# Arbitrage Restrictions and Multi-Factor Models of the Term Structure of Interest Rates.[1]

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**Abstract**

In this paper we investigate models of the term structure where the factors are interest rates. As an example, we derive a no-arbitrage model of the term structure in which any two futures (as opposed to forward) rates act as factors. The term structure shifts and tilts as the factor rates vary. The cross-sectional properties of the model derive from the solution of a two-dimensional autoregressive process for the short rate, which exhibits mean reversion and a lagged memory parameter. We show that the correlation of the factor rates is restricted by the no-arbitrage conditions of the model. Hence in a multiple-factor model it is not valid to independently choose both the mean reversion, volatility and correlation parameters, contrary to the approach of some models in the literature. The term-structure model, derived here, can be used to value options on bonds and swaps or to generate term structure scenarios for the risk management of portfolios of interest-rate derivatives.

# 1 Introduction

The term structure of nominal interest rates exhibits several patterns of changes over time. In some periods, it shifts up or down, perhaps in response to higher expectations of future inflation. In other periods, it tilts, with short rates rising and long rates falling, perhaps in response to a tightening of monetary policy. Sometimes its shape changes to an appreciable extent, affecting its curvature. Models of the term structure are of interest to practitioners and financial academics alike, both for the pricing of interest-rate sensitive derivative contracts, and for the measurement of the interest-rate risk arising from portfolios of these contracts. A desirable feature of these models is that they should capture at least the shifts and tilts of the term structure.

One early, intuitively appealing two-factor model which captured the above features of the empirical term structure was the long rate-spread model of Brennan and Schwartz (1979). Although this model has the attractive feature of modelling term structure movements in terms of two key rates, it is not presented in the "no-arbitrage" setting first proposed by Ho and Lee (1986). Today, it is recognised that a highly desirable, if not a necessary condition, for a model to satisfy is the no-arbitrage condition. In this paper, we develop a model that is consistent with the principle of no arbitrage and which yields a two- factor model similar to that of Brennan and Schwartz.

Fundamentally, the no-arbitrage condition, when applied to the term structure requires the price of a long-term bond to be related to the expected value, under the equivalent martingale measure (EMM), of the future relevant short-term bond prices. This requirement links the cross-sectional properties of the term structure at each point in time to the time-series properties of bond prices and interest rates. In this paper, we extend this analysis to a two-factor setting. In the context of our two-factor model, we are able to show that if the short rate follows a mean-reverting two-dimensional process (a process generated by two state variables), then the no-arbitrage condition implies a short rate-long rate model of the term structure, not dissimilar to that of Brennan and Schwartz. Also, in this model, the correlation between the long and short rates is restricted by the degree of mean reversion of the short rate and the relative volatilities of the long and short rates.

We suggest a time series model in which the conditional mean of the short rate follows a two-dimensional process, similar to that proposed by Hull and White (1994). This assumption allows us to nest the popular AR(1) single-factor model as a special case. It is also general enough to produce stochastic no-arbitrage term structures with shapes that capture most of those observed. A similar model in which the conditional mean of the short rate is stochastic has been suggested by Balduzzi, Das and Foresi (1995).

Recent literature, mainly inspired by the practical need to price various interest rate derivative contracts, has produced a bewildering variety of term structure models. In section 2 of this paper we discuss this literature, relate our model to previously proposed models and discuss the incremental contribution of our work. One of the most difficult aspects of term structure modelling is notation and definition of the relevant variables and parameters. For this reason, we devote much of section 3 to a description of the set-up of the problem, the variables and our notation. In this section, we also derive some general properties of two-factor models. In particular we show that if a price of a zero-coupon bond follows a two-dimensional process then its conditional expectation is generated by a two-factor model. In section 4, we analyse futures prices and rates and derive our main results for a two-factor lognormal interest-rate model. In section 5 we extend the analysis to forward prices and rates. The conclusions and possible applications of our model, to the valuation of interest rate options and to risk management are discussed in section 6.

## 2 Term Structure Models : The Literature

A basic decision that has to be made in term structure modelling is the choice of the assumption about the distributional properties of interest rates (and hence bond prices). One classification of the literature is according to whether interest rates are normally distributed or lognormally distributed and whether they evolve in discrete time or continuously. Gaussian interest rate models of the type first derived by Vasicek (1977) have been developed extensively by Jamshidian (1989), Hull and White (1993), Turnbull and Milne (1990) and applied to the valuation of a variety of interest rate and bond options. Also, the no- arbitrage models of Ho and Lee (1986) and one version of the general Heath,Jarrow and Morton (1990a, 1990b)

(HJM) model are discrete-time, additive binomial, models whose interest rates limit to normally distributed variables. An objection that has often been raised against this whole class of models is that they allow nominal rates to be negative, with positive probability. However, perhaps from a practical point of view, a more important drawback is that interest rates have higher variance when they are high than when they are low. Empirical evidence provided by Chan et al (1994) and Eom (1994) rejects the assumptions of this class of models in favour of the alternative assumption that variance is level dependent.

