sector j by time t_n:

$$P_j(t_n) = Pr\{\tau_j \le t_n\} = \sum_{i=1}^{n} Pr\{\tau_j = t_i\} = \sum_{i=1}^{n} p_j(t_i) \qquad (18)$$

Unconditional default probability term structures for each sector are an input to the model. They may be estimated from an internal model, from an external agency, or inferred from one period unconditional transition matrices, assuming a Markovian process.

In the development of the single-step models we noted that an obligor's credit worthiness index, Y_j, can be interpreted as the single-step standardized return of its asset levels. In the multi-step model the credit worthiness index of each obligor evolves through time. For a given obligor j, $A_j(t_i)$ is the level of the index. The return of the index up to time t_i is $y_j(t_i)$; i.e., $y_j(t_i) = ln\{A_j(t_i)/A_j(t_0)\}$. Finally, $Y_j(t_i)$ is the standardized return on the index

$$Y_j(t_i) = \frac{y_j(t_i) - \mu_j(t_i)}{\sigma_j(t_i)} \qquad (19)$$

where $\mu_j(t_i)$ and $\sigma_j(t_i)$ are respectively the mean and volatility of the index returns. $Y_j(t_i)$ has zero mean and unit volatility at every time step and is thus a canonical process. A canonical process is the process equivalent of a standard normal variable. In addition, we define the single period index returns:

$$\hat{y}_j(t_i) = ln\left(\frac{A(t_i)}{A(t_{i-1})}\right) = y_j(t_i) - y_j(t_{i-1})$$

and

$$\hat{Y}_j(t_i) = \frac{\hat{y}_j(t_i) - \hat{\mu}_j(t_i)}{\hat{\sigma}_j(t_i)}$$

where $\hat{Y}_j(t_1) \equiv Y_j(t_1)$ and $\hat{\mu}_j(t_i)$ and $\hat{\sigma}_j(t_i)$ are the mean and volatility of the single period returns.

We assume that the index is related to the scenario through a continuous multi-factor model. The model for each obligor j is given by

$$dy_j(t) = \sum_{k=1}^{q^c} \beta_{jk}\, dx_k^c(t) + \sigma_j dw_j \qquad (20)$$

where β_{jk} is the sensitivity of the index to the k-th credit driver; dw_j, $j = 1,\ldots, N$, are independent Wiener processes and σ_j is the volatility of the j-th idiosyncratic component. The canonical process $Y_j(t)$ can be derived from Equations 19 and 20. This process is the continuous analog to the single-step multi-factor model in Equation 2, except that the latter is standardized while the former is not.

In discrete time, the solution of Equation 20 can be written as

$$y_j(t_n) = \sum_{k=1}^{q^c} \beta_{jk}\, x_k^c(t_n) + \sigma_j \sum_{i=1}^{n} \sqrt{\Delta t_i}\,\varepsilon_i \tag{21}$$

where ε_i are independent and identically distributed standard normal variables and $\Delta t_i = t_i - t_{i-1}$. For ease of exposition we restrict attention to the case of uniform time steps, $\Delta t_i \equiv \Delta t, i = 1, 2, \ldots, M$.

Therefore, y_j is a stationary, independent-increments process and thus $\mu_j(t_i) \equiv \mu_j(\Delta t)$, $\sigma_j(t_i) \equiv \sigma_j(\Delta t)$ and $\sigma_j(t_i) = \sigma_j(\Delta t)\sqrt{i}$ (since $\sigma_j^2(t_i) = \sum_{k=1}^{i} \hat{\sigma}_j^2(t_k)$). Note that the random variables $\{\tilde{\varepsilon}_{ji} = \hat{Y}_j(t_i), i = 0, 1, \ldots, M-1\}$ are independent, identically distributed and normal with zero mean and unit variance.

For the default model, we assume a multi-step Merton model. Default occurs in the first time step i that the index of the firm falls below the **unconditional default threshold,** $\alpha_{ji} = \alpha_j(t_i)$. The prescription of the model proceeds in two steps. The first step in the model defines the unconditional default thresholds for each obligor, α_{ji}. The second step calculates the conditional default probabilities in each scenario.

