USING HULL-WHITE INTEREST-RATE TREES

John Hull and Alan White

Rotman School of Management, University of Toronto
105 St. George Street, Toronto, Ontario, Canada M5S 3E6

Telephone Numbers

John Hull: (416) 978-8615
Alan White: (416) 978-3689

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ABSTRACT

The Hull-White tree-building procedure was first outlined in the Fall 1994 issue of *Journal of Derivatives*. It is becoming widely used by practitioners. This procedure is appropriate for models where there is some function $x = f(r)$ of the short rate r that follows a mean-reverting arithmetic process. It can be used to implement the Ho-Lee model, the Hull-White model, and the Black-Karasinski model. Also, it is a tool that can be used for developing a wide range of new models.

In this article we provide more details on the way in which Hull-White trees can be used. We discuss the analytic results available when $x = r$ and make the point that it is important to distinguish between the Δt-period rate on the tree and the instantaneous short rate that is used in some of these analytic results. We provide an example of the implementation of the model using market data. We show how the model can be implemented so that it provides an exact fit to the initial volatility environment while at the same time explaining why we do not recommend this approach. We also discuss how to deal with such issues as variable time steps, cash flows that occur between nodes, barrier options, and path dependence.

USING HULL-WHITE INTEREST-RATE TREES

In a recent *Journal of Derivatives* article, Hull and White [1994a], we described a procedure for constructing trinomial trees for one-factor yield curve models of the form:

$$dx = q(t) - ax\,dt + s\ dz \qquad (1)$$

where r is the short rate, $x = f(r)$ is some function of r, a and   are constants, and $\theta(t)$ is a function of time chosen so that the model provides an exact fit to the initial term structure of interest rates. The model can be written

$$dx = a\left[\frac{q(t)}{a} - x\right]dt + s\ dz$$

This shows that, at any given time, x reverts toward $\theta(t)/a$ at rate a. Its variance rate per unit time is σ^2.

When $f(r) = r$, the model reduces to the Hull-White [1990] model.

$$dr = a[q(t) - r]dt + s\ dz \qquad (1A)$$

The attraction of the Hull-White model is its analytic tractability. As shown in Hull and White [1990, 1994a] bonds and European options at some future time t can be valued analytically in terms of the initial term structure and the value of r at time t. When $f(r)$ = log(r) and a and ### are allowed to be functions of time the model becomes Black and Karasinski [1991]. When $f(r)$ = log(r) and $a(t) = -\sigma'(t)/\sigma(t)$, and $\sigma'(t) = \partial\sigma/\partial t$, the model becomes the Black, Derman, and Toy [1990] model. In Section III below we describe how to extend the basic tree-building procedure to accommodate time-varying mean reversion and volatility.

The construction of the Hull-White tree involves two stages. The first stage involves defining a new variable x^* obtained from x by setting both $\theta(t)$ and the initial value of x equal to zero. The process for x^* is:

$$dx^* = -ax^*\,dt + s\ dz \qquad (2)$$

We construct a tree for x^* that has the form shown in Figure 1. The central node at each time step has $x^* = 0$. The vertical distance between the nodes on the tree is set equal to $\Delta x^* = \sqrt{3V}$ where V is the variance of the change in x in time Δt, the length of each time step. The probabilities at each node are chosen to match the mean and standard deviation[1] of the change in x^* for the process in equation (2). Defining the expected change in x^* as $M\,x^*$, at node $j\,\Delta x^*$ the up-, middle-, and down-branching probabilities are

[1]The expected value and variance of the change in x^* over some time Δt are

$$E[dx^*] = Mx^* = \left(e^{-a\Delta t} - 1\right)x^*; \qquad \mathrm{Var}[dx^*] = V = s^2\left(1 - e^{-2a\Delta t}\right)/2a$$

$$p_u = \frac{1}{6} + \frac{j^2M^2 + jM}{2}$$
$$p_m = \frac{2}{3} - j^2M^2 \qquad \text{(3A)}$$
$$p_d = \frac{1}{6} + \frac{j^2M^2 - jM}{2}$$

As indicated in Figure 1, we cope with mean reversion by allowing the branching to be nonstandard at the edge of the tree. At the top edge of the tree where the branching is non-standard the modified probabilities become

$$p_u = \frac{7}{6} + \frac{j^2M^2 + 3jM}{2}$$
$$p_m = -\frac{1}{3} - j^2M^2 - 2jM \qquad \text{(3B)}$$
$$p_d = \frac{1}{6} + \frac{j^2M^2 + jM}{2}$$

and at the bottom edge of the tree where the branching is non-standard the modified probabilities become

$$p_u = \frac{1}{6} + \frac{j^2M^2 - jM}{2}$$
$$p_m = -\frac{1}{3} - j^2M^2 + 2jM \qquad \text{(3C)}$$
$$p_d = \frac{7}{6} + \frac{j^2M^2 - 3jM}{2}$$

The second stage in the construction of the tree involves forward induction. We work forward from time zero to the end of the tree adjusting the location of the nodes at each time step in such a way that the initial term structure is matched. This produces a tree of the form shown in Figure 2. The size of the displacement is the same for all nodes at a particular time t, but is not usually the same for nodes at two different times. The effect of this second stage is to convert a tree for x^* into a tree for x.