In this paper we propose a model in which the rate of interest is lognormally distributed. This assumption has the advantage that the variance is dependent on the level of the rate. Thus rates are skewed to the right in our model. In practice many traders use the Black (1976) model to price interest rate caps, a model that also assumes lognormal interest rates. Also as discrete approximations, the Black, Derman and Toy (1990)(BDT) and Black and Karazinski (1990) models have similar assumptions. Our incremental contribution to this literature is that we provide a particularly simple two-factor extension of the BDT model. We also provide a set of sufficient conditions for the cross- sectional two-factor model to hold in a no-arbitrage setting.

Another categorization of models in the literature is that between equilibrium models and no-arbitrage models of the term structure. The former include Cox, Ingersoll and Ross (1985) and the extension to a two-factor model with stochastic volatility by Longstaff and Schwartz (1992). In contrast, there are the no-arbitrage models of Ho and Lee (1986), BDT, HJM and many others. Our model is in the no-arbitrage model category. The addition to the literature in this case is that we show that the no-arbitrage condition restricts the correlation of the factor interest rates in a multi-factor model. Closely connected to the no-arbitrage models, in fact a sub-category, are recent theories based on the pricing kernel. Constantinides (1992) assumes a process for the kernel and derives a single-factor model of the term structure. Backus and Zin (1993) develop a model in a Gaussian one-factor framework and show the relationship between the time series process of the pricing kernel, the process for the short interest rate and the term structure. Backus and Zin use a discrete time ARMA model of the pricing kernel. In this paper we directly model the 'risk-neutral' density of the short rate, rather than the pricing kernel. Hence our approach is somewhat different from theirs. However, in one aspect, we extend their approach by using a two-dimensional autoregressive process which leads, given no- arbitrage, to a two-factor characterization of the term structure.

A multi-factor model for the term structure has been proposed recently by Duffie and Kan (1994). Duffie and Kan analyse a class of 'affine' or linear models, assuming a vector process for the yields on zero-coupon bonds. Their non-stochastic volatility example reduces to a multi- variable Gaussian model in which any two rates can be interpreted as factors. In our model, we derive a somewhat similar result. In our case, any two futures rates can be employed as factors. However, our no-arbitrage model restricts the correlation of these chosen factor rates. Lastly, in Longstaff and Schwartz (1992), a two-factor model is derived in which the volatility of interest rates is the second factor. This model is capable of explaining the term risk premium. In this sense, it is similar in spirit to the model proposed here. However, in our model, two factors explain the term structure even when either the local expectations hypothesis holds or when volatility is non-stochastic. If stochastic volatility is an important explanatory variable, it may act in addition to our two factors. It could, therefore, be added as a third factor in a possible extension. In our model, the term structure shifts and tilts perhaps in response to expectations of future real interest rates and inflation rates. It does so even in a risk neutral world. Hence our incremental contribution is to derive a different set of conditions for a two-factor model to those of Longstaff and Schwartz.

# 3 Some general properties of two-factor models

## 3.1 Definitions and notation

We denote $P_t$ as the time $t$ price of a zero-coupon bond paying \$1 with certainty at time $t + m$, where $m$ is measured in years. The short-term interest rate is defined in relation to this $m$-year bond, where $m$ is fixed. The short-term interest rate for $m$-year money at time $t$ is some function of the price, $i_t = \phi(P_t)$. In this paper, we investigate alternative definitions of the interest rate function $\phi$.[1] The other difference between this spot rate

[1] The conventional definition of the interest rate is the continuously compounded rate when $\phi(P_t) = -\ln(P_t)/m$. In this paper, we consider the general case, but analyse in

and the interest rate in the paper of HJM is that $m$ is not necessarily a very short (instantaneous) period. However, as in HJM, $m$ does not vary.

We are concerned with interest rate contracts for delivery at a future date $T$. We denote the futures price at time $t$ for delivery of an $m$-year maturity bond at time $T$ as $P_{t,T}$. The corresponding futures rate, is denoted $F_{t,T}$, where $F_{t,T} = \phi(P_{t,T})$.

We now denote the logarithm of the futures rate as

$$f_{t,T} = \ln[F_{t,T}] \tag{1}$$

Note that under this notation, which is broadly consistent with HJM, $F_{t,t} = i_t$ and $f_{t,t} = \ln(i_t)$.

In Table 1 we summarize the notation used in the paper. The mean and annualized standard deviation of the (logarithm) of the spot rate are denoted

$$\mu(t,T,T) = E_t[f_{T,T}] \tag{2}$$

$$\sigma(t,T,T) = [\mathrm{var}_t[f_{T,T}]/(T-t)]^{\frac{1}{2}} \tag{3}$$

respectively.