In the discrete, multi-step model, the time of default τ_j is the first time the credit worthiness index falls below the unconditional threshold:

$$\tau_j = \min_{i=1,\ldots M}\{Y_j(t_i) < \alpha_{ji}\}$$

Thus, the probability that default occurs in time step n is the probability that the index falls below the threshold in time step n and exceeds the threshold in each preceding period:

$$\begin{aligned} p_j(t_n) &= Pr\{\tau_j = t_n\} \\ &= \{Pr Y_j(t_1) > \alpha_{j1}, Y_j(t_2) > \alpha_{j2}, \ldots, \\ &\qquad Y_j(t_{n-1}) > \alpha_{jn-1}, Y_j(t_n) < \alpha_{ji}\} \end{aligned}$$

The calculation of the threshold for obligor j in the first time step, t_1, is similar to that of the single-step models. As given in Equation 4, the single-step unconditional probability of default is the cumulative normal of the unconditional threshold. More formally, we can also write the probability of default in the first time step as

$$p_j(t_1) = \Phi\{\alpha_{j1}\} = \int_{-\infty}^{\alpha_{j1}} \phi(v)dv \tag{22}$$

where

$$\phi(v) = \frac{e^{-v^2/2}}{\sqrt{2\pi}}$$

The threshold α_{j1} is the standard normal quantile associated with the unconditional probability appearing on the left side of Equation 22:

$$\alpha_{j1} = \Phi^{-1}\{p_j(t_1)\} \tag{23}$$

Note the similarity between Equation 23 and Equation 5.

The probability that default occurs in the second time step, t_2, can be written as

$$\begin{aligned} p_j(t_2) &= Pr\{\tau = t_2\} = Pr\{Y_j(t_1) > \alpha_{j1}, Y_j(t_2) < \alpha_{j2}\} \qquad (24) \\ &= Pr\left\{Y_j(t_1) > \alpha_{j1}, \frac{Y_j(t_1) + \hat{Y}_j(t_2)}{\sqrt{2}} < \alpha_{j2}\right\} \\ &= Pr\left\{\tilde{\varepsilon}_{j1} > \alpha_{j1}, \tilde{\varepsilon}_{j1} + \tilde{\varepsilon}_{j2} < \tilde{\alpha}_{j2}\right\} \\ &= \iint_{\substack{u > \alpha_1 \\ u+v < \tilde{\alpha}_{j2}}} \phi(u)\phi(v)du\,dv \end{aligned}$$

$$= \int_{-\infty}^{\tilde{\alpha}_{j2}} \int_{\alpha_{j1}}^{\infty} \phi(u)\phi(v-u)\,du\,dv$$

where $\tilde{\alpha}_{j2} = \alpha_{j2}\sqrt{2}$. Default does not occur in the first time step, thus the limits of integration are $[\alpha_{j1}, \infty)$; default must occur in the second time step, thus the limits of integration are $(-\infty, \tilde{\alpha}_{j2}]$.

The unconditional threshold for the second time step, α_{j2}, is defined implicitly in Equation 24 from the probability $p_j(t_2)$ and the t_1-threshold α_{j1}.

More generally, for any time step n, the threshold α_{jn} is determined implicitly from the default probabilities and the thresholds at all previous time steps:

$$p_j(t_n) = Pr\{\tau = t_n\} \tag{25}$$

$$= \int_{-\infty}^{\tilde{\alpha}_{jn}} \int_{\tilde{\alpha}_{jn-1}}^{\infty} \dots \int_{\tilde{\alpha}_{j1}}^{\infty} \phi(v_n - v_{n-1})\phi(v_{n-1} - v_{n-2})$$

$$\dots\phi(v_2 - v_1)\phi(v_1)\,dv_1 \dots dv_{n-1}\,dv_n$$

where $\tilde{\alpha}_{ji} = \alpha_{ji}\sqrt{i}$.

Given the cumulative default probability curve for each sector, thresholds can be computed using a Monte Carlo method which solves recursively for the limits of the integrals in Equation 25.