The full details of the tree building procedure are given in Hull and White [1994a]. In a later article, Hull and White [1994b], we describe extensions where two interest rates are modeled simultaneously and where the tree building technology is used to construct two-factor models of a single term structure.

The purpose of this article is to provide more details on the basic Hull-White tree building procedure. We discuss how to use analytic results when $f(r) = r$. We provide sample results based on a real yield curve that the reader can use to test his or her own implementation of the model. We show how the tree-building procedure can be used for models such as Black and Karasinski [1991] where a and σ are functions of time, but point out some pitfalls of these models. We also discuss issues such as how the length of the time step can be changed, how cash flows that occur between time steps can be handled, and so on.

I. Analytic Results

Bond Prices:

When $f(r) = r$, the model in equation (1) is analytically very tractable. For example, as shown in Hull and White [1990, 1994a]

$$P(t,T) = A(t,T)e^{-B(t,T)r} \qquad (4)$$

where $P(t, T)$ is the price at some time t of a zero coupon bond maturing at time T, r is the short-term rate of interest at time t, and A and B are functions only of t and T. The function A is determined from the initial values of the discount bonds, $P(0, T)$.

$$\begin{aligned} A(t,T) &= \frac{P(0,T)}{P(0,t)} \exp\left[B(t,T)F(0,t) - s^{2}B(t,T)^{2}\left(1-e^{-2at}\right)/(4a)\right] \\ B(t,T) &= \left(1-e^{-a(T-t)}\right)/a \end{aligned} \qquad (5)$$

$F(0, t)$ is the instantaneous forward rate that applies to time t as observed at time zero. It can be computed from the initial price of a discount bond as $F(0,t) = -\P \log[P(0,t)]/ \P t$

The variable r in equation (4) is the instantaneous short rate while the interest rates on the Hull-White tree are Δt-period rates. The two should not be assumed to be interchangeable. Let R be the Δt period rate at time t, and r be the instantaneous rate at time t. Using equation (4):

$$e^{-R\Delta t} = A(t,t+\Delta t)e^{-B(t,t+\Delta t)r}$$

so that

$$r = \frac{R\Delta t + \log A(t,t+\Delta t)}{B(t,t+\Delta t)} \qquad (6)$$

To calculate points on the term structure given the Δt period rate R at a node of the Hull-White tree it is first necessary to use equation (6) to get the instantaneous short rate, r. Equation (4) can then be used to determine rates for longer maturities. When this procedure is followed, it can be shown that the prices of discount bonds that are computed are independent of the forward rate, $F(0, t)$.[2]

Expected Future Rates:

Inspection of equations (1) and (2) shows that $x(t)$ and $x^*(t)$ differ only by some function of time. Define this difference as

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[2]Since the forward rate is computed from the first derivative of the yield curve it is very sensitive to the exact shape of the yield curve. Slight variations in the yield curve create large changes in the computed forward rate. If the computed bond price did depend on the forward rate the results would be very sensitive to exactly how one computed the yield curve.

$$\alpha(t) = x(t) - x^*(t) \tag{7}$$

This is the difference between the location of comparable nodes in the *x* and *x** trees at time *t*. In particular it is the difference between the central or expected values of *x* and *x** at time *t*, and since the expected value of *x** is zero $\alpha(t)$ can be interpreted as the expected value of $x(t)$. As has been pointed out by Kijima and Nagayama [1994] and Pelsser [1994], $\alpha(t)$ can be calculated analytically for the model where $f(r) = r$. Differentiating equation (7) it follows from equations (1) and (2) that

$$\frac{\partial \alpha(t)}{\partial t} = \theta(t) - a\alpha(t)$$

or

$$\alpha(t) = \exp\left\{-at\left[r(0) + \int_0^t \theta(\theta)e^{a\theta}d\theta\right]\right\}$$

Substituting the analytic expression for $\theta(t)$ given in Hull and White [1990, 1994a] this reduces to

$$\alpha(t) = F(0,T) + \frac{\sigma^2}{2a^2\left(1-e^{-at}\right)^2} \tag{8}$$

The use of the analytic expression for α to determine the location of the central nodes in the tree avoids the need to obtain them from forward induction.[3] However, the resulting tree does not provide an exact fit to the initial term structure. This is because the tree is a discrete representation of the underlying continuous stochastic process. The advantage of the forward induction procedure is that the initial term structure is always matched exactly by the tree itself.

II. An Example

As an example of the implementation of the model we use the data in Table 1. This data, which is for the DM yield curve on July 8, 1994, was kindly provided to us by Antoon Pelsser of ABN Amro Bank.

[3]Forward induction is always necessary when $f(r) \neq r$ since there are no analytic results in that case.

Table 1 The DM zero coupon yield curve, July 8, 1994.					
Maturity	Days	Rate	Maturity	Days	Rate
3 days	3	5.01772	4 years	1461	6.73464
1 month	31	4.98284	5 years	1826	6.94816
2 month	62	4.97234	6 years	2194	7.08807
3 month	94	4.96157	7 years	2558	7.27527
6 month	185	4.99058	8 years	2922	7.30852
1 year	367	5.09389	9 years	3287	7.39790
2 years	731	5.79733	10 years	3653	7.49015
3 years	1096	6.30595			

Data points for maturities between those indicated are generated using linear interpolation.