Also in the case of futures rates, we define

$$\mu(0,t,T) = E_0[f_{t,T}] \tag{4}$$

$$\sigma(0,t,T) = [\mathrm{var}_0[f_{t,T}]/t]^{\frac{1}{2}} \tag{5}$$

Note that the mean and variance of the spot rate in equations (2) and (3)are statistics of a time $T$ measurable random variable. In equations (4) and (5) the statistics relate to a time $t$ measurable random variable.

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detail the simple case where

$$i_t = \phi(P_t) = [1 - P_t]/m$$

Note that here the interest rate is defined on a bankers' discount (or T-Bill basis). This has considerable analytical advantages over the conventional definition where the rate is defined on a continuously compounded basis.

**Table 1**
**Notation for the Mean and Volatility of Spot and Futures Rates**

| | (1) | | (2) | | (3) | |
|---|---|---|---|---|---|---|
| Time Period | 0 | | $t$ | | $T$ | |
| Spot prices and interest rates for $m$-year money | $\mu(0,t,t)$ | Unconditional logarithmic mean of $i_t$ | $P_t$ | Zero bond price at $t$ for delivery of \$1 at $(t+m)$ | $P_T$ | Zero bond price at time $T$ for delivery of \$1 at time $T+m$ |
| | $\sigma(0,t,t)$ | Unconditional (annualised) volatility of $i_t$ | $i_t$ $= F_{t,t}$ | $m$–year interest rate at time $t$ | $i_T$ $= F_{T,T}$ | $m$– year interest rate at time $T$ |
| Futures interest rates for bonds maturing at time $T+m$ | $\mu(0,t,T)$ | Mean of $f_{t,T}$ | | | | |
| | $\sigma(0,t,T)$ | Unconditional (annualised) volatility of $F_{t,T}$ | | | | |
| | | | $F_{t,T}$ | futures interest rate at $t$ for delivery at T ($m$-year money) | | |
| | | | $f_{t,T}$ | Logarithm of $F_{t,T}$ | | |
| | | | $\mu(t,T,T)$ | Conditional mean of $f_{T,T}$ | | |
| | | | $\sigma(t,T,T)$ | Conditional (annualised) volatility of $F_{T,T}$ | | |

### 3.2 General properties of two-factor models

If a variable follows a two-dimensional process similar to that assumed for interest rates by Hull and White (1994), the conditional expectation of the variable is governed by a two-factor cross-sectional model. We first establish this quite generally and then apply it to the case of bond prices and interest rates. In Appendix 1, we show:

**Lemma 1** *The variable $x_t$ follows the time series process*

$$x_t = (1-c)x_{t-1} + y_{t-1} + \epsilon_t$$

*where*

$$y_{t-1} = (1-\alpha)y_{t-2} + \nu_{t-1}$$

*if and only if, the conditional expectation of $x_{t+k}$ is of the form*

$$E_t(x_{t+k}) = a_k x_t + b_k E_t(x_{t+1})$$

*where*

$$b_k = \sum_{\tau=1}^{k} (1-c)^{k-\tau}(1-\alpha)^{\tau-1}$$

*and*

$$a_k = (1-c)^k - (1-c)b_k.$$

*Proof.* See appendix 1.

We now assume that some function of the zero-coupon bond price, $P_t$ follows the process assumed in Lemma 1. Specifically, let

$$x_t = f(P_t) - E_0[f(P_t)]$$

where $f(P_t)$ is any function and $E_0[.]$ is its expectation at time 0. It follows immediately from Lemma 1 that:

**Lemma 2** *A function of the price of an m-year zero-coupon bond $P_t$ follows a two-dimensional process :*

$$f(P_t) = E_0[f(P_t)] + (1-c)\{f(P_{t-1}) - E_0[f(P_{t-1})]\} + y_{t-1} + \epsilon_t \qquad (6)$$

*where*

$$y_{t-1} = (1-\alpha)y_{t-2} + \nu_{t-1}$$

*if and only if the conditional expectation of* $E_t[f(P_{t+k})$ *is given by*

$$E_t[f(P_{t+k})] - E_0[f(P_{t+k})] = a_k\,[f(P_t) - E_0[f(P_t)]] + b_k\,[E_t[f(P_{t+1})] - E_0[f(P_{t+1})]] \ labelprop1b$$

*where*

$$b_k = \sum_{\tau=1}^{k} (1-c)^{k-\tau}(1-\alpha)^{\tau-1}$$

*and*

$$a_k = (1-c)^k - (1-c)b_k. \square$$

## 4 Futures prices and futures rates in a no-arbitrage economy

In this section, we apply the results in the previous section, to derive futures prices and futures interest rates in a no-arbitrage setting. We assume here that the two-dimensional process for prices or rates defined above, holds under the equivalent martingale measure.

