Economic Implications of Using a Mean-VaR Model for Portfolio Selection: A Comparison with Mean-Variance Analysis*

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Abstract

While proposed by Baumol (1963) almost four decades ago, Value at Risk (VaR) has recently reemerged in the finance literature as a new paradigm to manage and control risk. We relate VaR to mean-variance analysis and examine the economic implications arising from a mean-VaR framework. When comparing two mean-variance efficient portfolios, the higher variance portfolio might have less VaR. Consequently, an efficient portfolio that globally minimizes VaR may not exist. Interestingly, the mean-VaR efficient set is a proper subset of the mean-variance efficient set and it may even be empty. A characterization of the existence of the minimum VaR portfolio suggests that one must be careful in choosing the confidence level at which VaR is determined. Extensions of our results to the case of non-normality are also provided. We find that, at least as an approximation, the use of a mean-VaR criteria is consistent with an expected utility maximization framework. Economic implications of a mean-VaR framework include the result that the risk exposure of a highly risk-averse agent, as measured by standard deviation, increases when he or she decides to use VaR as the relevant measure of risk.

JEL classification: G11; G12; D81

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1. Introduction

There is a general belief in the finance literature that Value at Risk (VaR) is a new approach to manage and control risk.[1] The concept of VaR, however, was first proposed by Baumol (1963) when examining a model referred as the Expected Gain-Confidence Limit Criterion, almost four decades ago.[2] More generally, the associated "safety first models" were initially analyzed by Roy (1952) and Telser (1955), among others.[3] In this paper we relate VaR to mean-variance analysis and examine the economic implications arising from a mean-VaR framework.

Our work is different from previous results in several respects. First, we provide an analytical characterization of the mean-VaR efficient frontier. Second, we investigate the implications of using VaR as the relevant measure of risk in an agent's optimal portfolio. Third, we study equilibrium implications resulting from a mean-VaR framework. Finally, we examine how a mean-VaR criteria relates to an expected utility maximization framework.

Given a prespecified confidence level and a particular time horizon, a portfolio's VaR is the maximum loss one expects to suffer at that confidence level by holding that portfolio over that time horizon. More precisely, the VaR at the $100t\%$ confidence level of a risky portfolio for a specific time period is the rate of return v such that the probability of that portfolio having a rate of return of $-v$ or less is $1-t$.

Initially we suppose that security rates of return have a multivariate normal distribution. As Hull and White (1998, pp. 10) pointed out, this is a popular assumption when calculating a portfolio's VaR. For example, they note that it is common to assume that changes in market variables such as equity, bond, and commodity prices are normally distributed when using RiskMetrics.[4] Furthermore, Duffie and Pan (1997, pp. 11-12) show that computing the daily VaR of the S&P500 at the 95 and 99% confidence levels under the assumption of normality results in an estimate that is a good approximation to the value that is obtained using the historical distribution of its daily returns during the period 1986-1996. Nevertheless, we also extend our results to the case of non-normality.

First, we assume that no riskfree security exists in the economy. When security rates of return have a multivariate normal distribution, it is not surprising that the mean-VaR boundary corresponds to the mean-variance boundary.[5] However, we show that if the confidence level is sufficiently small, then the minimum VaR portfolio at that confidence level may not exist. We provide a characterization of the existence of the minimum VaR portfolio and derive the closed-form solution for the VaR minimization problem. Interestingly, we show that using VaR as a measure of risk results in an efficient frontier that is smaller than the mean-variance efficient frontier and, surprisingly, we show that it may even preclude the existence of mean-VaR efficient portfolios in the economy.

Using our characterization of the minimum VaR portfolio, we obtain several interesting results. We show that the minimum VaR portfolio converges to the minimum variance portfolio as the confidence level at which VaR is determined converges to 100%. Consequently, the mean-VaR efficient frontier converges to the mean-variance efficient frontier as the confidence level at which VaR is determined converges to 100%. We find that when the confidence level chosen to compute VaR decreases to a sufficiently small number, the expected rate of return of the minimum VaR portfolio converges to infinity and the set of mean-VaR efficient portfolios converges to the empty set. We then show that a portfolio is mean-variance efficient if and only if it globally minimizes VaR for some confidence level.

Second, we assume that a riskfree security exists in the economy. We find that if the confidence level at which VaR is computed is sufficiently small, then the mean-VaR efficient frontier is empty. We show that if the confidence level at which VaR is computed is sufficiently large, then the mean-VaR efficient frontier coincides with the mean-variance efficient frontier. This result contrasts with the case when no riskfree security exists in the economy, where the mean-VaR efficient frontier is always a proper subset of the mean-variance efficient frontier when the confidence level at which VaR is determined is less than 100%.

Since in many applications security rates of return have leptokurtosis ('fat tails'), skewness, or both, it is useful to study the case of non-normality. We extend the above results to the case when security rates of return have a multivariate t-distribution to address the case of leptokurtosis. When the normality or t-distribution assumption is not made, an application of Chebyshev's inequality shows that our results provide relevant information in terms of VaR upper bounds and reinforces the idea that one must be careful in choosing confidence levels to determine VaR in a way that makes sense (i.e., so that the problem of minimizing the Chebyshev upper bound on VaR has a solution).

We then examine whether a mean-VaR criteria is consistent with the typical preference specifications used in modern financial economics. Interestingly, we find that, at least as an approximation, the framework examined in this paper arises from the portfolio choice problem of a risk-averse agent who maximizes expected utility. In particular, we show that a "mean-Chebyshev upper bound on VaR" framework follows from the approximation to expected utility proposed by Levy and Markowitz (1979).

We then investigate economic implications arising from a mean-VaR framework. First, we study the implications of a mean-VaR framework in an agent's optimal portfolio. Interestingly, we find that when no riskfree security exists in the economy, the change of an agent's risk exposure, as measured by standard deviation, arising from the decision to use VaR as the relevant measure of risk depends upon his or her degree of risk aversion. A highly risk-averse agent who decides to use VaR as the relevant measure of risk will increase his or her risk exposure.[6] In contrast, the risk exposure of a slightly risk-averse agent who decides to use VaR as the relevant measure of risk may decrease, remain constant, or increase. When there is a riskfree security in the economy, the risk exposure of an agent who decides to use VaR as the relevant measure of risk may also decrease, remain constant, or increase.

Second, we study the equilibrium implications arising from a mean-VaR framework. We show that an agent's decision to use VaR as the relevant measure of risk may affect equilibrium security prices. In equilibrium, the mean-VaR efficient set is non-empty for every agent. Therefore, if an agent chooses a confidence level such that at given security prices the mean-VaR efficient set is empty, then those prices cannot be equilibrium prices. Hence, security prices have to adjust so that the mean-VaR efficient set is non-empty for the agent who chooses the lowest confidence level to compute VaR. We note that the CAPM [see Sharpe (1964), Lintner (1965), and Mossin (1966)] result holds if agents have mean-VaR preferences. Finally, we find that in equilibrium, there is a linear relation between expected rate of return and VaR for efficient portfolios and derive the price of VaR, which we refer to as the reward-to-VaR ratio.

Our work is related to Basak and Shapiro (1999) and Ahn, Boudoukh, Richardson, and Whitelaw (1999) in that these two studies also investigate an optimization framework that takes VaR into consideration. While the first study provides a rigorous analysis of expected utility maximization with a constraint involving VaR, the second examines the problem of minimizing VaR with put options given a restriction in the maximal expenditure spent on such options. In contrast, we provide a characterization of the mean-VaR efficient set and examine the economic implications arising from a mean-VaR framework.[7]

This paper is organized as follows. In Section 2 we relate VaR to mean-variance analysis. Section 3 addresses the case of non-normality. In Section 4 we relate a mean-VaR criteria to expected utility maximization. In Section 5 we study the economic implications arising from using a mean-VaR framework. Section 6 concludes. All proofs are given in the Appendix.

2. The Mean-VaR Boundary and Efficient Frontier

Suppose that no riskfree security exists and let $n \geq 2$ be the number of risky securities in the economy. Assume that the rates of return of these securities have a multivariate normal distribution where $\mu \in \mathbb{R}^n$ is the vector of expected rates of return and Σ is the variance-covariance positive definite matrix of rates of return. Let $W \equiv \{w \in \mathbb{R}^n : \sum_{j=1}^{n} w_j = 1\}$ be the set of portfolios with well-defined expected rates of return (w_j has the interpretation of the proportion of wealth invested in security j). We do not rule out the possibility of short sales in that w_j is allowed to be negative. For any $w \in W$, define r_w as the random rate of return of portfolio w, and let $E[r_w]$ and $\sigma[r_w]$ denote the expected rate of return and the standard deviation of the rate of return of portfolio w.[8] Formally, we define VaR as follows.

Definition 1. Given a time period, the VaR at the $100t\%$ confidence level of a risky portfolio is the rate of return v such that $F(-v) = 1-t$, where $t \in (1/2, 1)$ and $F(.)$ is the cumulative distribution function of the portfolio's rate of return at the end of that period.

For any $t \in (1/2, 1)$, let $t^* \in (0, \infty)$ be such that $\Phi(-t^*) = 1-t$, where $\Phi(.)$ is the standard normal cumulative distribution function. Let $\mathbb{X}$ denote the space of random variables with normal distributions. Using Definition 1, the function $V : (1/2, 1) \times \mathbb{X} \rightarrow \mathbb{R}$, defined by

$$V[t, r] = t^*\sigma[r] - E[r], \ \forall(t, r) \in (1/2, 1) \times \mathbb{X} \tag{1}$$

gives the VaR at $100t\%$ confidence level for a portfolio with a normally distributed rate of return r. The measure of risk suggested by Baumol, the "lower confidence limit" for that portfolio, is given by $E[r]-k\sigma[r]$, where $k \in \mathbb{R}$, i.e., minus VaR for the appropriate confidence level.[9] The equivalency between the Baumol's measure of risk and VaR is evident.