The zero curve was used to price a 3-year[4] (= 3 × 365 day) put option on a zero coupon bond that will pay $100 in 9 years (= 9 × 365 days). Interest rates were assumed to follow the Hull-White (equation (1A)) model. The strike price was $63, and the parameters a and σ were chosen to be $a = 0.1$, and $\sigma = 0.01$. These two parameters determine the volatility of the discount bond for option pricing purposes. The values that were chosen were roughly representative of the values that are observed in the market. The tree was constructed out to the end of the life of the option. The zero-coupon bond prices at the final nodes were calculated analytically as described in the previous section.

To illustrate the process consider the construction of a 3-step tree. First, we must determine the time and rate step sizes, and where non-standard branching (if any) takes place. The size of the time step is Δt = 3 × 365 days / 3 / 365 days/year = 1.0 years. As shown in Hull and White [1994a] the expected change in r^* and the variance of the change in r^* in time Δt are given by

$$E[dr^*] = Mr^* = \left(e^{-a\Delta t} - 1\right)r^*; \qquad \mathrm{Var}[dr^*] = V = s^2\left(1 - e^{-2a\Delta t}\right)/2a$$

For the given parameter values M = –0.095162582 and $\sqrt{V}$ = 0.009520222. Since the step size $\Delta r = \sqrt{3V}$, Δr = 0.016489508. Finally, as shown in Hull and White [1994a] non-standard branching takes place at nodes $\pm j^*$ where j^* is the smallest integer greater than $-0.184/M$. In this case j^* is 2. The data defining the initial tree is shown in Table 2.

[4]The fundamental unit of time in this example is one day. For convenience we define 1 year as 365 days, which is approximately the length of a real year, and quote rates and volatilities per year. The data in Table 1 is quoted on this basis. Thus the 10-year rate of 7.49015% is actually a rate of 0.0205210% per day. This rate applies for 3,653 days or about 10.0082 years. This convention may seem cumbersome but is necessary to avoid the ambiguity associated with the variable length of a calendar year.

Table 2 Data defining a 3-step tree in r^*.					
j	rate = $j\,\Delta r$	p_u	p_m	p_d	Eq'n
2	0.032979	0.899291	0.011093	0.089616	3B
1	0.016490	0.123613	0.657611	0.218776	3A
0	0.0	0.166667	0.666667	0.166667	3A
–1	–0.016490	0.218776	0.657611	0.123613	3A
–2	–0.032979	0.089616	0.011093	0.899291	3C

The rates at each node in the tree at each time step are now shifted up by some amount, α, chosen so that the revised tree correctly prices discount bonds. Since there are nodes at the 1-, 2-, and 3-year points we need the discount bond prices corresponding to these dates as well as the 4-year price, one time-step beyond the option maturity. When the option price is calculated, the 9-year bond price will be required as well. This data, interpolated from the data in Table 1 is shown in Table 3. Table 3 also shows the value of α required to fit the bond prices at each time step. An efficient procedure for implying the value of α is given in Hull and White [1994a]. For reference purposes the instantaneous forward rate and the instantaneous values of α (based on equation (8)) are also shown.

Table 3 The amount, α, by which the interest rates at each time step must be raised in order to replicate the bond prices computed from the zero coupon discount rates. The instantaneous forward rate and the instantaneous value of α are also shown.						
Time Step i	$t = i\,\Delta t$ Years	Zero Rate (%)	Discount Bond Price	α (%)	Forward Rate (%)	$\alpha(t)$ - Eq'n (8) (%)
0	0.0	5.017720	1.000000	5.09275	5.017720	5.017720
1	1.0	5.092755	0.950348	6.50257	5.299942	5.304470
2	2.0	5.795397	0.890557	7.33932	7.206143	7.222572
3	3.0	6.304557	0.827673	8.05381	7.830417	7.864004
4	4.0	6.733466	0.763885			
	9.0	7.397410	0.513879			

Combining the α's from Table 3 with the rates and probabilities in Table 2 produces the complete tree. The tree is shown in Table 4, which shows the Δt period rates at each node of the tree and the probabilities of branching from one node to the next. Table 5 shows how this tree can be used to compute the price of a 2-year discount bond. At each step the bond price is computed as the discounted value of the expected value at the next time-step. Calculations of the type shown in Table 5 are used to determine what value of α is needed at each time step in order to replicate the discount bond prices. Table 6 shows the calculations required to compute the discount bond prices at the option maturity, 3 years. Finally, Table 7 shows the discounting of the option value back through the tree.