Cox, Ingersoll and Ross (1981) and Jarrow and Oldfield (1981) established the proposition that the futures price, of any asset, is the expected value of the future spot price, where the expected value is taken with respect to the equivalent martingale measure (EMM). In this section we apply this result to determine the futures price of zero- coupon bonds, assuming that the bond prices are generated by a two- factor model. Since there is a one-to-one relationship between zero- coupon bond prices and short-term interest rates, and also a one-to-one relationship between futures bond prices and futures rates, we can then go on to derive a model for futures interest rates.

Initially, we make no distributional assumptions. We assume only a) the existence of a no-arbitrage economy in which the EMM exists, and b) that a function of the time $t$ price of an $m$-year zero-coupon bond, $f(P_t)$, follows a two-dimensional process of the general form assumed in Lemma 1, and c) that a market exists for trading futures contracts on $f(P_t)$. We first have:

**Lemma 3** *Assume that equation (6) holds for $f(P_t)$ under the EMM, then*

$$f_t(P_{t+k}) - f_t(P_{t+k}) = a_k\left[f(P_t) - f_0(P_t)\right] + b_k\left[f_t(P_{t+1}) - f_0(P_{t+1})\right]$$

*where*

$$b_k = \sum_{\tau=1}^{k}(1-c)^{k-\tau}(1-\alpha)^{\tau-1}$$

*and*

$$a_k = (1-c)^k - (1-c)b_k.$$

*where $f_t(P_{t+k})$ is the time $t$ futures price of $f(P_{t+k})$.*

Proof: From CIR (1981), proposition 2, and Satchell, Stapleton and Subrahmanyam (1997), the futures price of any payoff is its expected value, under the EMM. Applying this result to $f(P_{t+k})$, and applying lemma 2, yields (2).□

The rather general result in lemma 3 is of interest because of two special cases. First, we consider the case where the futures contract is on the zero-coupon bond itself. Second, we take the case of a futures contract on an interest rate, which is a function of the zero-coupon bond price. A simple relation between futures prices exists if spot prices follow a two-dimensional process. In this case we have, as an implication of lemma 3:

## Example 1: A Linear Process for the Zero-Bond Price

**Proposition 1** *The price of an m-year zero-coupon bond $P_t$ follows a two-dimensional process under the equivalent martingale measure (EMM):*

$$P_t = E_0(P_t) + (1-c)[P_{t-1} - E_0(P_{t-1})] + y_{t-1} + \epsilon_t$$

*where*

$$y_{t-1} = (1-\alpha)y_{t-2} + \nu_{t-1}$$

*if and only if, the kth futures price $P_{t,t+k}$ is given by*

$$P_{t,t+k} - P_{0,t+k} = a_k[P_t - P_{0,t}] + b_k[P_{t,t+1} - P_{0,t+1}]$$

*where*

$$b_k = \sum_{\tau=1}^{k} (1-c)^{k-\tau}(1-\alpha)^{\tau-1}$$

*and*

$$a_k = (1-c)^k - (1-c)b_k.$$

Proof This follows as a special case of lemma 3 with $f(P_t) = P_t$. $\square$

Proposition 1 is the key to understanding the conditions under which the term structure follows a two-factor process. Essentially, if futures prices of long delivery futures are given by the cross-sectional model in Proposition 1, then forward prices, and also futures and forward rates will follow two-factor models. The relationship for interest rates, however, is in general complex, since the function $i_t = \phi(P_t)$ is, in general, non-linear. In the special case where interest rates are defined on a banker's discount basis, i.e. where $i_t = [1 - P_t]/m$, we have a very simple two- factor model for interest rates which follows directly from Proposition 1. We can establish:

**Corollary 1** *The m-year interest rate defined by* $i_t = [1 - P_t]/m$ *follows a two- dimensional process under the equivalent martingale measure (EMM):*

$$i_t = E_0[i_t] + (1-c)[i_{t-1} - E_0(i_{t-1})] + y'_{t-1} + \epsilon'_t$$

*where*

$$y'_{t-1} = (1-\alpha)y'_{t-2} + \nu'_{t-1}$$

*if and only if, the kth futures rate* $F_{t,t+k}$ *is given by*

$$F_{t,t+k} - F_{0,t+k} = a_k[F_{t,t} - F_{0,t}] + b_k[F_{t,t+1} - F_{0,t+1}]$$

*where*

$$b_k = \sum_{\tau=1}^{k} (1-c)^{k-\tau}(1-\alpha)^{\tau-1}$$

*and*

$$a_k = (1-c)^k - (1-c)b_k.$$

<u>Proof</u> If the price $P_t$ follows the process in Proposition 1 then the m-year interest rate defined by $i_t = [1 - P_t]/m$ follows the process