2.1. The Mean-VaR Boundary

We now give two definitions and then note that they are equivalent when $t > 1/2$.

Definition 2. A portfolio $\overline{w} \in W$ belongs to the mean-VaR boundary at the $100t\%$ confidence level if and only if, for some $\overline{E} \in \mathbb{R}$, $\overline{w}$ solves the following problem:

$$\min_{w \in W} t^*\sigma[r_w] - E[r_w] \tag{2}$$

$$s.t. \quad E[r_w] = \overline{E}. \tag{3}$$

The set of mean-VaR boundary portfolios does not depend on t as long as $t > 1/2$, which we assume.[10] Hence, when referring to the mean-VaR boundary we omit the confidence level at which VaR is computed.

Definition 3. A portfolio $\overline{w} \in W$ belongs to the mean-variance boundary if and only if, for some $\overline{E} \in \mathbb{R}$, $\overline{w}$ solves the following problem:

$$\min_{w \in W} \sigma^2[r_w] \tag{4}$$

subject to equation (3).

Using equation (3) in the minimization problem (2), a portfolio belongs to the mean-VaR boundary if and only if it belongs to the mean-variance boundary since $t^* > 0$.[11] Merton (1972) showed that portfolio w belongs to the mean-variance boundary if and only if

$$\frac{\sigma^2[r_w]}{1/C} - \frac{(E[r_w] - A/C)^2}{D/C^2} = 1, \tag{5}$$

where A, B, C, and D are constants defined by

$$A = \imath^\top \Sigma^{-1} \mu, \tag{6}$$

$$B = \mu^\top \Sigma^{-1} \mu, \tag{7}$$

$$C = \imath^\top \Sigma^{-1} \imath, \tag{8}$$

$$D = BC - A^2, \tag{9}$$

with B, C and D being positive, and $\imath \in \mathbb{R}^n$ denoting the n-dimensional unit vector $(1, ..., 1)$.

Using equations (1) and (5), a portfolio w is in the mean-VaR boundary if and only if

$$\frac{\left[\left(V[t, r_w] + E[r_w]\right)/t^*\right]^2}{1/C} - \frac{\left(E[r_w] - A/C\right)^2}{D/C^2} = 1. \tag{10}$$

The representation of mean-VaR boundary portfolios in mean-VaR space corresponds to a simple transformation of the representation of mean-variance boundary portfolios in mean-standard deviation space [see Figures 1(a) and 1(b)].[12]

For any portfolio $w \in W$, $\lim_{t \to 1/2} V[t, r_w] = -E[r_w]$ since $t^* \to 0$ as $t \to 1/2$. That is, as the confidence level converges to 50%, the VaR of any portfolio converges to minus its expected rate of return.[13] Hence, the mean-VaR boundary converges to a line with slope of minus one that intersects the origin as $t \to 1/2$ [see Figure 2(a)].

For any portfolio $w \in W$, $\lim_{t \to 1} V[t, r_w] = +\infty$ since $t^* \to +\infty$ as $t \to 1$. In order to obtain a convergence result we note that $\lim_{t \to 1} V[t, r_w]/t^* = \lim_{t^* \to +\infty} \sigma[r_w] - E[r_w]/t^* = \sigma[r_w]$, indicating that the effect of the expected rate of return on the VaR of a risky portfolio decreases as the confidence level at which VaR is computed increases. Therefore, the representation of mean-VaR boundary portfolios in mean-VaR/t* space "converges" to the traditional mean-standard deviation hyperbola as $t \to 1$ [see Figure 2(b)].

2.2. The Mean-VaR Efficient Frontier

We now define mean-VaR efficiency.

Definition 4. A portfolio $w \in W$ belongs to the mean-VaR efficient frontier at the $100t\%$ confidence level if and only if no portfolio $v \in W$ exists such that $E[r_v] \geq E[r_w]$ and $V[t, r_v] \leq V[t, r_w]$, where at least one of the inequalities is strict.

Definition 4 is a "VaR version" of the standard mean-variance efficiency notion:

Definition 5. A portfolio $w \in W$ belongs to the mean-variance efficient frontier if and only if no portfolio $v \in W$ exists such that $E[r_v] \geq E[r_w]$ and *$\sigma[r_v] \leq \sigma[r_w]$, where at least one of the inequalities is strict.*

2.2.1. The Minimum VaR Portfolio

Parallel to the characterization of the minimum variance portfolio we now characterize the minimum VaR portfolio.

Lemma 1. If the minimum VaR portfolio at the $100t\%$ confidence level exists, then it is mean-variance efficient.

Recall that for a multivariate normally distributed rate of return r we have $V[t, r] = t^*\sigma[r] - E[r]$. Hence, there are two effects on VaR when we move along the mean-variance efficient frontier: (i) the *standard deviation effect* that occurs through the term $t^*\sigma[r]$; and (ii) the *mean effect* that occurs through the term $E[r]$. Therefore, if the confidence level at which VaR is determined is not large enough so that the standard deviation effect outweighs the mean effect, then the problem of minimizing VaR has no solution.

Assuming it exists, let $m_t \in W$ denote the minimum VaR portfolio at the $100t\%$ confidence level. The following result provides a characterization of the existence of the minimum VaR portfolio and a closed-form solution to the VaR minimization problem.

Proposition 1. The minimum VaR portfolio at the $100t\%$ confidence level exists if and only if $t > \Phi(\sqrt{D/C})$. Furthermore, if $t > \Phi(\sqrt{D/C})$, then $m_t \in W$ is given by

$$m_t = g + h\left[\frac{A}{C} + \sqrt{\frac{D}{C}\left(\frac{(t^*)^2}{C\,(t^*)^2 - D} - \frac{1}{C}\right)}\right], \tag{11}$$

where g and h are n-dimensional vectors defined by $g = \frac{1}{D}[B\,(\Sigma^{-1}\iota) - A(\Sigma^{-1}\mu)]$ and $h = \frac{1}{D}[C(\Sigma^{-1}\mu) - A\,(\Sigma^{-1}\iota)]$.

Proposition 1 shows that one must be careful in choosing confidence levels to calculate VaR in a way that makes sense (i.e., so that minimizing VaR is an obtainable objective). If t is sufficiently small, then the VaR minimization problem does not have a solution.

Using Proposition 1 and equation (1), if the minimum VaR portfolio exists, then its VaR is given by

$$V[t, r_{m_t}] = t^*\sqrt{\frac{(t^*)^2}{C\,(t^*)^2 - D}} - \left[\frac{A}{C} + \sqrt{\frac{D}{C}\left(\frac{(t^*)^2}{C\,(t^*)^2 - D} - \frac{1}{C}\right)}\right]. \tag{12}$$

The following result indicates that the solution to the VaR minimization problem is always distinct from the solution to the variance minimization problem.

Corollary 1. If the minimum VaR portfolio exists at a given confidence level $t < 1$, then it lies above the minimum variance portfolio on the mean-variance efficient frontier.

Corollary 1 shows that when $t < 1$, the mean effect is always present in the VaR minimization problem. That is, when $t < 1$ we may decrease VaR through an increase in the mean (and, therefore, in the standard deviation) of the minimum variance portfolio as we move along the mean-variance efficient frontier.[14] This result can be explained as follows. Using equation (5), note that for the minimum variance portfolio, denoted by $m_\sigma \in W$, we have $\partial\sigma[r_w]/\partial E[r_w]|_{w=m_\sigma} = 0$. Hence, an increase in the mean of the minimum variance portfolio as we move along the mean-variance efficient frontier produces a relatively small increase in standard deviation and, therefore, the mean effect outweighs the standard deviation effect.

2.2.2. A Characterization of Mean-VaR Efficiency

Proposition 2 is a characterization of mean-VaR efficiency.[15]

Proposition 2. (i) If $t > \Phi(\sqrt{D/C})$, then a portfolio is mean-VaR efficient at that confidence level if and only if it belongs to the mean-VaR boundary and it has an expected rate of return greater than or equal to the expected rate of return of the minimum VaR portfolio at that confidence level. (ii) If $t \leq \Phi(\sqrt{D/C})$, then no mean-VaR efficient portfolio exists at that confidence level.

Using Propositions 1 and 2, for any $t > 1/2$, there exist a vector of expected rates of return $\mu \in \mathbb{R}^n$ and a variance-covariance matrix of rates of return Σ generating an empty mean-VaR efficient frontier at the $100t\%$ confidence level. Corollary 2 follows immediately from Proposition 2.

Corollary 2. The minimum variance portfolio is mean-VaR inefficient at any confidence level $t < 1$.

An illustration of Corollary 2 is provided in Figure 1(b). Propositions 1 and 2 are also useful to derive the following convergence results.

2.2.3. Convergence Results

Corollary 3 shows that the solution of the VaR minimization problem converges to the solution of the variance minimization problem as $t \to 1$.

Corollary 3. The minimum VaR portfolio converges to the minimum variance portfolio as $t \to 1$.

Corollary 3 can be explained as follows. As the confidence level at which VaR is determined increases, the standard deviation effect also increases due to the associated increase in t^*. Hence, as the confidence level at which VaR is determined converges to 100%, the solution of the VaR minimization problem converges to the solution of the variance minimization problem since the mean effect becomes small and virtually disappears in the limit.

Corollary 4. The set of mean-VaR efficient portfolios is a proper subset of the set of mean-variance efficient portfolios when $t < 1$, *but the former converges to the latter as* $t \to 1$.[16]

Corollary 4 shows that using VaR as a measure of risk reduces the set of efficient portfolios in the economy when compared with the mean-variance efficiency criteria, and that the mean-VaR efficient frontier converges to the mean-variance efficient frontier as $t \to 1$.