Table 4							
The 4 time steps in the interest rate tree. The probability of transiting from node (i, j) to nodes (i+1, j+1), (i+1, j), and (i+1, j–1) are normally $p_u(j)$, $p_m(j)$, and $p_d(j)$ respectively. When $j = \pm 2$ the alternative branching schemes are used.							
	Transition Probabilities			Node Rates, R, (%)			
j	p_u	p_m	p_d	$i = 0$	$i = 1$	$i = 2$	$i = 3$
2	0.8993	0.0111	0.0896			10.6372	11.3517
1	0.1236	0.6576	0.2188		8.1515	8.9883	9.7028
0	0.1667	0.6667	0.1667	5.0928	6.5026	7.3393	8.0538
–1	0.2188	0.6576	0.1236		4.8536	5.6904	6.4049
–2	0.0896	0.0111	0.8993			4.0414	4.7559

Table 5						
Computing the price of a bond that pays \$1 at time 2 Δt (2 years). Each value is calculated as $v_{i,j} = (p_u v_{i+1,j+1} + p_m v_{i+1,j} + p_d v_{i+1,j-1}) \exp(-R_{i,j} \Delta t)$.						
	Transition Probabilities			Bond Price		
j	p_u	p_m	p_d	$i = 0$	$i = 1$	$i = 2$
2	0.8993	0.0111	0.0896			1.0
1	0.1236	0.6576	0.2188		0.9217	1.0
0	0.1667	0.6667	0.1667	0.8906	0.9370	1.0
–1	0.2188	0.6576	0.1236		0.9526	1.0
–2	0.0896	0.0111	0.8993			1.0

Table 6

Computing the option payoff at each terminal node (i = 3) on the tree. The Δt-period rate, R, is the rate that applies from 3 to 4 years. The instantaneous rate, r, is computed using equation (6). The forward rate at time 3 years was computed to be 0.078304. On the basis of this equation (5) gives

$$A(3, 4) = 0.994229,\ A(3, 9) = 0.881944,$$
$$B(3, 4) = 0.951626,\ B(3, 9) = 4.511884.$$

The bond price, P(3, 9), is computed with equation (4) and the option payoff is 100 Max[0.63 – P(3, 9), 0].

j	Δt-period rate, R	Instantaneous rate, r	Bond Price	Option Payoff
2	0.113517	0.113206	0.529196	10.080445
1	0.097028	0.095878	0.572229	5.777133
0	0.080538	0.078550	0.618761	1.123884
–1	0.064049	0.061222	0.669078	0.0
–2	0.047559	0.043895	0.723486	0.0

Table 7

Discounting the option price back through the tree. At the third step the option value is as given in Table 6. The computed value, at earlier steps is

$$v_{i,j} = (p_u v_{i+1,j+1} + p_m v_{i+1,j} + p_d v_{i+1,j-1}) \exp(-R_{i,j} \Delta t)$$

where $R_{i,j}$, the rate at node j and time step i, is $\alpha_i + j\,\Delta r$. Note that when $j = \pm 2$, non-standard branching applies. When $j = 2$ the computed value is

$$v_{i,j} = (p_u v_{i+1,j} + p_m v_{i+1,j-1} + p_d v_{i+1,j-2}) \exp(-R_{i,j} \Delta t)$$

and when $j = -2$ the computed value is

$$v_{i,j} = (p_u v_{i+1,j+2} + p_m v_{i+1,j+1} + p_d v_{i+1,j}) \exp(-R_{i,j} \Delta t).$$

				Time step, i			
j	p_u	p_m	p_d	0	1	2	3
2	0.8993	0.0111	0.0896			8.2987	10.0804
1	0.1236	0.6576	0.2188		4.1977	4.8362	5.7771
0	0.1667	0.6667	0.1667	1.8734	1.7854	1.5910	1.1239
–1	0.2188	0.6576	0.1236		0.4885	0.2323	0.0000
–2	0.0896	0.0111	0.8993			0.0967	0.0000
			α (%)	5.0928	6.5026	7.3393	8.0538

The results of pricing this put option for trees of different size are shown in Table 8. This example provides a good test of one's implementation of the model because the gradient of the zero curve changes sharply immediately after the expiration of the option. Small errors in the construction and use of the tree are liable to have a big effect on the option values obtained. For example, when 100 time steps are used, the value of the option is reduced by about $0.25 if the Δt-period rate is assumed to be the instantaneous rate.

Table 8

Value of a 3-year Put Option on a 9-year, $100, Zero-coupon Bond. The Strike price is $63. The volatility parameters are a = 0.1 and σ = 0.01.

Steps	Tree Based Value	Analytic Value
10	1.8491	1.8093
30	1.8179	1.8093
50	1.8060	1.8093
100	1.8128	1.8093
200	1.8089	1.8093
500	1.8090	1.8093

III. Making Volatility Parameters Time Dependent

When a and σ are functions of time the model in equation (1) becomes

$$dx = [q(t) - a(t)x]dt + s(t)dz \qquad (9)$$

The three functions of time in this diffusion equation each play a separate role. The function $\theta(t)$ is chosen so that the prices of all discount bonds are matched at the initial time. The other two functions provide two extra degrees of freedom that allow us to match the initial volatility of all zero coupon rates and the volatility of the short rate at all future times. The tree can then be tuned to price not only the zero-coupon bonds, but also a set of interest-rate derivatives at their current market prices. The initial volatility of all rates depends on $\sigma(0)$ and $a(t)$. The volatility of the short rate at future times is determined by $\sigma(t)$. Unless $\sigma(t)$ and $a(t)$ are constants the volatility term structure is non-stationary.