$$i_t = E_0[i_t] + (1-c)[i_{t-1} - E_0(i_{t-1})] + y'_{t-1} + \epsilon'_t$$

where $\epsilon'_t = -m\epsilon_t$ and

$$y'_{t-1} = (1-\alpha)y'_{t-2} + \nu'_{t-1}$$

, with $y'_t = -my_t$ and $\nu'_t = -m\nu_t$. Defining the futures rate similarly as

$$F_{t,T} = [1 - P_{t,T}]/m$$

and substituting for the futures prices in Proposition 1 yields the corollary. □

Corollary 1 shows that the kth futures rate is related to the spot rate innovation and the first futures rate innovation, where these innovations are relative to the time 0 futures rate. This simple result stems directly from the definition of the interest rate as a linear function of the zero-bond price, together with the assumption of a linear process for the zero-bond price.

## Example 2: A Linear Process for the Exponential Rate

In general the relationship between futures rates will be complex. However, one further special case of the two-factor model, where rates are defined on an exponential basis, yields tractable solutions. We now assume that the interest rate is defined by $i_t = -ln(P_t)$. Applying lemma 3, we then have:

**Proposition 2** *The the m-year interest rate $r_t = -lnP_t$ follows a two-dimensional process under the equivalent martingale measure (EMM):*

$$r_t = E_0(r_t) + (1-c)[r_{t-1} - E_0(r_{t-1})] + y_{t-1} + \epsilon_t$$

*where*

$$y_{t-1} = (1-\alpha)y_{t-2} + \nu_{t-1}$$

*if and only if, the kth futures rate $E_t(r_{t+k})$ is given by*

$$E_t(r_{t+k}) - E_0(r_{t+k}) = a_k[r_t - E_0(r_t)] + b_k[E_t(r_{t+1}) - E_0(r_{t+1})]$$

*where*

$$b_k = \sum_{\tau=1}^{k} (1-c)^{k-\tau}(1-\alpha)^{\tau-1}$$

*and*

$$a_k = (1-c)^k - (1-c)b_k.$$

Proof This follows as a special case of lemma 3 with $f(P_t) = -lnP_t$. $\square$

## 5 A logarithmic two-factor model of the short-term interest rate

In this section we develop in detail the properties of a two-factor model of interest rates, where the logarithm of the short rate follows a two dimensional process, similar to those assumed in the previous section. Again we make use of the lemma 2. We assume that the logarithm of the short-term interest rate, defined on a bankers' discount basis is mean reverting and of the form

$$f_{t,t} = \mu(0,t,t) + [f_{t-1,t-1} - \mu(0,t-1,t-1)](1-c) + y_{t-1} + \varepsilon_t \tag{7}$$

where

$$y_t = y_{t-1}(1-\alpha) + \nu_{t-1}$$

where time is measured in periods of length $n$ years. In (7), $c$ is the rate of mean reversion per period, $\nu_t$ and $\epsilon_t$ are mutually and intertemporally independent variables.

Equation (7) assumes a spot rate process which is essentially an extension of the Vasicek (1977) process. In its simplest form with $\nu_{t-\tau} \equiv 0$, the spot rate follows the process

$$f_{t,t} = \mu(0,t,t) + [f_{t-1,t-1} - \mu(0,t-1,t-1)](1-c) + \epsilon_t \tag{8}$$

Here, the logarithm of the spot rate is a mean-reverting process with a mean reversion coeficient of $c$ per period. The process is 'calibrated' to current expectations of future rates, $\mu(0,t,t)$. The process in equation (8) is not complex enough, however, to mirror actual movements of the term structure. We need to capture changes in expected spot rates that are unrelated to current realisations of the spot rate itself. This is achieved by adding a second dimension to the process. Hence, we assume that the short rate follows the two- dimensional autoregressive process in equation (7). Equation (7) allows for an independent shift in the conditional expectation of $f_{t,t}$. For example, we have with $\alpha = 1$

$$\mu(t-1,t,t) = \mu(0,t,t) + [f_{t-1,t-1} - \mu(0,t-1,t-1)](1-c) + \nu_{t-1} \quad (9)$$

Hence, the conditional expectation of the time $t$ spot rate depends on two time $t-1$ measurable stochastic variables, $\varepsilon_{t-1}$ which determines $f_{t-1,t-1}$ and $\nu_{t-1}$ which further shifts the expectation of $f_{t,t}$. If $\alpha < 1$ the effect of a shock to expectations persists to later spot rates. $\alpha$ measures the degree of decay in expectations. If $\alpha = 1$, there is no decay at all. In this case the conditional expectation of $f_{t,t}$ is affected equally by all realisations of $\nu$ between time 0 and time $t$. Given that the short rate follows the process in equation (7) the expectations of the spot rate $f_{t+k,t+k}$ at time $t$ can be found by successive substitution. We find