Corollary 5. The expected rate of return of the minimum VaR portfolio converges to infinity and the set of mean-VaR efficient portfolios converges to the empty set as $t \downarrow \Phi(\sqrt{D/C})$.

The first result of Corollary 5, that the expected rate of return of the minimum VaR portfolio converges to infinity when the confidence level at which VaR is determined converges to $\Phi(\sqrt{D/C})$, shows that the mean effect determines the solution of VaR minimization problem for small confidence levels. That is, we may decrease VaR through an increase in the expected rate of return along the mean-variance efficient frontier when the confidence level at which VaR is determined is small since the mean effect dominates the standard deviation effect.

Embedding VaR as a constraint in an expected utility maximization framework, Basak and Shapiro (1999) show that agents that use VaR to manage risk often choose to have a larger exposure to risky assets than agents that do not manage risk. The first result of Corollary 5 says that agents that minimize VaR at relatively small confidence levels will choose portfolios with large expected rates of return and, therefore, are exposed to very high levels of risk as measured by standard deviation.

The second result of Corollary 5 says that using VaR at a small confidence level reduces the set of mean-VaR efficient portfolios in the economy and it may preclude their existence.

2.2.4. A Characterization of Mean-Variance Efficiency Using the Minimum VaR Portfolio

Using our results about the minimum VaR portfolio, we have the following interesting characterization of the mean-variance efficient frontier.

Proposition 3. A portfolio $w \in W \setminus \{m_\sigma\}$ belongs to the mean-variance efficient frontier if and only if w is the minimum VaR portfolio for the confidence level given by:

$$\Phi\left(\sqrt{D/C + \frac{D^2/C^3}{(E[r_w] - A/C)^2}}\right).$$

Proposition 3 says that any mean-variance efficient portfolio other than the minimum variance portfolio globally minimizes VaR for some confidence level. Using Propositions 1-3, note that any mean-variance efficient portfolio $w \in W \setminus \{m_\sigma\}$ is mean-VaR inefficient (efficient) at any confidence level lower than (greater than or equal to) $\Phi\left(\sqrt{D/C + \frac{D^2/C^3}{(E[r_w]-A/C)^2}}\right)$.

2.3. Adding a Riskfree Security

Our previous analysis of the mean-VaR boundary assumes that no riskfree security exists in the economy. Suppose now that there are n risky securities and a riskfree security with rate of return $r_f > 0$. Let $W_f \equiv \{w \in \mathbb{R}^{n+1} : \sum_{j=1}^{n+1} w_j = 1\}$, where w_{n+1} is the proportion of wealth invested in the riskfree security. Then, $\overline{w} \in W_f$ belongs to the mean-VaR boundary at the $100t\%$ confidence level if and only if, for some $\overline{E} \in \mathbb{R}$, $\overline{w}$ solves:

$$\min_{w \in W_f} t^* \sigma[r_w] - E[r_w] \tag{13}$$

$$s.t. \quad E[r_w] = \overline{E}. \tag{14}$$

Using equation (14) in the minimization problem (13) and noting that $t^* > 0$, a portfolio belongs to the mean-VaR boundary if and only if it belongs to the mean-variance boundary. Merton showed that when a riskfree security exists in the economy, a portfolio $w \in W_f$ belongs to the mean-variance boundary if and only if

$$\sigma^2[r_w] = (E[r_w] - r_f)^2 / H, \tag{15}$$

where H is a positive constant defined by $H = Cr_f^2 - 2Ar_f + B$. Accordingly, a portfolio $w \in W_f$ belongs to the mean-VaR boundary if and only if

$$[(V[t, r_w] + E[r_w])/t^*]^2 = (E[r_w] - r_f)^2 / H. \tag{16}$$

Rearranging equation (16), we obtain

$$V[t, r_w] = \begin{cases} \frac{E[r_w](t^* - \sqrt{H}) - t^* r_f}{\sqrt{H}} & \text{if } E[r_w] \geq r_f \\ \frac{E[r_w](-t^* - \sqrt{H}) + t^* r_f}{\sqrt{H}} & \text{if } E[r_w] < r_f, \end{cases} \tag{17}$$

i.e., the two lines that characterize the relation between VaR and expected rate of return for mean-VaR boundary portfolios. Note, however, that the absolute value of the slopes of these two lines differ. This contrasts with the mean-variance case where the absolute value of the slopes of the two lines that characterize the relation between expected rate of return and standard deviation of the rate of return coincide [see Huang and Litzenberger (1988, pp. 76-80)]. Figures 3(a) and 3(b) illustrate the location of the mean-standard deviation and mean-VaR boundaries, respectively, when there is a riskfree security in the economy.

We now characterize mean-VaR efficiency when a riskfree security exists in the economy.

Proposition 4. (i) If $t > \Phi(\sqrt{H})$, then a portfolio belongs to the mean-VaR efficient frontier at that confidence level if and only if it belongs to the mean-variance efficient frontier. (ii) If $t \le \Phi(\sqrt{H})$, then no mean-VaR efficient portfolio exists at that confidence level.

If a riskfree security exists in the economy and VaR is computed at a sufficiently large confidence level, then the mean-VaR efficient frontier coincides with the mean-variance efficient frontier. This contrasts with the case when no riskfree security exists in the economy and the mean-VaR efficient frontier is a proper subset of the mean-variance efficient frontier.

Using equation (17) and Proposition 4, note that the minimum VaR portfolio at the $100t\%$ confidence level exists if and only if $t > \Phi(\sqrt{H})$. If $t > \Phi(\sqrt{H})$, then the minimum VaR portfolio at the $100t\%$ confidence level is the riskfree security [see Figure 3(b)].

It can be shown that $H \ge D/C$, and that $H > D/C$ if and only if $r_f \ne A/C$.[17] Hence, given a fixed confidence level it is easier to obtain an empty mean-VaR efficient set when there is a riskfree security in the economy. In particular, if $H > D/C$, then for any confidence level $t \in (\Phi(\sqrt{D/C}), \Phi(\sqrt{H}))$, the mean-VaR efficient set is empty if the riskfree security is considered even though it is not empty in the absence of the riskfree security.

3. VaR and Non-Normality

We now examine a generalization of our results to the case of non-normality.[18] We note that our results can easily be extended to the case of multivariate t-distributed rates of return (this allows security rates of return to have 'fat tails').[19] We then show that when no distributional assumption is made (this allows security rates of return to have 'fat tails,' skewness, or both), our results still hold from the perspective of a Chebyshev upper bound on VaR.

3.1. t-Distributed Rates of Returns

Assume that the rates of return of n risky securities have a multivariate t-distribution, denoted by $\tau_n(\mu, \Omega, v)$, where $\mu \in \mathbb{R}^n$ is the vector of expected rates of return, Ω is the $(n \times n)$

scale matrix, and $v > 2$ is the number of degrees of freedom. Then, the variance-covariance matrix of the rates of return is $\Sigma \equiv [v/(v-2)]\Omega$.

For any $t \in (1/2, 1)$, let $t^*(v) \in (0, \infty)$ be such that $F_v(-t^*(v)) = 1-t$, where $F_v(.)$ is the cumulative distribution function of a Student t-distribution with $v > 2$ degrees of freedom. Note that if $Z \sim \tau_n(\mu, \Omega, v)$, then $a^\top Z \sim \tau_1(a^\top \mu, a^\top \Omega a, v)$ for every $a \in \mathbb{R}^n$. Hence, for any portfolio $w \in W$, we have $r_w \sim \tau_1(w^\top \mu, w^\top \Omega w, v)$, and $V[t, r_w, v]$, portfolio w's VaR, is

$$V[t, r_w, v] = t^*(v)[(v-2)/v]^{1/2}\sigma[r_w] - E[r_w].$$

That is, for a fixed v, portfolio w's VaR is a linear function of $\sigma[r_w]$ and $E[r_w]$. Therefore, our previous results still hold if rates of return have a multivariate t-distribution, and $V[t, r_w, v]$ and $t^*(v)[(v-2)/v]^{1/2}$ are used instead of $V[t, r_w]$ and t^*, respectively.

3.2. VaR and Chebyshev's Inequality

Analytical results involving VaR seem improbable under other general distributional assumptions. However, using Chebyshev's inequality [see, for example, Fristedt and Gray (1997, pp. 62)], a general statement can be made when both the first and second moments of the distribution of security rates of return are finite. Chebyshev's inequality says that

$$P\{|x - E[x]| \geq k\} \leq (\sigma[x]/k)^2 \tag{18}$$

for any random variable x with finite mean $E[x]$ and standard deviation $\sigma[x]$, and any $k \in \mathbb{R}_{++}$. Let $w \in W$ be a portfolio. Using equation (18) with $x = r_w$ and $k = t^*\sigma[r_w]$,

$$P\{|r_w - E[r_w]| \geq t^*\sigma[r_w]\} \leq (1/t^*)^2. \tag{19}$$

Using equation (19), we have $P\{r_w \leq -t^*\sigma[r_w] + E[r_w]\} \leq (1/t^*)^2$ or, equivalently,

$$V[1 - (1/t^*)^2, r_w] \leq t^*\sigma[r_w] - E[r_w]. \tag{20}$$

Hence, Chebyshev's inequality provides an useful upper bound on VaR when no distributional assumption is imposed. For example, the Chebyshev upper bound on $V[.95, r_w]$ is $4.4721\sigma[r_w] - E[r_w]$, whereas under normality $V[.95, r_w] = 1.6449\sigma[r_w] - E[r_w]$.[20]

Equation (20) can also be used to extend our previous results to the case when no distributional assumption is made. For example, our characterization of the VaR minimization problem identifies the portfolio that globally minimizes the Chebyshev upper bound on VaR among all feasible portfolios. Moreover, it shows that one must be careful in choosing a confidence level to compute VaR so that upper bounds on VaR exist. This result is similar to the one that was derived under normality. If one chooses a sufficiently small confidence level, then minimizing VaR is not an obtainable objective since the problem of minimizing the Chebyshev upper bound on VaR does not have a solution.