We define the **conditional probability of default,** $p_j(t_n;\mathbf{x}^c)$, as the probability that default of an obligor in sector j occurs in the n-th time step conditional on the realization of the credit drivers up to time t_n:

$$p_j(t_n;\mathbf{x}^c) = Pr\left\{\tau_j = t_n \middle| \mathbf{x}^c(t_i), i = 1, \dots, n\right\} \tag{26}$$

The computation of the conditional default probabilities is as follows: for the first time step, the conditional probability of default is, as in the previous models, given by

$$p_j\left(t_1;\mathbf{x}^c\right) = Pr\left\{Y_j(t_1) < \alpha_{j1} \middle| \mathbf{x}^c(t_1)\right\} \tag{27}$$

$$= Pr\left\{\frac{\left[\sum_{k=1}^{q^c} \beta_{jk}\, x_k^c(t_1) + \sigma_j\sqrt{\Delta t}\,\varepsilon_1\right] - \mu_j(t_1)}{\sigma_j(t_1)} < \alpha_{j1} \,\middle|\, \mathbf{x}^c(t_1)\right\}$$

$$= Pr\left\{\varepsilon_1 < \frac{\sigma_j(t_1)\alpha_{j1} + \mu_j(t_1) - \sum_{k=1}^{q^c} \beta_{jk}\, x_{k1}^c}{\sigma_j\sqrt{\Delta t}}\right\}$$

$$= \Phi\left(\frac{\bar{\alpha}_{j1} - \sum_{k=1}^{q^c} \beta_{jk}\, x_{k1}^c}{\sigma_j\sqrt{\Delta t}}\right)$$

The threshold, adjusted by the drift and volatility of the index returns, is $\bar{\alpha}_{ji} = \sigma_j(t_i)\alpha_{ji} + \mu_j(t_i)$. Note that Equation 27 is equivalent to Equation 6.

For the second time step, the conditional probability is given by

$$p_j\left(t_2;\mathbf{x}^c\right) = Pr\left\{Y_j(t_1) > \alpha_{j1}, Y_j(t_2) < \alpha_{j2} \middle| \mathbf{x}^c(t_1), \mathbf{x}^c(t_2)\right\}$$

$$= Pr\left\{\varepsilon_1 > \frac{\bar{\alpha}_{j1} - \sum_{k=1}^{q^c} \beta_{jk}\, x_{k1}^c}{\sigma_j\sqrt{\Delta t}}, \varepsilon_1 + \varepsilon_2 > \frac{\bar{\alpha}_{j2} - \sum_{k=1}^{q^c} \beta_{jk}\, x_{k2}^c}{\sigma_j\sqrt{\Delta t}}\right\}$$

Then, in general, for time step n

$$p_j\left(t_n;\mathbf{x}^c\right) = \left\{Pr \bigcap_{i=1}^{n-1} [Y_j(t_i) > \alpha_{ji}], Y_j(t_n) < \alpha_{jn} \middle| \mathbf{x}^c(t_i), i = 1, \dots, n\right\} \tag{28}$$

where

$$u_{ji} = \frac{\bar{\alpha}_{ji} - \sum_{k=1}^{q^c} \beta_{jk}\, x_{ki}^c}{\sigma_j\sqrt{\Delta t}}$$

Equation 28 can be restated as

$$p_j\left(t_n;x^c\right) = Pr\left\{\bigcap_{i=1}^{n-1}\left[\sum_{l=1}^{i}\varepsilon_l > u_{ji}\right], \sum_{l=1}^{n}\varepsilon_l < u_{jn}\right\} \quad (29)$$

The right side of Equation 29 can be computed using numerical integration. Details of the computation of the multi-step conditional default probabilities are presented in Appendix 3.

Part 3. Obligor exposures and recoveries in a scenario

As in the single-step stochastic model, PCR_SS, obligor exposures are stochastic. However, in this model, the exposure to obligor j is dependent on the path of the market risk factors up to time t_i, $V_{ji}(\boldsymbol{x}^m) \equiv V_j(t_i) = f(\boldsymbol{x}^m(t_k)), 0 \le k \le i$. Since exposures at various times are summed, each is already discounted to today. Thus, discounted exposures express the capital that must be held today to cover future defaults (unadjusted for recoveries).