**Proposition 3** *If the logarithm of the spot rate follows the process*

$$f_{t,t} - \mu(0,t,t) = [f_{t-1,t-1} - \mu(0,t-1,t-1)](1-c) + y_{t-1} + \varepsilon_t, \quad \forall t, \quad (10)$$

*where*

$$y_t = y_{t-1}(1-\alpha) + \nu_{t-1}$$

*then the expectation of the logarithm of the interest rate $i_{t+k}$ at time $t$ is*

$$\begin{aligned} \mu(t,t+k,t+k) - \mu(0,t+k,t+k) &= a_k[f_{t,t} - \mu(0,t,t) \\ &+ b_k[\mu(t,t+1,t+1) - \mu(0,t+1,t+1)] \end{aligned}$$

Proof The Proposition follows directly from Lemma 2 where $f(P_t) = ln(i_t)$ and $i_t = \phi(P_t)$ for any interest rate function $\phi$. □

To appreciate the meaning of Proposition 3 we will look at various limiting cases. First, if $\alpha = 0, c = 0$, the process for the short rate is a two-dimensional random walk. The expectation at $t$ of $f_{t+k,t+k}$ is in this case

$$\mu(t, t+k, t+k) = \mu(0, t+k, t+k) + f_{t,t} - \mu(0, t, t) + \nu_t \qquad (11)$$

The expectation in (11) is affected both by the degree to which $f_{t,t}$ exceeded its expected value, $\mu(0, t, t)$ and by the independent shift factor $\nu_t$. Note that in this case, the shift in the expectation is the same for each $k$. If $c = 0$, we have

$$\begin{aligned} \mu(t, t+k, t+k) &= \mu(0, t+k, t+k) + f_{t,t} - \mu(0, t, t) \\ &+ \sum_{\tau=0}^{t-1} \nu_{t-\tau}(1-\alpha)^{\tau} \left[ \frac{1-(1-\alpha)^k}{\alpha} \right] \end{aligned} \qquad (12)$$

and with $\alpha = 0$

$$\mu(t, t+k, t+k) = \mu(0, t+k, t+k) + f_{t,t} - \mu(0, t, t) + \sum_{\tau=0}^{t-1} \nu_{t-\tau} k \qquad (13)$$

In this case, each of the shift factors affects the expectation. Also, the shift in the expectation depends on $k$. Also, if $c > 0, \alpha = 1$, we find

$$\mu(t, t+k, t+k) = \mu(0, t+k, t+k) + [f_{t,t} - \mu(0, t, t)](1-c)^k + \nu_t(1-c)^{k-1} \quad (14)$$

Finally, with $c > 0, \alpha = 0$

$$\mu(t, t+k, t+k) = \mu(0, t+k, t+k) + [f_{t,t} - \mu(0,t,t)](1-c)^k + \sum_{\tau=0}^{t-1} \nu_{t-\tau} \left[ \frac{1-(1-c)^k}{c} \right] \tag{15}$$

## 5.1 Futures and forward prices rates in the logarithmic rate model

We now look further at the properties of such a model. However, if the logarithm of the rate mean reverts in such a manner, we need rather stronger assumptions if we are to derive a cross-sectional model for futures rates. First, we assume that the logarithm of the short-term rate follows the two-dimensional process under the EMM. Also, as in Hull and White (1994), we assume that the rate is lognormal. In contrast to Hull and White, we assume that it is the interest rate, defined on a banker's discount basis, that is lognormal.

**Lemma 4** *In a no-arbitrage economy, if the spot interest rate, defined on a 'banker's discount' basis, is lognormally distributed under the martingale measure, the $k$ period futures rate at time $t$ for an $m-$ year loan is*

$$f_{t,t+k} = \mu(t, t+k, t+k) + \frac{kn}{2} \sigma^2(t, t+k, t+k) \tag{16}$$

*where $n$ is the length, in years, of the period $t$ to $t+1$*

*Proof.* From the no-arbitrage condition, the futures price is equal to the expectation under the equivalent martingale measure, $P_{t,t+k} = E_t(P_{t+k})$. Hence, using the definition of the interest rates $i_{t+k}$ , the futures price is given by $1 - mE_t(i_{t+k})$. It then follows immediately from the definition of the futures rate that $F_{t,t+k} = E_t(i_{t+k})$. Since by assumption, $i_{t+k}$ is lognormal, under the martingale measure, with a conditional logarithmic mean and annualised volatility, of $\mu(t, t+k, t+k)$ and $\sigma(t, t+k, t+k)$, we have

$$\begin{aligned} \mathrm{E}_t(i_{t+k}) &= \exp\left[\mu(t,t+k,t+k) + \frac{kn}{2}\sigma^2(t,t+k,t+k)\right] \\ &= F_{t,t+k} = \exp f_{t,t+k} \end{aligned}$$