4. Expected Utility and VaR

We now show that, at least as an approximation, the mean-VaR framework examined in this paper arises naturally from the portfolio choice problem of a risk-averse agent who maximizes expected utility. We begin by addressing the case when security rates of return have either a multivariate normal or t-distribution. Then, we consider the general case when no distributional assumption is made.

4.1. Normally or t-Distributed Rates of Return

Given a confidence level and arbitrary preferences over wealth, the mean-VaR model can be motivated with the assumption of multivariate normally or t-distributed rates of return. The reason is that under these assumptions, the rate of return of any portfolio can be completely described by its mean and VaR for a fixed confidence level.

4.2. General Case

We now consider the general case when no assumption is imposed on the distribution of security rates of return other than its first and second moments are finite. Suppose that an agent has a strictly increasing von Neumann-Morgenstern utility function $u : \mathbb{R}_+ \to \mathbb{R}$ that displays non-increasing absolute risk aversion.[21] Without loss of generality, we normalize the agent's wealth at time 0 to be equal to 1. The agent's portfolio choice problem is:

$$\max_{w \in W} \quad E[u(1 + r_w)]$$
$$s.t. \quad r_w \geq -1.$$

Levy and Markowitz (1979) proposed the following approximation to expected utility

$$E[u(1 + r_w)] \approx f(E[r_w], \sigma[r_w], k), \tag{21}$$

where $k > 0$ and $f : \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_{++} \to \mathbb{R}$ is defined by

$$f(E, \sigma, k) = u(1 + E) + \frac{u(1 + E + k\sigma) + u(1 + E - k\sigma) - 2u(1 + E)}{2k^2} \tag{22}$$

for every $(E, \sigma, k) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_{++}$. Since negative consumption is precluded, using equation (22) as an approximation to expected utility for a given $k > 0$ requires the portfolio set to be restricted to only include portfolios w satisfying $E[r_w] - k\sigma[r_w] > -1$. Using several utility functions, Levy and Markowitz presented evidence showing that the approximation given by equation (22) is accurate.

Let $V \equiv k\sigma - E$. Using equation (22), we obtain

$$\begin{aligned} f(E, \sigma, k) &= u(1 + E) + \frac{u(1 + 2E + V) + u(1 - V) - 2u(1 + E)}{2k^2} \\ &\equiv g(E, V, k). \end{aligned}$$

Note that if $k \geq 1$, then $\partial g(E, V, k)/\partial E > 0$. Moreover,

$$\frac{\partial g(E, V, k)}{\partial V} = \frac{u'(1 + 2E + V) - u'(1 - V)}{2k^2} < 0 \tag{23}$$

since $2E + V > -V$ and $u''(\cdot) < 0$. Therefore, any portfolio that is a solution to the problem of maximizing approximate expected utility, as given by equation (22), is efficient according to a $(E, k\sigma - E)$ criteria, where $k \geq 1$. Consequently, assuming that the approximation given by equation (22) is accurate as attested by Levy and Markowitz, our results in a mean-VaR framework also hold in economic environments where the assumptions of normally or t-distributed rates of return are not made. More precisely, given the interpretation of the parameter $t^*\sigma - E$ explained in Subsection 3.2., a "mean-Chebyshev upper bound on VaR" framework arises, at least as an approximation, from the portfolio choice problem of a risk-averse agent who maximizes expected utility.

5. Economic Implications of a Mean-VaR Framework

We now examine the economic implications arising from using a mean-VaR framework.[22] Since in practice VaR has become a standard tool for controlling risk, a natural approach is to assume that an agent, previously with mean-variance preferences, begins to use VaR as the relevant measure of risk, either by regulation or by choice.[23] We study the implications of a mean-VaR framework both in an agent's portfolio choice problem and in equilibrium prices.

5.1. Portfolio Choice Problem

5.1.1. Optimal Portfolio According to a Mean-VaR Criteria

How does an agent's optimal portfolio change when he or she, previously having mean-variance preferences, decides to use VaR as the relevant measure of risk? Interestingly, we show that the change in an agent's risk exposure depends upon his or her degree of risk aversion and whether a riskfree security exists in the economy. Throughout, by the change in an agent's risk exposure we mean the difference between the standard deviation of the agent's optimal portfolio according to a mean-VaR criteria and the standard deviation of the agent's optimal portfolio according to a mean-variance criteria. Furthermore, we assume that the agent decides to use VaR at a confidence level such that the minimum VaR portfolio exists at that confidence level (if this assumption is not made, then the agent's portfolio choice problem does not have a solution).

Assume that no riskfree security exists in the economy. First, we consider the case of a highly risk-averse agent. Since the agent is highly-risk averse, according to a mean-variance criteria, his or her optimal portfolio lies on the mean-variance efficient frontier relatively close to the minimum variance portfolio. Hence, unless the confidence level at which the agent determines VaR is extremely high, his or her optimal portfolio according to a mean-variance criteria is mean-VaR inefficient [see Figures 1(a) and 1(b)]. Hence, the agent's optimal portfolio according to a mean-VaR criteria lies on the mean-variance efficient frontier above the agent's optimal portfolio according to a mean-variance criteria. Consequently, the agent's risk exposure increases if he or she decides to use VaR as the relevant measure of risk.[24]

Second, we consider the case of a slightly risk-averse agent. In contrast to the previous case, the agent's optimal portfolio according to a mean-variance criteria is mean-VaR efficient even at a relatively small confidence level. Consequently, a slightly risk-averse agent who decides to use VaR as the relevant measure of risk may choose a portfolio that coincides with his or her optimal portfolio according to a mean-variance criteria. Depending upon the specification of the agent's (mean-VaR and mean-variance) preferences, the agent's risk exposure may decrease, remain constant, or increase.[25]

Suppose now that there is a riskfree security in the economy. Depending upon the confidence level at which VaR is computed, it follows from Proposition 4 that the mean-VaR efficient frontier is either empty or it coincides with the mean-variance efficient frontier. If the confidence level is chosen so that the mean-VaR efficient frontier is non-empty, then, depending upon the specification of the agent's (mean-VaR and mean-variance) preferences, the agent's risk exposure may decrease, remain constant, or increase.

5.1.2 A Lower Bound on the Change in an Agent's Risk Exposure

We have shown that when no riskfree security exists in the economy, a highly risk-averse agent, previously having mean-variance preferences, will increase his or her risk exposure if he or she decides to use VaR as the relevant measure of risk. We now provide a lower bound on the change in an agent's risk exposure arising from the decision to use VaR instead of standard deviation.

For simplicity, we consider an agent with mean-variance preferences represented by the utility function $U : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ defined by

$$U(E, \sigma) = E - \frac{1}{2} a\sigma^2, \ \forall (E, \sigma) \in \mathbb{R} \times \mathbb{R}_+, \tag{24}$$

where $a > 0$. Using equation (24), the agent's optimal portfolio is mean-variance efficient. Since along the mean-variance efficient frontier $E = A/C + \sqrt{(D/C)\,(\sigma^2 - 1/C)}$, a necessary and sufficient condition for σ to be the agent's optimal level of standard deviation is:

$$\frac{\partial \left[A/C + \sqrt{(D/C)\,(\sigma^2 - 1/C)} \right]}{\partial \sigma} = a\sigma. \tag{25}$$

It follows from equation (25) that the agent's optimal portfolio has

$$\sigma = \sqrt{(1 + D/a^2)/C}. \tag{26}$$

In the Appendix, we show that if $t > \Phi(\sqrt{D/C})$, then $\sigma[r_{m_t}] > \sigma$ if and only if $t < \Phi(\sqrt{a^2/C + D/C})$.

Suppose that an agent chooses a confidence level $t \in (\Phi(\sqrt{D/C}), \Phi(\sqrt{a^2/C + D/C}))$. Since the agent's optimal portfolio according to a mean-VaR criteria lies on the mean-VaR efficient frontier, $\sigma[r_{m_t}] - \sigma$ is a lower bound on the change of an agent's risk exposure arising from his or her decision to use VaR at the $100t\%$ confidence level as the relevant measure of risk. This lower bound is useful since it does not depend on a particular specification for an agent's mean-VaR preferences other than the confidence level at which he or she decides to compute VaR. If the agent decides to minimize VaR at the $100t\%$ confidence level, then $\sigma[r_{m_t}] - \sigma$ is a precise measure of the change of an agent's risk exposure. Note that a necessary but not sufficient condition for an agent decreasing his or her risk exposure is that $\sigma[r_{m_t}] - \sigma < 0$. A further assumption on the agent's mean-VaR preferences is needed to precisely quantify the change in his or her risk exposure.

Assuming the data used in Figure 1, Table 1 reports $\sigma[r_{m_t}] - \sigma$ for several values of the parameters t and a. For a slightly risk-averse agent ($a = 3$), the lower bound on the change of his or her risk exposure is negative. For a moderately risk-averse agent ($a = 5$, 10), the decision to minimize VaR at a relatively small (large) confidence level implies that he or she will increase (decrease) his risk exposure. For a highly risk-averse agent ($a = 25$), the decision to use VaR even at a relatively high confidence level (99%) implies that his or her risk exposure will increase. Finally, for an infinitely risk-averse agent ($a = \infty$), the decision to use VaR (when $t < 1$) will increase his or her risk exposure.