Exposures for each obligor are obtained through a multi-step simulation of all outstanding transactions, accounting for all netting agreements, mitigation and collateral. Aziz and Charupat (1998) present examples of the computation of these exposures. The table of obligor exposures over every scenario, $V_j(\boldsymbol{x}^m, t_i)$ is referred to as the Multi-step Exposure MtF Table.

The economic loss if obligor j defaults in time step i, is the exposure of obligor j at time step i, net of recoveries, where it is also assumed that recoveries in the event of default, γ_{ji}, are stochastic:

$$L_{ji}(\boldsymbol{x}) = V_{ji}(\boldsymbol{x}^m)\,P\left(1 - \gamma_{ji}(\boldsymbol{x}^m, \boldsymbol{x}^c)\right) \quad (30)$$

The probability of this default is $p_j(t_i, \boldsymbol{x}^c)$. Thus, for every time step i, the distribution of conditional obligor losses is given by

$$L_{ji}(\boldsymbol{x}) = \begin{cases} V_{ji}(\boldsymbol{x}^m)\cdot\left(1 - \gamma_{ji}(\boldsymbol{x}^m, \boldsymbol{x}^c)\right) & \text{with prob. } p_j\left(t_i;\boldsymbol{x}^c\right) \\ 0 & \text{with prob. } 1 - p_j\left(t_i;\boldsymbol{x}^c\right) \end{cases} \quad (31)$$

The table of conditional obligor losses, $L_{ji}(\boldsymbol{x})$, is referred to as the Multi-step MtF Table of Obligor Losses.

Part 4. Conditional portfolio loss distribution in a scenario

At each time step, obligor defaults are independent, conditional on a scenario. The losses in a given scenario are simply the sum of the losses at each time step in that scenario.

If the portfolio contains a very large number of obligors, each with a small marginal contribution, the LLN dictates that conditional portfolio losses at each time step converge to the sum of the expected losses of each obligor:

$$L(t_i, \boldsymbol{x}) = \sum_{j=1}^{N} V_{ji}(\boldsymbol{x}^m)\,P\left(1 - \gamma_{ji}(\boldsymbol{x}^m, \boldsymbol{x}^c)\right)\cdot p_j\left(t_i;\boldsymbol{x}^c\right) \quad (32)$$

Expected portfolio losses in a given scenario are the sum of the expected losses in each time step:

$$L(\boldsymbol{x}) = \sum_{i=1}^{M} L(t_i, \boldsymbol{x}) \quad (33)$$

Part 5. Aggregation of losses in all scenarios

Unconditional portfolio losses are obtained by averaging the conditional losses over all scenarios:

$$Pr\{L_P < \lambda\} = \int_{\boldsymbol{x}} Pr\{L(\boldsymbol{x}) < \lambda\}\,dF(\boldsymbol{x}) \quad (34)$$

where $F(\boldsymbol{x})$ is the probability distribution in the scenario space. This integral is generally computed using Monte Carlo techniques.

The third column of Table A1 (Appendix 1) summarizes the features of the PCR_MS model. In the third columns of Tables A2 to A4 we summarize the components of the PCR_MS model associated with the framework.

Concluding remarks

We have presented a new multi-step Portfolio Credit Risk model that integrates exposure simulation and advanced portfolio credit risk methods. The integrated model, PCR_MS, overcomes a major limitation currently shared by portfolio models in accounting for the credit risk of portfolios whose exposures depend on the level of the market.

Specifically, the model presented in this paper is an improvement over current portfolio models in three main aspects. First, it defines explicitly the joint evolution over time of market risk factors and credit drivers. Second, it models directly stochastic exposures through simulation, as in Counterparty Credit Exposure models. Finally, it extends the Merton model of default to multiple steps. Although the latter seems conceptually straightforward, the resulting mathematical model is not trivial. Moreover, expressing the model so that it is amenable to an efficient solution is essential.

The model is computationally efficient because it combines a Mark-to-Future (MtF) framework of counterparty exposures and a conditional default probability framework. The computational benefits are threefold:

- First, the number of scenarios for which expensive portfolio valuations are made is minimized.
- Second, the model is based on the same computations used for monitoring counterparty exposures and placing limits at the desks. A MtF framework allows users to exploit these computations and use them for both counterparty exposures and portfolio credit risk. This is not only important for computational purposes, but also leads to more consistent enterprise risk measurement.
- Third, advanced Monte Carlo or analytical techniques that take advantage of the problem structure can be used to solve the problem faster and more accurately than standard Monte Carlo methods.