□

Lemma 4 states that lognormality of the futures rate follows from lognormality of the spot rate. This is because the conditional logarithmic mean of the spot rate, $\mu(t,t+k,t+k)$ is normally distributed and the conditional variance, $\sigma(t,t+k,t+k)$, of the spot rate is a constant. Lemma 4 also restricts the correlation of the spot and the futures rates. Combining the results of Proposition 3 and 4 we can write the logarithm of the $k$th futures rate as

$$\begin{aligned} f_{t,t+k} &= \mu(0,t+k,t+k) + [f_{t,t} - \mu(0,t,t)](1-c)^k \\ &+ \sum_{\tau=0}^{t-1} \nu_{t-\tau}\alpha^{\tau} \sum_{\tau=1}^{k}(1-c)^{k-\tau}\alpha^{\tau-1} \\ &+ \frac{kn}{2}\sigma^2(t,t+k,t+k) \end{aligned} \tag{17}$$

The conditional variance of the futures rate is

$$\begin{aligned} \sigma^2(t-1,t,t+k)/n &= (1-c)^{2k}\mathrm{var}_{t-1}[f_{t,t}] \\ &+ \left[\sum_{\tau=1}^{k}(1-c)^{k-\tau}\alpha^{\tau-1}\right]^2 \mathrm{var}_{t-1}(\nu_t) \end{aligned} \tag{18}$$

Also, since the variance of the spot rate is

$$\sigma^2(t-1,t,t)/n = \mathrm{var}_{t-1}[f_{t,t}] \tag{19}$$

it follows that the covariance of the spot and the $k$ th futures rate is

$$\begin{aligned} \mathrm{cov}_{t-1}[f_{t,t}, f_{t,t+k}] &= (1-c)^k \mathrm{var}_{t-1}[f_{t,t}] \\ &= (1-c)^k \sigma^2(t-1,t,t)/n \end{aligned} \tag{20}$$

and the correlation of the spot and futures rates is therefore

$$\rho(t-1,t,t+k) = \frac{(1-c)^k \sigma(t-1,t,t)}{\sigma(t-1,t,t+k)} \tag{21}$$

This expression for the correlation of the short rate and the $k$th futures rate illustrates an important implication of the no-arbitrage model. Given the volatilities of the spot and futures rates, we are not able to independently choose both the correlation and the degree of mean reversion. The no-arbitrage model restricts the correlation between the two factors to be a function of the degree of mean reversion of the short rate.

We can now establish an important property of the $k$th futures rate that allows us to solve for the cross-sectional term structure of interest rates. We have:

**Lemma 5** *Given the conditions of Lemma 4, the logarithmic mean of the* $k$th *futures rate is related to the conditional logarithmic mean of the spot interest rate at* $t+k$ *by*

$$\mu(0,t,t+k) = \mu(0,t+k,t+k) + \frac{kn}{2}\sigma^2(t,t+k,t+k) \tag{22}$$

The lemma relates the means of the futures and corresponding spot rates. The extra term reflects the fact that from Lemma 4 the futures rate itself is lognormal with volatility $\sigma(0,t,t+k)$.

*Proof.* See Appendix 1.

We can now derive the main result of the paper. This is a two-factor cross-sectional relationship between interest rates at time $t$. The following proposition follows from Proposition 3, 4, and 5. We show now that the two-dimensional time-series process assumed in the statement of Proposition 3 is necessary and sufficient to generate a two–factor term structure. We have

**Proposition 4** *In a no-arbitrage economy in which the short rate of interest follows a lognormal process of the form*

$$f_{t,t} = \mu(0,t,t) + [f_{t-1,t-1} - \mu(0,t-1,t-1)](1-c) + y_{t-1} + \epsilon_t$$

*where*

$$y_{t-1} = (1-\alpha)y_{t-2} + \nu_{t-1}$$

*the term structure of futures rates at time* $t$ *is generated by a two– factor model. The* $k$th *futures rate is given by*

$$\begin{aligned} f_{t,t+k} &= \mu(0,t,t+k) + a_k[f_{t,t} - \mu(0,t,t)] \\ &\quad + b_k[f_{t,t+1} - \mu(0,t,t+1)] \end{aligned} \tag{23}$$

*where*

$$b_k = [(1-c)^{k-1} + \ldots + \alpha^{k-1}]$$

*and*

$$a_k = (1-c)^k - (1-c)b_k.$$

Also, a short rate process in the form of (8) is necessary for the two-factor model in equation (23).