5.1.3. Confidence Level Implied by Mean-Variance Optimal Portfolios

Suppose that an agent, previously having the utility function given by equation (24), now has mean-VaR preferences represented by the utility function $U : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

$$U(E, V) = E - \frac{1}{2} b V^2, \ \forall (E, V) \in \mathbb{R} \times \mathbb{R}, \tag{27}$$

where $b > 0$ and VaR is computed at a fixed confidence level t.

Assuming the data used in Figure 1, Table 2 reports the values of the confidence level so that the agent's optimal portfolio in a mean-VaR framework coincides with his or her optimal portfolio in a mean-variance framework. Results are reported for several values of the parameters a and b. Those values also have the following interpretation. Given a and b, if an agent chooses to determine VaR at a confidence level smaller (larger) than the confidence level reported in Table 2, then the agent's decision to use VaR as the relevant measure of risk will increase (decrease) his or her risk exposure.

5.2. Equilibrium

5.2.1 Security Prices

What happens to equilibrium security prices if an agent uses VaR as the relevant measure of risk at a given confidence level? Clearly, given security prices, if the mean-VaR efficient set is empty at that confidence level, then the agent's portfolio choice problem does not have a solution. Therefore, in equilibrium, the mean-VaR efficient set must be non-empty for the agent who considers the smallest confidence level in the computation of VaR. Hence, equilibrium security prices when agents use VaR as the relevant measure of risk may be differ from equilibrium security prices when agents have mean-variance preferences.[26]

5.2.2 Equilibrium under Normality and Mean-VaR Preferences

We now show that the CAPM holds when security rates of return have a multivariate normal distribution and agents have mean-VaR preferences. We then obtain the relation between expected rate of return and VaR in equilibrium. Then we define the reward-to-VaR ratio, i.e., the price of VaR. Finally, we derive the Capital Market Line in mean-VaR space.

Let $U : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a mean-VaR utility function strictly increasing in expected rate of return and strictly decreasing in VaR. Observe that if security rates of return are multivariate normally distributed, then for any $\sigma \in \mathbb{R}_+$,

$$U(E_2, t^*\sigma - E_2) > U(E_1, t^*\sigma - E_1), \ \forall E_2 > E_1, \tag{28}$$

and, for any $E \in \mathbb{R}$,

$$U(E, t^*\sigma_2 - E) < U(E, t^*\sigma_1 - E), \ \forall \sigma_2 > \sigma_1. \tag{29}$$

Therefore, U is strictly increasing in expected rate of return and strictly decreasing in standard deviation. Hence, if: (1) agents have mean-VaR preferences, (2) the riskfree security exists, and (3) the rates of return of the risky securities have a multivariate normal distribution, then the CAPM result holds in equilibrium. Using Proposition 4, equilibrium security prices are such that $\Phi(\sqrt{H})$ is smaller than the confidence level at which every agent computes VaR. Note that the CAPM result also holds in equilibrium in an economy where some agents have mean-variance preferences and others have mean-VaR preferences.

Proposition 5. In equilibrium, the relation between expected rate of return and VaR of efficient portfolios is given by

$$E[r_p] = r_f \left(\frac{V[t, r_M] + E[r_M]}{V[t, r_M] + r_f} \right) + \left(\frac{E[r_M] - r_f}{V[t, r_M] + r_f} \right) V[t, r_p], \ \forall \ V[t, r_p] \geq -r_f. \tag{30}$$

It follows from Proposition 5 that in equilibrium there is a linear relation between expected rate of return and VaR for efficient portfolios. Using equation (30) and assuming that $V[t, r_M] > -r_f$, note that the intercept of the capital market line in mean-VaR space is above r_f since in equilibrium $E[r_M] > r_f$.

Defining the VaR of a riskfree portfolio at any confidence level to be minus its rate of return,[27] it follows from equation (30) that

$$E[r_p] = r_f \left(\frac{V[t, r_M] + E[r_M]}{V[t, r_M] - V[t, r_f]} \right) + \left(\frac{E[r_M] - r_f}{V[t, r_M] - V[t, r_f]} \right) V[t, r_p]. \tag{31}$$

Using equation (31), the market price of risk as measured by VaR (i.e., the additional expected rate return that an agent requires for each additional unit of VaR that he or she bears) is given by the reward-to-VaR ratio $(E[r_M] - r_f)/(V[t, r_M] - V[t, r_f])$. This measure corresponds in a mean-variance framework to the Sharpe ratio of the market portfolio, $(E[r_M] - r_f)/\sigma[r_M]$.

6. Conclusion

In this paper we relate VaR with mean-variance analysis and study the economic implications arising from a mean-VaR framework. We begin by investigating mean-VaR efficiency when security rates of returns have a multivariate normal distribution. Initially, we assume that no riskfree security exists in the economy. A characterization of the existence of the minimum VaR portfolio shows that one must be careful in choosing the confidence level to determine VaR so that minimizing VaR is an obtainable objective. If the confidence level at which VaR is computed is small, then the minimum VaR portfolio may not exist and, consequently, the mean-VaR efficient set may be empty.

Our characterization of the minimum VaR portfolio allows us to analytically describe the mean-VaR efficient frontier and to derive several results. As the confidence level at which VaR is computed increases: (i) the minimum VaR portfolio converges to the minimum variance portfolio; and (ii) the mean-VaR efficient frontier converges to the mean-variance efficient frontier. Conversely, as the confidence level at which VaR is computed decreases: (i) the expected rate of return on the minimum VaR portfolio converges to infinity; and (ii) the set of mean-VaR efficient portfolios converges to the empty set. We also show that a portfolio is mean-variance efficient if and only if it globally minimizes VaR for some confidence level. We find that if there is a riskfree security in the economy and the confidence level at which VaR is computed is sufficiently small, then minimizing VaR is also not an obtainable objective.

Since in many instances security rates of returns have 'fat tails,' we provide a useful extension of the above results to the case when security rates of return have a multivariate t-distribution. We then generalize the result to the case when the normality or t-distribution assumption is not imposed, thereby allowing security rates of return to have both skewness and 'fat tails,' by using Chebyshev's inequality. Assuming that security rates of return have finite first and second moments, a Chebyshev upper bound on VaR can easily be derived. This simple calculation makes an interesting point on how careful one must be in choosing confidence levels to compute VaR, as the problem of minimizing the Chebyshev upper bound on VaR may not have a solution. Furthermore, the Chebyshev upper bound on VaR provides relevant information in terms of a VaR upper bound for the appropriate confidence level when a distributional assumption is not made.

We find that a mean-VaR criteria, at least as an approximation, is consistent with a portfolio choice problem of a risk-averse agent who maximizes expected utility. We then derive economic implications arising from a mean-VaR framework. We find that when no riskfree security exists in the economy, a highly risk-averse agent will increase his or her risk exposure if he or she decides to use VaR as the relevant measure of risk. In contrast, the risk exposure of a slightly risk-averse agent may decrease, remain constant, or increase if he or she decides to use VaR as the relevant measure of risk. When there is a riskfree security in the economy, the risk exposure of an agent who decides to use VaR as a relevant measure of risk may also decrease, remain constant, or increase.

Equilibrium implications arising from a mean-VaR framework include the results that an agent's decision to use VaR as the relevant measure of risk may affect equilibrium security prices and that the CAPM holds if agents have mean-VaR preferences. Finally, we find that in equilibrium, there is a linear relation between expected rate of return and VaR for efficient portfolios and derive the price of VaR, which we refer to as the reward-to-VaR ratio.

Appendix

Proof of Lemma 1. Assume that the minimum VaR portfolio at the $100t\%$ confidence level, denoted m_t, exists. Suppose by way of a contradiction that it is not mean-variance efficient. Using Definition 5, there exists $v \in W$ such that $E[r_v] \geq E[r_{m_t}]$, $\sigma[r_v] \leq \sigma[r_{m_t}]$, where at least one of the inequalities is strict. Hence, we have $t^*\sigma[r_v] - E[r_v] < t^*\sigma[r_{m_t}] - E[r_{m_t}]$ and using equation (1), $V[t, r_v] < V[t, r_{m_t}]$, which contradicts the assumption that m_t is the minimum VaR portfolio at the $100t\%$ confidence level. ■

Proof of Proposition 1. First, we show that $t > \Phi(\sqrt{D/C})$ or, equivalently, that $t^* > \sqrt{D/C}$ is a necessary and sufficient condition for the existence of the minimum VaR portfolio. Note that using equation (5), we have

$$E[r_w] = A/C + \sqrt{(D/C)(\sigma^2[r_w] - 1/C)} \tag{32}$$

for any mean-variance efficient portfolio $w \in W$. We need to solve

$$\min_{w \in W} V[t, r_w]. \tag{33}$$

From Huang and Litzenberger (1988, pp. 66-67) the minimum variance portfolio, denoted m_σ, has $\sigma[r_{m_\sigma}] = 1/\sqrt{C}$. Hence, using Lemma 1 and equations (1) and (32), we can solve

$$\min_{\sigma \in [1/\sqrt{C}, \infty)} t^*\sigma - \left[A/C + \sqrt{(D/C)\,(\sigma^2 - 1/C)}\right] \tag{34}$$

to determine the VaR of the minimum VaR portfolio. Note that

$$\frac{\partial \left\{t^*\sigma - \left[A/C + \sqrt{(D/C)\,(\sigma^2 - 1/C)}\right]\right\}}{\partial \sigma} = t^* - \frac{\sigma\sqrt{D/C}}{\sqrt{\sigma^2 - 1/C}}. \tag{35}$$

Clearly, $\overline{\sigma} = 1/\sqrt{C}$ does not solve the minimization problem (34) since

$$\lim_{\sigma \to 1/\sqrt{C}} t^* - \frac{\sigma\sqrt{D/C}}{\sqrt{\sigma^2 - 1/C}} = -\infty.$$

Using equation (35), a necessary condition for σ_{m_t} to solve the minimization problem (34) is

$$t^* - \frac{\sigma_{m_t}\sqrt{D/C}}{\sqrt{\sigma_{m_t}^2 - 1/C}} = 0. \tag{36}$$

Simplifying equation (36) we obtain $t^*\sqrt{\sigma_{m_t}^2 - 1/C} = \sigma_{m_t}\sqrt{D/C}$ and, therefore,

$$\sigma_{m_t} = \sqrt{(t^*)^2 / [C\,(t^*)^2 - D]}. \tag{37}$$

It follows from equation (37) that $C\,(t^*)^2 - D > 0$ or, equivalently, $t^* > \sqrt{D/C}$ is a necessary condition for the existence of the minimum VaR portfolio at the $100t\%$ confidence level.