We have restricted this paper to a default mode model. It is not conceptually difficult to extend the model to account for migration losses as well. Note that since credit migrations are simply changes in expectations of future defaults, a multi-step model partially and indirectly captures migration losses. Future work will address these issues in detail.

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Appendix 1. Model summary tables

This appendix contains tables summarizing the three models presented.

	PCR_SD	PCR_SS	PCR_MS
Time steps	single-step	single-step	multi-step
Exposures and recoveries	deterministic	stochastic	stochastic
Credit events	default	default	default
Default model	Merton	Merton	Merton
Simulation factors	credit drivers	credit drivers market factors	credit drivers market factors
Factor distributions	credit drivers: standardized normal returns	credit drivers: standardized normal returns market factors: normal returns	credit drivers: normal returns market factors: general
Input data	(unconditional) one year default probabilities for each obligor/sector unconditional threshold	(unconditional) one year default probabilities for each obligor/sector unconditional threshold	term structure of (unconditional) default probabilities for each obligor/sector unconditional thresholds
Output results	Exposure MtF Table Obligor Losses MtF Table	Exposure MtF Table Obligor Losses MtF Table	Multi-step Exposure MtF Table Multi-step MtF Table of Obligor Losses

Table A1: Feature summary of Portfolio Credit Risk models

	PCR_SD	PCR_SS	PCR_MS
Credit risk factors	credit driver returns; standardized normal $Z_k(t) \sim N(0,1)$	credit driver returns; standardized normal $Z_k(t) \sim N(0,1)$	Equation 16 credit driver returns; normal $dx_k^c(t) = \mu_k^c dt + \sum_{l=1}^{q^c} \sigma_{kl}^c dw_l^c$
Market risk factors	not applicable	market factor returns; normal $x^m(t) \sim N(\mu, Q)$	Equation 15 market factor returns; general $dr_k^m(t) = \mu_k(r, t)dt + \sum_{l=1}^{q} \sigma_{kl}(r, t)dw_l$
Scenarios	single-step	single-step	multi-step

Table A2: Part 1 – Definition of risk factors and scenarios

	PCR_SD	PCR_SS	PCR_MS
Unconditional default probability	Equation 1 $p_j(t) = Pr\{\tau_j \le t\}$	Equation 1 $p_j(t) = Pr\{\tau_j \le t\}$	Equation 17 $p_j(t_i) = Pr\{\tau_j = t_i\}$ Equation 18 $P_j(t_n) = Pr\{\tau_j \le t_n\}$
Credit worthiness index	Equation 2 $Y_j(t) = \sum_{k=1}^{q^c} \beta_{jk} Z_k(t) + \sigma_j \varepsilon_j$ where $\sigma_j = \sqrt{1 - \sum_{k=1}^{q^c} \beta_{jk}^2}$	Equation 2 $Y_j(t) = \sum_{k=1}^{q^c} \beta_{jk} Z_k(t) + \sigma_j \varepsilon_j$ where $\sigma_j = \sqrt{1 - \sum_{k=1}^{q^c} \beta_{jk}^2}$	Equation 21 $y_j(t_n) = \sum_{k=1}^{q^c} \beta_{jk} x_k^c(t_n) + \sigma_j \sum_{i=1}^{n} \sqrt{\Delta t}\, \varepsilon_i$
Unconditional default threshold	Equation 5 $\alpha_j = \Phi^{-1}(p_j)$	Equation 5 $\alpha_j = \Phi^{-1}(p_j)$	Equation 25, solve for $\alpha_{jn} = \tilde{\alpha}_{jn} / \sqrt{n}$ $p_j(t_n) = \int_{-\infty}^{\tilde{\alpha}_{jn}} \int_{\tilde{\alpha}_{jn-1}}^{\infty} \dots \int_{\tilde{\alpha}_{j1}}^{\infty} \phi(v_n - v_{n-1}) \phi(v_{n-1} - v_{n-2})$ $\dots \phi(v_2 - v_1) \phi(v_1) dv_1 \dots dv_{n-1} dv_n$
Conditional default probabilities	Equation 3 $p_j(t;\mathbf{Z}) = Pr\{\tau_j \le t \mid \mathbf{Z}(t)\}$ Equation 66 $p_j(\mathbf{Z}) = \Phi\left(\dfrac{\alpha_j - \sum_{k=1}^{q^c} \beta_{jk} Z_k}{\sigma_j}\right)$	Equation 3 $p_j(t;\mathbf{Z}) = Pr\{\tau_j \le t \mid \mathbf{Z}(t)\}$ Equation 66 $p_j(\mathbf{Z}) = \Phi\left(\dfrac{\alpha_j - \sum_{k=1}^{q^c} \beta_{jk} Z_k}{\sigma_j}\right)$	Equation 26 $p_j(t_n;\mathbf{x}^c) = Pr\left\{\tau_j = t_n \middle\vert \mathbf{x}^c(t_i), i = 1, \dots, n\right\}$ Equation 29 $p_j\left(t_n;\mathbf{x}^c\right) = Pr\left\{\bigcap_{i=1}^{n-1}\left[\sum_{l=1}^{i} \varepsilon_l > u_{ji}\right], \sum_{l=1}^{n} \varepsilon_l < u_{jn}\right\}$ where $u_{ji} = \bar{\alpha}_{ji} - \left(\sum_{k=1}^{q^c} \beta_{jk} x_{ki}^c\right) / \sigma_j^s \sqrt{\Delta t}$