Our tree building procedure can be extended to accommodate the model in equation (9). Analogously to the constant a and σ case we first build a tree for x^* where

$$dx^* = -a(t)x^*dt + s(t)dz$$

We first choose the times at which nodes will be placed, $t_0, t_1, t_2, \ldots, t_n$, where $t_0 = 0$ and $t_i = i\,\Delta t$ for $i = 0, \ldots, n$. The vertical (x^* dimension) spacing between adjacent nodes at time t_{i+1} is then set equal to $\sqrt{3V_i}$ where

$$V_i = s(t_i)^2\left(1 - e^{-2a(t_i)\Delta t}\right)/2a(t_i)$$

Suppose that the value of x^* at the jth node at time t_i is $x^*_{i,j}$. The mean and standard deviation of x^* at time t_{i+1} conditional on $x^* = x^*_{i,j}$ at time t_i are approximately $x^*_{i,j} + M_i\, x^*_{i,j}$ and $\sqrt{V_i}$, where

$$M_i = \left(e^{-a(t_i)\Delta t} - 1\right)$$

We match these by branching from $x^*_{i,j}$ to one of $x^*_{i+1,k-1}$, $x^*_{i+1,k}$, and $x^*_{i+1,k+1}$ where k is chosen so that $x^*_{i+1,k}$ is as close as possible to $x^*_{i,j} + M_i x^*_{i,j} \Delta t$. We then calculate the displacements, $\alpha(t)$, necessary for the tree to match the initial term structure.

The $a(t)$ and $\sigma(t)$ can be set in advance of the numerical procedure. Alternatively, it is not difficult to devise a numerical procedure that chooses $a(t)$ and $\sigma(t)$ so that the initial prices of caps or swap options (or both) are matched. When used for $x = \log(r)$ this type of tree building procedure has the advantage over Black and Karasinski [1991] that the length of the time step is under the control of the user.[5]

It seems appealing to take advantage of all the degrees of freedom in a model to exactly fit initial market data. However, the resulting non-stationarity in the volatility term structure may have many untoward and unexpected effects. To illustrate this we use the $x = r$ model:

$$dr = [q(t) - a(t)r]dt + s(t)dz$$

and show the effect of matching cap prices.

Caps are usually priced using Black's model, under which the price at time zero of a caplet expiring at T on a rate that applies from T to $T + \tau$ is

$$C = tPe^{-R(T+t)}[F(T, T+t)N(d_1) - XN(d_2)]$$

where P is the notional principal, R is the zero coupon rate with a maturity $T + \tau$, $F(T, T + \tau)$ is the forward rate for the period T to $T + \tau$ and X is the cap rate.

$$d_1 = \frac{\log(F(T, T+t)/X)}{v(T)\sqrt{T}} + \frac{v(T)\sqrt{T}}{2}$$

$$d_2 = d_1 - v(T)\sqrt{T}$$

where $v(T)$ is the volatility for the caplet expiring at T.

The data set that we will use for calibration consists of the market prices of at-the-money caps that are reset monthly (τ = 1 month). The particular $v(T)$ function we assume for illustration purposes is shown in Figure 3.[6] This has a similar shape to the $v(T)$ function commonly observed in the market. We assume the term structure is flat at 7% continuously compounded.

In order to match the Black volatilities we first used them in conjunction with Black's model to calculate caplet prices. We then matched the caplet prices in two ways:

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[5] We will explain in the next section how the length of the time step can be changed by the user in the Hull-White tree building procedure.

[6] This volatility curve is $v(T) = [1 + bT + c(1 - e^{-dT})]v(0)$ for $T \le 5$, $b = -0.1$, $c = 0.5$, $d = 0.8$, and $v(0) = 0.2$. The curve was extended beyond $T = 5$ by assuming that the gradient of $v(T)T$ when $T > 5$ equals its gradient when $T = 5$.

1. We fixed the short rate standard deviation, σ, and allowed the reversion rate, a, to be a function of time; and
2. We fixed a and allowed σ to be a function of time.

Figure 4 shows the value of $a(t)$ required to fit the market data when σ is fixed[7] at 1.4% and the value of $\sigma(t)$ required to fit the market data when a is fixed at 5%. It can be seen that the implied $a(t)$ and $\sigma(t)$ exhibit severe non-stationarity. Although by construction this non-stationarity leads to caplets being priced correctly, it is liable to lead to unacceptable results when used to price other instruments.

Any instrument whose price depends on the future volatility structure, rather than today's volatility structure, is liable to be mispriced by a model with time dependent volatility parameters. One example of such a security is an American-style call option where the decision to exercise at some future date depends on the volatility structure at that date. Another example is a caption, an option to buy a cap, where the decision to exercise the option at it expiry depends on the value of the cap at that time.

This example illustrates the types of problems that can arise when a model is implemented in such a way that the volatility structure is not stationary. It is a problem that afflicts all Markov interest-rate models including the Black, Derman and Toy, and Black and Karasinski models. By fitting a one-factor Markov interest-rate model to today's option prices, we make it exactly reflect the initial volatility structure. However, we are also unwittingly making a statement about how the volatility term structure will evolve in the future. Using all the degrees of freedom in the model to fit the volatility exactly constitutes an over parameterization of the model. It is our opinion that there should be no more than one time varying parameter used in Markov models of the term structure evolution and that this should be used to fit the initial term structure.