We now show that $t^* > \sqrt{D/C}$ is a sufficient condition for the existence of the minimum VaR portfolio at the $100t\%$ confidence level by showing that we are minimizing a convex function. Using equation (35), we have

$$\begin{aligned}
\frac{\partial^2 \left\{t^*\sigma - \left[A/C + \sqrt{(D/C)(\sigma^2 - 1/C)}\right]\right\}}{\partial \sigma^2} &= \frac{\partial\left(t^* - \frac{\sigma\sqrt{D/C}}{\sqrt{\sigma^2 - 1/C}}\right)}{\partial \sigma} \\
&= -\frac{\sqrt{D/C}\sqrt{\sigma^2 - 1/C} - \frac{\sigma^2\sqrt{D/C}}{\sqrt{\sigma^2 - 1/C}}}{\sigma^2 - 1/C} \\
&= \frac{1/C\sqrt{D/C}}{(\sigma^2 - 1/C)^{3/2}} > 0, \forall \sigma \in (1/\sqrt{C}, \infty). \qquad (38)
\end{aligned}$$

Using equation (38), $t^* > \sqrt{D/C}$ is a sufficient condition for the existence of the minimum VaR portfolio at the $100t\%$ confidence level. This completes the first part of our proof.

Second, we show that equation (11) characterizes the minimum VaR portfolio. For any $E \in \mathbb{R}$ there exists [see Huang and Litzenberger (1988, pp. 64-65)] a unique mean-variance boundary portfolio $w_E \in W$ with expected rate of return E:

$$w_E = g + hE. \tag{39}$$

Using equations (32) and (37), we have

$$E[r_{m_t}] = \frac{A}{C} + \sqrt{\frac{D}{C}\left(\frac{(t^*)^2}{C\,(t^*)^2 - D} - \frac{1}{C}\right)}. \tag{40}$$

Equation (11) follows from equations (39) and (40). This completes the second part of our proof. ■

Proof of Corollary 1. Suppose that the minimum VaR portfolio exists at a given confidence level $t < 1$. Using Lemma 1, the minimum VaR portfolio lies on the mean-variance efficient frontier. From Huang and Litzenberger (1988, pp. 66-67) the minimum variance portfolio has $E[r_{m_\sigma}] = A/C$. Using equation (40), we have $E[r_{m_t}] > E[r_{m_\sigma}]$ since $(t^*)^2/[C(t^*)^2 - D] > 1/C$. Hence, if the minimum VaR portfolio exists at a given confidence level $t < 1$, then it lies above the minimum variance portfolio on the mean-variance efficient frontier. ■

Proof of Proposition 2. First, we show (i). If $t > \Phi(\sqrt{D/C})$, then $t^* > \sqrt{D/C}$. It is enough to show that $\partial^2 V[t, r_w]/\partial E[r_w]^2 > 0$ for every mean-VaR boundary portfolio $w \in W$. Let $w \in W$ be an arbitrarily chosen mean-VaR boundary portfolio. Then, using equation (5), we have

$$\sigma[r_w] = \sqrt{\frac{1}{C} + \frac{(E[r_w] - A/C)^2}{D/C}}. \tag{41}$$

It follows from equations (1) and (41) that

$$\begin{aligned}
\frac{\partial V[t, r_w]}{\partial E[r_w]} &= \frac{\partial(t^*\sigma[r_w] - E[r_w])}{\partial E[r_w]} \\
&= t^* \frac{\partial \sqrt{\frac{1}{C} + \frac{(E[r_w]-A/C)^2}{D/C}}}{\partial E[r_w]} - 1 \\
&= t^* \frac{\frac{E[r_w]-A/C}{D/C}}{\sqrt{\frac{1}{C} + \frac{(E[r_w]-A/C)^2}{D/C}}} - 1.
\end{aligned} \tag{42}$$

Using equation (42), we obtain

$$\begin{aligned}
\frac{\partial^2 V[t, r_w]}{\partial E[r_w]^2} &= t^* \frac{\frac{\sqrt{\frac{1}{C} + \frac{(E[r_w]-A/C)^2}{D/C}}}{D/C} - \frac{\frac{(E[r_w]-A/C)^2}{(D/C)^2}}{\sqrt{\frac{1}{C} + \frac{(E[r_w]-A/C)^2}{D/C}}}}{\frac{1}{C} + \frac{(E[r_w]-A/C)^2}{D/C}} \\
&= \frac{t^*/D}{\left[\frac{1}{C} + \frac{(E[r_w]-A/C)^2}{D/C}\right]^{3/2}} > 0,
\end{aligned} \tag{43}$$

which completes the first part of our proof.

Second, we show (ii). If $t \le \Phi(\sqrt{D/C})$, then $t^* \le \sqrt{D/C}$. Using equation (42), we have $\partial V[t, r_w]/\partial E[r_w] < 0$ for any mean-VaR boundary portfolio $w \in W$, i.e., the slope of the mean-VaR boundary is everywhere negative. Hence, for any portfolio $w \in W$ there is a portfolio $v \in W$ with both higher expected rate of return and lower VaR. This completes the second part of our proof. ■

Proof of Corollary 2. It follows immediately from Corollary 1 and (i) in Proposition 2. ■

Proof of Corollary 3. Since $t^* \to \infty$ as $t \to 1$, note that

$$\lim_{t\to 1} (t^*)^2 / [C (t^*)^2 - D] = 1/C. \tag{44}$$

Thus, it follows from equations (40) and (44) that

$$\begin{aligned} \lim_{t\to 1} E[r_{m_t}] &= \lim_{t^*\to\infty} \frac{A}{C} + \sqrt{\frac{D}{C}\left(\frac{(t^*)^2}{C (t^*)^2 - D} - \frac{1}{C}\right)} \\ &= E[r_{m_\sigma}], \end{aligned} \tag{45}$$

i.e., the minimum VaR portfolio converges to the minimum variance portfolio as $t \to 1$. ■

Proof of Corollary 4. First, we show that the set of mean-VaR efficient portfolios is a proper subset of the set of mean-variance efficient portfolios when $t < 1$. If $t \leq \Phi(\sqrt{D/C})$, then the minimum VaR portfolio does not exist. Using (ii) in Proposition 2, the set of mean-VaR efficient portfolios is empty and, therefore, trivially a proper subset of the set of mean-variance efficient portfolios. So, assume that $t > \Phi(\sqrt{D/C})$. Using (i) in Proposition 2, the set of mean-VaR efficient portfolios at the confidence level t consists of the set of portfolios which belong to the mean-variance boundary and have an expected rate of return greater than or equal to the expected rate of return of the minimum VaR portfolio at that confidence level. Since $E[r_{m_t}] > E[r_{m_\sigma}]$, the set of mean-VaR efficient portfolios is a proper subset of the set of mean-variance efficient portfolios. This completes the first part of our proof.

Second, we verify that, as $t \to 1$, the mean-VaR efficient frontier converges to the mean-variance efficient frontier. Note that the desired claim follows immediately from (i) in Proposition 2 and equation (45). This completes the second part of our proof. ■

Proof of Corollary 5. First, we show that the expected rate of return of the minimum VaR portfolio converges to infinity as $t \downarrow \Phi(\sqrt{D/C})$. Using equation (40), we have

$$\lim_{t\downarrow\Phi(\sqrt{D/C})} E[r_{m_t}] = \lim_{t^*\downarrow\sqrt{D/C}} \frac{A}{C} + \sqrt{\frac{D}{C}\left(\frac{(t^*)^2}{C (t^*)^2 - D} - \frac{1}{C}\right)} = \infty \tag{46}$$

since $(t^*)^2/[C(t^*)^2 - D] \to \infty$ as $t^* \downarrow \sqrt{D/C}$. This completes the first part of our proof.

Second, we verify that the set of mean-VaR efficient portfolios converges to the empty set as $t \downarrow \Phi(\sqrt{D/C})$. Observe that the desired claim follows immediately from (i) in Proposition 2 and equation (46). This completes the second part of our proof. ■

Proof of Proposition 3. First, note that the "if" part follows immediately from Lemma 1. Second, we show the "only if" part. Let $w \in W \backslash \{m_\sigma\}$ be arbitrarily chosen. We need to show that w globally minimizes the VaR for the confidence level $\Phi\left(\sqrt{D/C + \frac{D^2/C^3}{(E[r_w]-A/C)^2}}\right)$. Since portfolio w is mean-variance efficient, we have $w = g + hE[r_w]$. It follows from Proposition 1 that if there exists t^* such that

$$E[r_w] = \frac{A}{C} + \sqrt{\frac{D}{C}\left(\frac{(t^*)^2}{C(t^*)^2 - D} - \frac{1}{C}\right)}, \tag{47}$$

then w is the minimum VaR portfolio for the confidence level $\Phi(t^*)$. Using elementary algebra and equation (47), we obtain

$$t^* = \sqrt{D/C + \frac{D^2/C^3}{(E[r_w] - A/C)^2}}, \tag{48}$$

which completes the "only if" part of our proof. ■

Proof of Proposition 4. First, we show (i). If $t > \Phi(\sqrt{H})$, then $t^* > \sqrt{H}$. Using equation (17), the slope of the mean-VaR boundary is positive if $E[r_w] > r_f$ and negative if $E[r_w] < r_f$. Hence, no portfolio $v \in W_f$ with $E[r_v] < r_f$ is mean-VaR efficient and any mean-variance efficient portfolio $w \in W_f$ with $E[r_w] \geq r_f$ belongs to the mean-VaR efficient frontier. This completes the first part of our proof.