Table A3: Part 2 – Joint default models

	PCR_SD	PCR_SS	PCR_MS
Obligor exposure	V_j	$V_j(\boldsymbol{x}^m)$	$V_{ji}(\boldsymbol{x}^m)$
Obligor recovery	γ_j	$\gamma_j(\boldsymbol{x}^m, \mathbf{Z})$	$\gamma_{ji}(\boldsymbol{x}^m, \boldsymbol{x}^c)$
Obligor losses in the event of default	Equation 7 $L_j(\mathbf{Z}) = V_j \cdot (1 - \gamma_j)$	Equation 11 $L_j(\boldsymbol{x}^m, \mathbf{Z}) = V_j(\boldsymbol{x}^m) \cdot (1 - \gamma_j(\boldsymbol{x}^m, \mathbf{Z}))$	Equation 30 $L_{ji}(\boldsymbol{x}) = V_{ji}(\boldsymbol{x}^m)\,\text{Þ}\left(1 - \gamma_{ji}(\boldsymbol{x}^m, \boldsymbol{x}^c)\right)$
Conditional portfolio losses (LLN)	Equation 9 $L(\mathbf{Z}) = \sum_{j=1}^{N} E\{L_j(\mathbf{Z})\} = \sum_{j=1}^{N} V_j \cdot (1 - \gamma_j) \cdot p_j(\mathbf{Z})$	Equation 13 $L(\boldsymbol{x}^m, \mathbf{Z}) = \sum_{j=1}^{N} V_j(\boldsymbol{x}^m) \cdot \left(1 - \gamma_j(\boldsymbol{x}^m, \mathbf{Z})\right) \cdot p_j(\mathbf{Z})$	Equation 32 $L(t_i, \boldsymbol{x}) = \sum_{j=1}^{N} V_{ji}(\boldsymbol{x}^m)\,\text{Þ}\left(1 - \gamma_{ji}(\boldsymbol{x}^m, \boldsymbol{x}^c)\right) \cdot p_j\left(t_i; \boldsymbol{x}^c\right)$ Equation 33 $L(\boldsymbol{x}) = \sum_{i=1}^{M} L(t_i, \boldsymbol{x})$
Unconditional portfolio credit losses	Equation 10 $Pr\{L_P < \lambda\} = \int_{\mathbf{Z}} Pr\{L(\mathbf{Z}) < \lambda\} dF(\mathbf{Z})$	Equation 14 $Pr\{L_P < \lambda\} = \int_{(\boldsymbol{x}^m, \mathbf{Z})} Pr\left\{L(\boldsymbol{x}^m, \mathbf{Z}) < \lambda\right\} dF(\boldsymbol{x}^m, \mathbf{Z})$	Equation 34 $Pr\{L_P < \lambda\} = \int_{\boldsymbol{x}} Pr\{L(\boldsymbol{x}) < \lambda\} dF(\boldsymbol{x})$