IV. Other Issues

There are a number of other practical issues to consider when implementing Hull-White trees for valuing interest rate derivatives. In this section we review a number of these and indicate how they can be handled.

In our description of the tree-building procedure in Hull and White [1994a] it was assumed that the length of the time step is constant. In practice, it is sometimes desirable to change the length of the time step.[8] Changing the length of the time step is straightforward. When drawing the tree for x^*, we first choose the times at which nodes will be placed, $t_0, t_1, t_2, \ldots, t_n$, where $t_0 = 0$. Defining $\Delta t_i = t_{i+1} - t_i$ for $i = 0, \ldots, n - 1$,

[7]The choice of the fixed value for σ in this example, and the following choice of the fixed value for a are arbitrary. However, the implied values of $a(t)$ and $\sigma(t)$ are representative of the type of non-stationarity that results from the given volatility structure. The best fixed value of σ (or a) to use might be the one that minimizes the variance of the implied $a(t)$ (or $\sigma(t)$).

[8]Consider for example the situation where the lognormal model is used to value a European 6-month option on a 5-year bond. It might be appropriate to use a longer Δt between 6 months and 5 years than during the first six months. This is because the part of the tree between 6 months and 5 years is used only to value the underlying bond.

the vertical (x^* dimension) spacing between adjacent nodes at time t_{i+1} is then set equal to $\sqrt{3V_i}$ where

$$V_i = s^2\left(1 - e^{-2a\Delta t_i}\right)/2a$$

From this point, the construction is similar to the procedure followed when the volatility parameters are a function of time. Suppose that the value of x^* at the jth node at time t_i is $x^*_{i,j}$. The mean and standard deviation of x^* at time t_{i+1} conditional on $x^* = x^*_{i,j}$ at time t_i are approximately $x^*_{i,j} + M_i x^*_{i,j}$ and $\sqrt{V_i}$, where

$$M_i = \left(e^{-a\Delta t_i} - 1\right)$$

We match these by branching from $x^*_{i,j}$ to one of $x^*_{i+1,k-1}$, $x^*_{i+1,k}$, and $x^*_{i+1,k+1}$ where k is chosen so that $x^*_{i+1,k}$ is as close as possible to $x^*_{i,j} + M_i x^*_{i,j} \Delta t_i$. Note that whenever the size of the time step changes, $\Delta t_i \neq \Delta t_{i+1}$, the vertical ($x^*$ dimension) spacing between nodes increases by $\sqrt{\Delta t_{i+1}/\Delta t_i}$. This means that the branching is nonstandard at points when the length of the time step changes. Figure 5 illustrates the tree that is constructed when the time step increase by a factor of three after two time steps.

The tree for x is constructed from the tree for x^* to match the initial zero coupon yield curve as described in Hull and White [1994a]. Note that, when the length of the time step changes from Δt_i to Δt_{i+1}, the interest rates considered at the nodes automatically change from the Δt_i period rates to the Δt_{i+1} rates.

Another issue in the construction of the tree concerns cash flows that occur between nodal dates. Suppose a cash flow occurs at time τ when the immediately preceding nodal date is t_i and the immediately following nodal date is t_{i+1}. One approach is to discount the cash flow from time τ to the nodes at time t_i using estimates of the $\tau - t_i$ rates prevailing at the nodes at time t_i.[9] Another approach is to assume that a proportion $(\tau - t_i)/(t_{i+1} - t_i)$ of the cash flow occurs at time t_{i+1} while the remainder occurs at time t_i.[10] A final approach is to avoid the problem altogether by changing the length of the time step so that every payment date is also a nodal date.

Barrier options present a further problem in the use of the tree because convergence tends to be slow when nodes do not lie exactly on barriers. In the case of an interest rate option the barrier is typically expressed in terms of a bond price or a particular rate. When $x = r$, analytic results can be used to express the barrier as a function of the Δt-period rate. Nonstandard branching can then be used to ensure that nodes always lie on the barrier. Ritchken (JOD Winter 1995) describes such an approach, and shows that a substantial improvement in performance is possible with it. An alternative approach that has more general applicability is to extend the idea suggested by Derman et al [1995] to interest rate

[9] In the case of the Hull-White, $x = r$ model these rates can be calculated analytically.

[10] This approach has the effect of apportioning the cash flow to nodal dates while ensuring that the expected time when the cash flow occurs is correct.

trees. This approach involves using a procedure to correct values of the derivative calculated at nodes close to a barrier.

A final problem in the use of interest rate trees is path dependence. This can sometimes be handled in the way described by Hull and White [1993]. The requirements for the Hull-White method to work are:

1. The value of the derivative at each node must depend on just one function of the path for the short rate r (e.g., the maximum, minimum, or average value);
2. In order to update the path function as we move forward through the tree we need to know only the previous value of the function and the new value of r.

Hull and White show how their approach can be used for index amortizing swaps and mortgage-backed securities. The relevant path function in each case is the remaining principal.