Second, we show (ii). If $t < \Phi(\sqrt{H})$, then $t^* < \sqrt{H}$. Using equation (17), the mean-VaR boundary has a negative slope when $E[r_w] \neq r_f$. If $t = \Phi(\sqrt{H})$, then $t^* = \sqrt{H}$, indicating that the mean-VaR boundary consists of a line with a negative slope when $E[r_w] < r_f$ and a vertical line when $E[r_w] > r_f$. Hence, if $t \leq \Phi(\sqrt{H})$, then for any portfolio $w \in W_f$ there is a portfolio $v \in W_f$ with higher expected rate of return and lower than or the same VaR as portfolio w. This completes the second part of our proof. ■

Proof of Proposition 5. Let p denote the portfolio with weight $w_p > 0$ invested in the market portfolio M and weight $1 - w_p$ in the riskfree security, i.e., $r_p = r_f + w_p(r_M - r_f)$. Using Definition 1, we have $P\{r_p \leq -V[t, r_p]\} = 1 - t$ or, equivalently, using the definition of r_p

$$P\{r_f + w_p(r_M - r_f) \leq -V[t, r_p]\} = 1 - t. \tag{49}$$

Rearranging equation (49) we have

$$P\{r_M \leq r_f - (V[t, r_p] + r_f)/w_p\} = 1 - t. \tag{50}$$

Using Definition 1 and equation (50), we have $V[t, r_M] = -r_f + (V[t, r_p] + r_f)/w_p$ or, equivalently,

$$V[t, r_p] = -r_f + w_p(V[t, r_M] + r_f). \tag{51}$$

It follows from equation (51) that $w_p = (V[t, r_p] + r_f) \,/\, (V[t, r_M] + r_f)$ and, therefore, using the definition of r_p we obtain

$$r_p = r_f + \left(\frac{V[t, r_p] + r_f}{V[t, r_M] + r_f}\right)(r_M - r_f). \tag{52}$$

Rearranging equation (52) and taking expectations we obtain equation (30). ■

Claim. Suppose that $t > \Phi(\sqrt{D/C})$. Then, $\sigma[r_{m_t}] > \sigma$ if and only if $t < \Phi(\sqrt{a^2/C + D/C})$.

Proof. Assume that $t > \Phi(\sqrt{D/C})$. Then, $t^* > \sqrt{D/C}$. Using equations (26) and (33), we have $\sigma[r_{m_t}] > \sigma$ if and only if

$$(t^*)^2 /[C\,(t^*)^2 - D] > 1/C \cdot (1 + D/a^2).$$

Since $C\,(t^*)^2 - D > 0$, the above inequality is equivalent to

$$(t^*)^2 > (1 + D/a^2)[(t^*)^2 - D/C].$$

The above inequality holds if and only if $t^* < \sqrt{a^2/C + D/C}$, i.e., $t < \Phi(\sqrt{a^2/C + D/C})$. ■

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Footnotes

[1]See Jorion (1997), Dowd (1998), Hull (1998), and Linsmeier and Pearson (2000) for an introduction to VaR, and Duffie and Pan (1997) for a review article.

[2]The efficiency criterion suggested by Baumol was meant to preclude "unacceptable" mean-variance efficient portfolios which are dominated by portfolios with both higher expected rate of return and lower VaR.

[3]There is an extensive literature on other downside measures of risk. To cite just a few, see Hogan and Warren (1972, 1974), Fishburn (1977), and Bawa and Lindenberg (1978).

[4]RiskMetrics is a data set provided by J.P. Morgan that is available at its website free of charge to anyone (www.riskmetrics.com/rm/index.cgi).

[5]The term "boundary" is used here to refer to the set of portfolios that minimize risk for various levels of expected rate of return. The term "frontier" is often used with the same meaning [see, for example, Huang and Litzenberger (1988, pp. 63-65)].

[6]Regulation may lead investors to use VaR as the relevant measure of risk. The Securities and Exchange Commission requires registrants to provide quantitative information about market risk with VaR being one of the disclosure alternatives (see Regulation S-K, Item 305, available online at www.sec.gov/smbus/forms/regsk.htm). Furthermore, the Basle Committee on Banking Supervision allows banks to use VaR when determining their capital-adequacy requirements arising from their exposure to market risk.

[7]Artzner, Delbaen, Eber, and Heath (1999) investigate whether VaR is a "coherent" measure of risk. They conclude that VaR is not a coherent measure of risk since it fails to satisfy the "subadditivity property" in that the VaR of a portfolio with two securities may be larger than the sum of the VaR of each of the securities in the portfolio. Interestingly, the subadditivity property holds under normality [see Artzner, Delbaen, Eber, and Heath (1999, Remark 3.6., pp. 217)].

[8]For any n-dimensional multivariate normally distributed random vector Z, any $\nu \in \mathbb{R}^n$, and any symmetric positive definite $(n \times n)$ matrix Ω, $Z \sim N(\nu, \Omega)$ implies that $a^\top Z \sim$

$N(a^\top \nu, a^\top \Omega a)$ for every $a \in \mathbb{R}^n$. Therefore, $r_w \sim N(w^\top \mu, w^\top \Sigma w)$.

[9]It would appear more appropriate for VaR to be defined in this manner, which in most cases would result in a negative value. Custom, however, is to refer to VaR as a positive number, giving rise to Definition 1 and equation (1). The results that follow hold under either definition with appropriate changes of sign.

[10]This assumption rules out the following degenerate cases. First, if $t = 1/2$, then any portfolio belongs to mean-VaR boundary since $t^* = 0$. Second, if $t < 1/2$ and $n > 2$, then for any portfolio $w \in W$, we can find a portfolio $v \in W$ with the expected rate of return of portfolio w and a higher standard deviation than the one of portfolio w. Hence, no portfolio belongs to the mean-VaR boundary if $t < 1/2$ and $n > 2$.

[11]This result also holds if there are short sales constraints, e.g., if $w_j \geq 0$ for any $j \in \{1, ..., n\}$.

[12]This transformation implies that the representation of the mean-VaR boundary portfolios in mean-VaR space does not result in a conic section, as in the case of the representation of mean-variance boundary portfolios in mean-variance (standard deviation) space where a parabola (hyperbola) is obtained.

[13]At that confidence level an agent with mean-VaR preferences is in fact risk neutral.

[14]The extent to which it is possible to reduce VaR through an increase in standard deviation depends upon the confidence level at which VaR is determined and the parameters that determine the form of the mean-variance boundary, that is, μ and Σ.

[15]According to Telser (1955), given a prespecified probability $t \in (1/2, 1)$ and a rate of return r, the optimal portfolio for an investor solves

$$\max_{w \in W} E[r_w]$$
$$s.t. \quad P[r_w < r] \leq 1 - t.$$

Assuming that rates of return are multivariate normally distributed, $P[r_w < r] \leq 1 - t$ is equivalent to $V[t, r_w] \leq -r$. Hence, if $t > \Phi(\sqrt{D/C})$ and we solve the above maximization problem subject to $V[t, r_w] \leq -r$ for every $r \in (-\infty, -V[t, r_{m_t}]]$, then we derive the set of mean-VaR efficient portfolios at the $100t\%$ confidence level. As explained in the Introduction, while this formulation is equivalent to the problem of finding the mean-VaR efficient frontier when rates of return are multivariate normally distributed, Telser (and Baumol) did not develop the mathematical characterization of the mean-VaR efficient frontier. Furthermore, an examination of how an expected utility maximization framework relates to a mean-VaR criteria and an investigation of the economic implications arising from a mean-VaR framework have not previously been provided.

[16]Baumol (1963), while not referring to the VaR model *per se*, pointed this out with his comparable expected gain-confidence level model.

[17]Note that $H \geq (>) D/C$ is equivalent to $Cr_f^2 - 2Ar_f + A^2/C \geq (>) 0$. Since $\partial(Cr_f^2 - 2Ar_f + A^2/C)/\partial r_f = 2Cr_f - 2A$ and $\partial^2(Cr_f^2 - 2Ar_f + A^2/C)/\partial r_f^2 = 2C > 0$, $Cr_f^2 - 2Ar_f + A^2/C$ attains its minimum value (i.e., zero) if and only if $r_f = A/C$. Consequently, $H \geq D/C$, and $H > D/C$ if and only if $r_f \neq A/C$.

[18]See Hull and White (1998) for a procedure to estimate VaR when the normality assumption is not made.

[19]See Blattberg and Gonedes (1974) who presented evidence in favor of the t-distribution as a model for stock returns.

[20]Note that $(1/4.4721)^2 \approx 0.05$.

[21]Examples of utility functions satisfying those assumptions include the negative exponential utility function, which displays constant absolute risk aversion, and the logarithmic utility function, which displays decreasing absolute risk aversion.

[22]Clearly, although in this section we refer to a mean-VaR framework, our analysis easily extends to a 'mean-Chebyshev upper bound on VaR' framework.

[23]An alternative interpretation for the results of this section is a comparison between the risk exposure of an agent with mean-VaR preferences and the risk exposure of an agent with mean-variance preferences.

[24]If an agent is an infinitely risk-averse, in the sense that his or her indifference curves are

vertical in mean-standard deviation space, then his or her optimal portfolio is the minimum variance portfolio. According to a mean-VaR criteria, his or her optimal portfolio (assuming it exists) lies on the mean-VaR efficient frontier above the minimum variance portfolio. Consequently, the agent will increase his or her risk exposure.