Table A4: Parts 3, 4 and 5 – Obligor exposure, recovery and losses

Appendix 2. The canonical credit worthiness process

Consider a single obligor. The canonical credit worthiness process, $Y(t)$, is described by

$$Y(t) = \frac{w_t}{\sqrt{t}}$$

where w_t is a one-dimensional Wiener process. In the continuous time Merton model, the time of default τ is the first time when the process falls below a boundary $\alpha(t)$:

$$\tau = \inf_t\{Y(t) < \alpha(t)\}$$

Let the function $P(t)$, with $0 \le P(t) \le 1$ for all $t > 0$ and $\frac{dP}{dt} \ge 0$, represent the probability of default before time T; i.e.,

$$P(t) = Pr\{\tau < t\}$$

This continuous time problem is depicted in Figure A1.

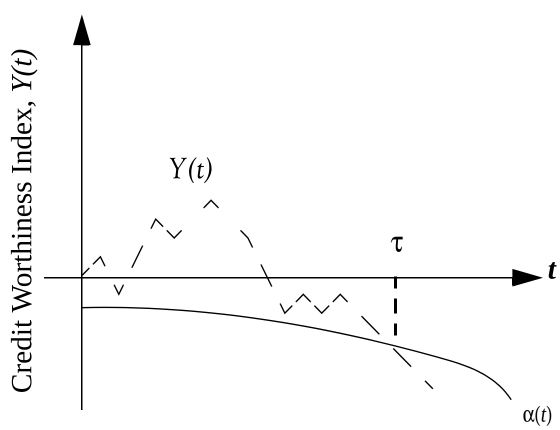

Figure A1: Continuous time default model

Appendix 3. Computation of multi-step conditional default probabilities

In general, for time step n, the conditional default probabilities of a given obligor are given by Equations 28 and 29:

$$p_j\left(t_n;\mathbf{x}^c\right) = Pr\left\{\bigcap_{i=1}^{n-1}[Y_{ji} > \alpha_{ji}], Y_{jn} < \alpha_{jn}\middle|\mathbf{x}^c(t_i), i = 1, \ldots, n\right\}$$

$$= Pr\left\{\bigcap_{i=1}^{n-1}\left[\sum_{l=1}^{i}\varepsilon_l > u_{ji}\right], \sum_{l=1}^{n}\varepsilon_l < u_{jn}\right\}$$

where

$$u_{ji} = \left(\bar{\alpha}_{ji} - \sum_{k=1}^{q}\beta_{jk}x_{ki}\right)/(\sigma_j\sqrt{\Delta t})$$

For simplicity, the index j, denoting a given obligor, is removed from the notation.

Denote by

$$A_n = \left\{\bigcap_{i=1}^{n-1}\left\{\sum_{l=1}^{i}\varepsilon_l > u_i\right\}, \sum_{l=1}^{n}\varepsilon_l < u_n\right\}$$

and

$$B_n = \bigcap_{i=1}^{n}\left\{\sum_{l=1}^{i}\varepsilon_l > u_i\right\}$$

Then it follows that

$$Pr(B_{n-1}) = Pr(A_n) + Pr(B_n)$$

The probability $Pr(B_n)$ is a function of n variables, $G_n(u_1,\ldots, u_n)$. The function $G_n(u_1,\ldots, u_n)$ satisfies the relation

$$G_{n+1}(u_1,\ldots,u_{n+1}) = G_n(u_1, \ldots,u_n)\,\Psi(u_{n+1} - u_n) \quad \text{(A1)}$$

$$+ \int_{u_n}^{\infty} G_n(u_1, u_2,\ldots,u_{n-1},u)\,\phi(u_{n+1} - u)\,du$$

where $\Psi(t) = \bar{\Phi}(t) = Pr\{\varepsilon_{n+1} > t\}$. The integrals in Equation A1 are evaluated using numerical integration techniques.