V. Summary

The Hull-White tree building procedure is a flexible approach to constructing trees for a wide range of different one-factor models of the term structure. The tree is constructed in such a way that it is exactly consistent with the initial term structure. In this article we have shown how the basic procedure presented in our earlier paper can be extended. Some of these extensions involve the use of analytic results and some involve changing the geometry of the tree to reflect special features of the derivative under consideration. We have devoted some time in this article to a discussion of what happens when the volatility parameters are made time-dependent. It not difficult to extend the Hull-White tree to incorporate time-dependent parameters so that the prices of caps or swap options (or both) are matched. However, this is liable to result in unacceptable assumptions about the evolution of volatilities.

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Figure 1

The initial tree ($\theta(t) = 0$ and $x(0) = 0$)

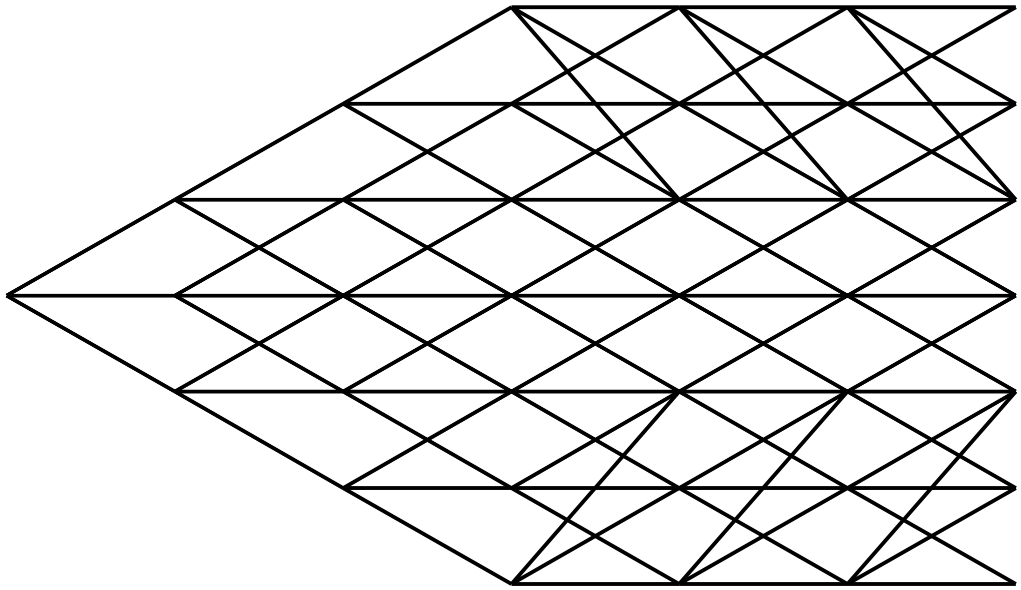

Figure 2

The final tree for x

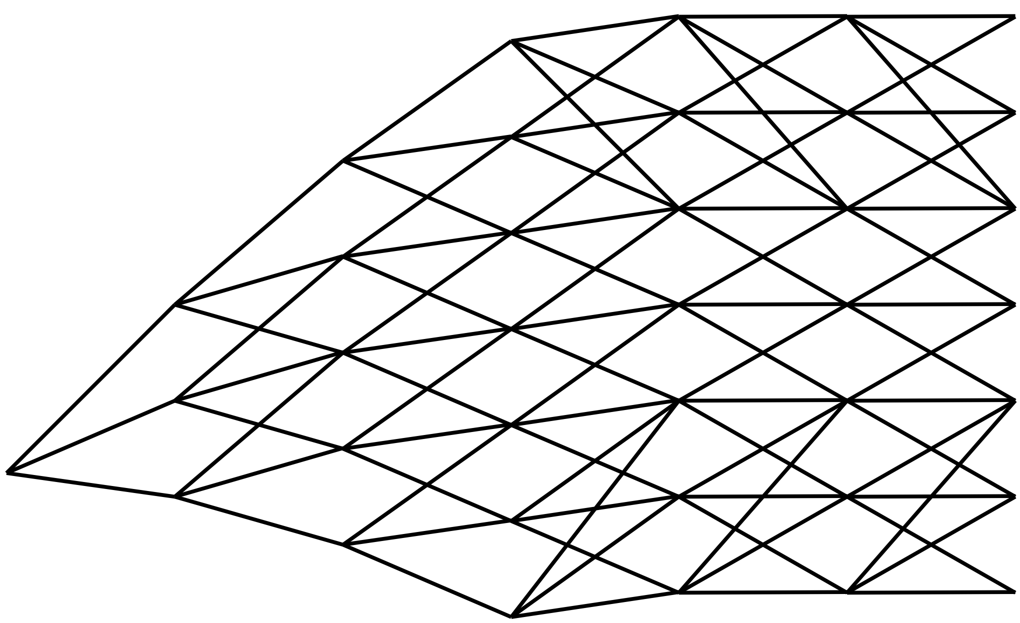

Figure 3

Black's volatility for at-the-money caplets that are reset monthly

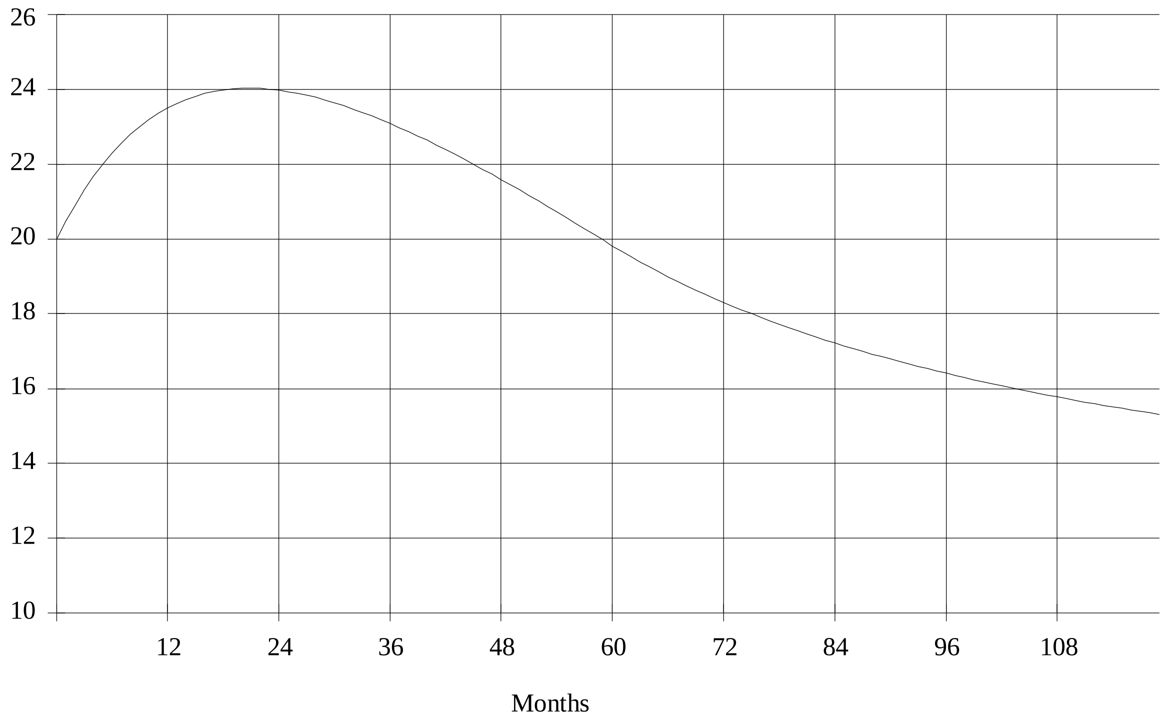

Figure 4

Value of $a(t)$ when $\sigma = 1.4\%$ (left-hand scale), and the value of $\sigma(t)$ when $a = 5\%$ (right-hand scale) required to replicate the caplet prices computed from the Black volatilities in Figure 3. The Black volatilities from Figure 3 are included for reference purposes.

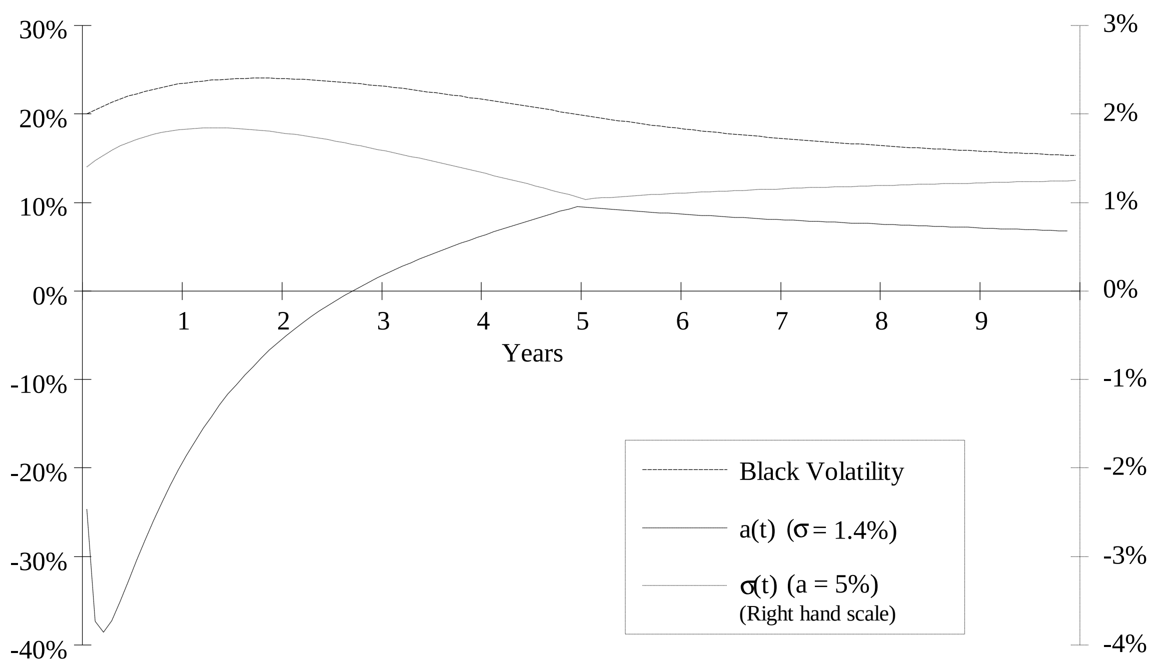

Figure 5

The tree for x^* when length of time step changes