[25]However, if the agent chooses a relatively small confidence level, so that the minimum VaR portfolio lies above his or her optimal portfolio according to a mean-variance criteria, then the reasoning used in the case of a highly risk-averse agent applies. By using VaR the agent will increase his or her risk exposure.

[26]This analysis does not depend on the existence of a riskfree security in the economy.

[27]Since the VaR of any portfolio is the maximum loss a portfolio is expected (at some confidence level) to suffer over a prespecified time horizon, a riskfree portfolio with rate of return $r_f > 0$ is expected (at any confidence level) to gain r_f, or to "suffer" a loss of $-r_f$. Moreover, if we define the VaR the riskfree security to be $-r_f$, then the capital market line in mean-VaR space is continuous.

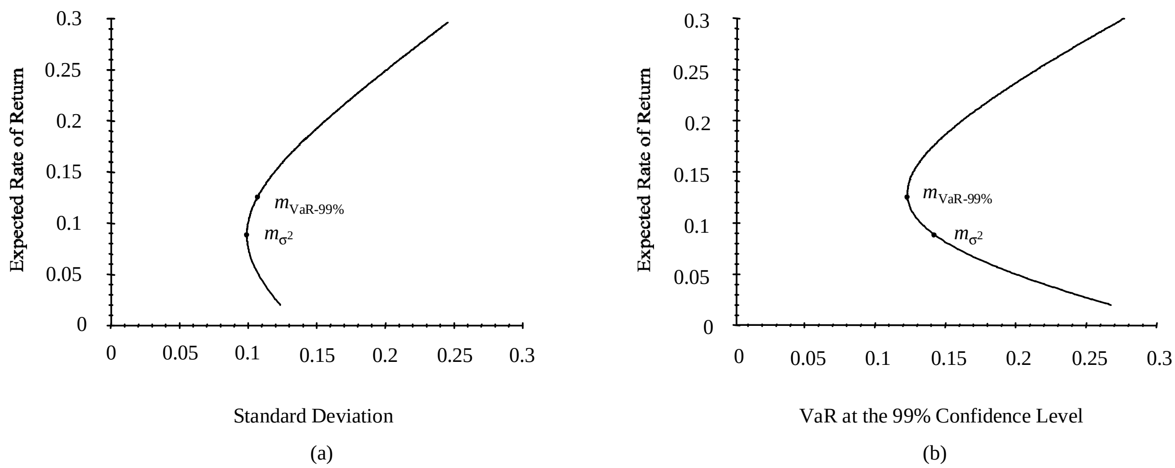

Figure 1: (a) The mean-standard deviation boundary assuming that there are only two securities, A and B, bivariate normally distributed, with $E[r_A] = 0.1$, $\sigma[r_A] = 0.1$, $E[r_B] = 0.25$, $\sigma[r_B] = 0.2$ and $\rho[r_A, r_B] = 0.6$; (b) the mean-VaR boundary at the 99% confidence level assuming that there are only two securities as in (a). In this case note that while the minimum VaR portfolio at the 99% confidence level (plotted in the figures as the point $m_{VaR-99\%}$) is the mean-variance efficient portfolio with expected rate of return equal to 12.87%, the minimum variance portfolio (plotted in the figures as the point m_{σ^2}) has an expected rate of return equal to 8.85%. Hence, all the mean-variance efficient portfolios with expected rates of return in the interval $[8.85\%, 12.87\%)$ are inefficient if VaR at the 99% confidence level is used as a measure of risk.

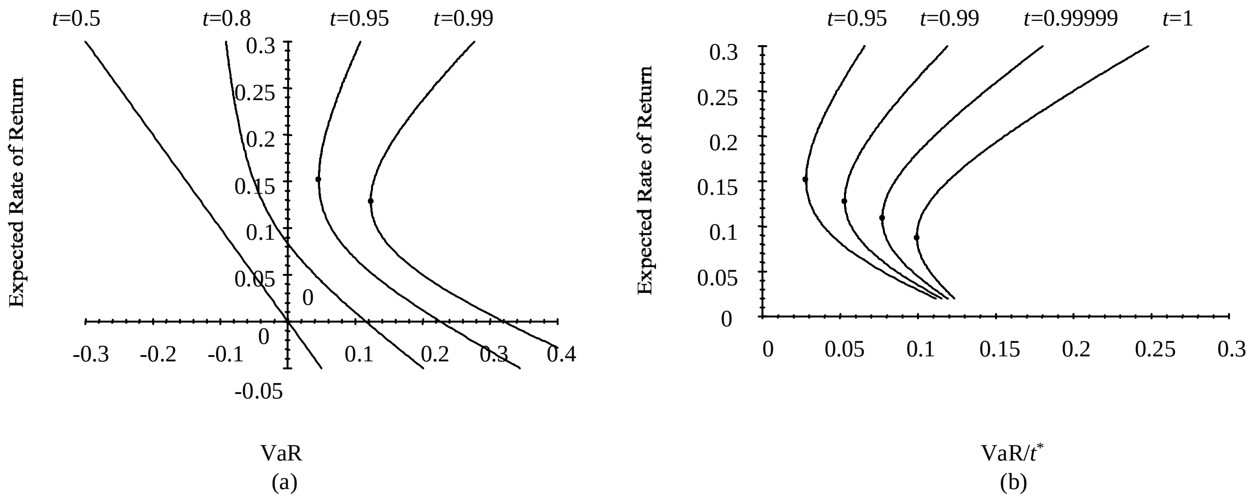

Figure 2: (a) As $t \rightarrow 0.5$, the mean-VaR boundary converges to a line with slope of minus one that intersects the origin. Observe that the minimum VaR portfolio at a sufficiently small confidence level does not exist (note, for example, that the mean-VaR boundary does not bend back if $t = 0.8$). The dotted points represent the minimum VaR portfolio at the 95 and 99% confidence levels; (b) as $t \rightarrow 1$, the mean-VaR frontier in mean-VaR/t^* space converges to the traditional mean-standard deviation hyperbola. Figure 2 assumes the data used in Figure 1. The dotted points represent the minimum VaR portfolio at various confidence levels.

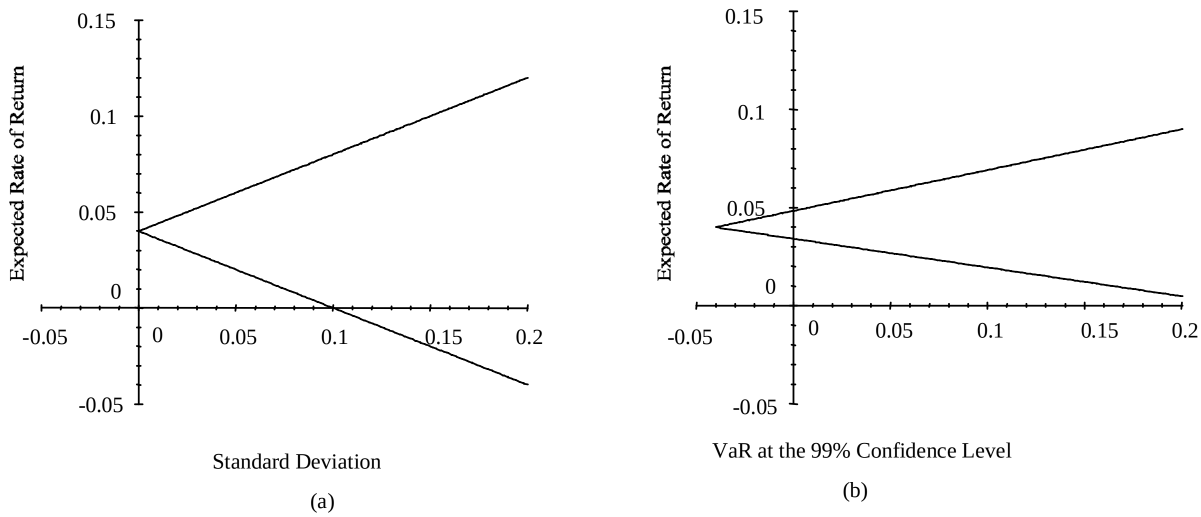

Figure 3: (a) The mean-standard deviation boundary with a riskfree security, assuming that the portfolio of risky securities w_T is mean-variance efficient, $r_{w_T} \sim N(0.12, 0.04)$, and $r_f = 0.04$. (b) the mean-VaR boundary at the 99% confidence level with a riskfree security and a portfolio of risky securities as in (a). Observe that the minimum VaR portfolio is the riskfree security, which has a negative VaR.

$t\backslash a$	3	5	10	25	∞
85%	-10.05%	1.42%	8.90%	11.91%	12.58%
90%	-18.13	-6.66	0.83	3.83	4.50
95%	-20.53	-9.05	-1.57	1.43	2.11
99%	-21.73	-10.26	-2.78	0.23	0.90

Table 1: The change of an agent's risk exposure as measured by standard deviation, assuming that he or she, previously having mean-variance preferences represented by the utility function defined by equation (24), decides to minimize VaR. Such a change is reported for several values of the parameters a and t. It also represents a lower bound on the change in an agent's risk exposure that decides to use VaR as the relevant measure of risk. Table 1 assumes the data used in Figure 1.

$b\backslash a$	3	5	10	25
3	98.08%	99.27%	99.92%	99.99%
5	96.76	98.43	99.69	99.99
10	94.76	96.86	98.99	99.99
25	92.30	94.57	97.48	99.92

Table 2: The confidence level at which the optimal portfolios arising from the utility functions defined by equations (24) and (27) coincide for several values of the parameters a and b. Table 2 assumes the data used in Figure 1.