*Proof.* From Proposition 3

$$\begin{aligned} \mu(t,t+k,t+k) - \mu(0,t+k,t+k) &= a_k[f_{t,t} - \mu(0,t,t) \\ &+ b_k[\mu(t,t+1,t+1) - \mu(0,t+1,t+1)] \end{aligned}$$

is a necessary and sufficient condition, since the proposition holds for any measure. Substituting the results of Lemmas 4 and 5 then yields the statement in the proposition

Proposition 4 relates the $k$th futures rate to the spot rate $f_{t,t}$ and the first futures rate, $f(t+1,t+1)$. If $m = 91/365$, for example, this means that the $k$th three–month futures rate is related to the spot three–month rate and the one period futures, three–month rate. In a recent contribution, Duffie and Kan (1993) have pointed out that if the model is linear in two such rates, it can always be expressed in terms of any two forward rates. In our context, it may be more practical to express the $k$th futures rate as a function of the spot rate and the $n$th futures rate. Hence, we derive the following implication of Proposition 4:

**Corollary 2** *Suppose we choose any two futures rates as factors, where $N_1$ and $N_2$ are the maturities of the factors then the following linear model holds:*

$$\begin{aligned} f_{t,t+k} &= \mu(0,t,t+k) + A_k(N_1,N_2)[f_{t,t+N_1} - \mu(0,t,t+N_1)] \\ &+ B_k(N_1,N_2)[f_{t,t+N_2} - \mu(0,t,t+N_2)] \qquad (24) \end{aligned}$$

*where*

$$B_k(N_1,N_2) = (a_k b_{N_1} - b_k a_{N_1})/(a_{N_2} b_{N_1} - b_{N_2} a_{N_1}),$$

$$A_k(N_1,N_2) = (-a_k b_{N_1} + b_k a_{N_1})/(a_{N_2} b_{N_1} - b_{N_2} a_{N_1}),$$

*and*

$$b_k = [(1-c)^{k-1} + \ldots + \alpha^{k-1}],$$

*and*

$$a_k = (1-c)^k - (1-c)b_k.$$

Corollary 2 follows by solving equation (23) for $k = N_1$, and $k = N_2$ and then substituting back into equation (23).

**Corollary 3 The Random Walk Case**

*Suppose that $c = 0$ and the logarithm of the interest rate follows a random walk. In this case, the $k$th futures is*

$$f_{t,t+k} = \mu(0,t,t+k) + \left(\frac{N-k}{N}\right)[f_{t,t} - \mu(0,t,t)] + \left(\frac{k}{N}\right)[f_{t,t+N} - \mu(0,t,t+N)]. \tag{25}$$

**Proof**

Corollary 3 follows directly from Corollary 2 with

$$b_{k,N} = \frac{k}{N},$$

and hence,

$$a_{k,n} = \frac{N-k}{N}.$$

Here, the $k$th futures is affected by changes in the $N$th futures according to how close $k$ is to $N$. Equation (25) is a simple two–factor 'duration' type model.

**Corollary 4 The Stochastic Process for the Futures Rates**

*Given that the spot rate follows the process assumed in Proposition 4 (sufficiency) then the $k$th futures rate follows the process*

$$\begin{aligned} f_{t,t+k} - \mu(0,t,t+k) &= (1-c)[f_{t-1,t+k-1} - \mu(0,t-1,t+k-1)] \\ &\quad -(1-c)V_{t-1}[K] + (V_{t-1} + \varepsilon_t)(1-c)^k + V_t[K] \end{aligned}$$

*where*

$$\begin{aligned} V_t &= \sum_{\tau=0}^{t-1} \nu_{t-\tau}\alpha^{\tau} \\ K &= \sum_{\tau=1}^{k} (1-c)^{k-\tau}\alpha^{\tau-1} \end{aligned}$$

**Proof** See Appendix 3.

Hence, if the spot rate follows a two-dimensional process, so does the futures rate. Corollary **??**is the closest relationship we derive to that between the Hull-White type process, which we assume for the spot rate, and the HJM type process assumed for the forward rate. However, in the corollary it is the futures, rather than the forward rate that follows the two- dimensional process. Also, HJM assume that the forward rate for a given final bond maturity, $T$, follows a given process. In corollary **??** the futures rate is always for a given maturity of loan, $m$. The exact relationships between our model and the HJM assumptions are complex, and results must await further research.

### 5.2 Forward rates and zero-coupon bond yields

We have derived the distribution of futures prices and futures rates, at time $t$, using an assumption about the process for short rates and the no-arbitrage condition. The term structure of futures rates at a point in time is closely related to the term structure of forward rates. The latter are required to completely describe the yields on zero-coupon bonds and to value an arbitrary set of cash flows at time $t$. The general relationship between futures prices and forward prices is well known from Cox, Ingersoll, and Ross (1981)(CIR). The CIR result states that the difference between the forward and the futures price of an asset, depends on the covariance, under the equivalent martingale measure, of the asset futures price and the money market accumulation factor. We can apply the CIR result to find an equivalent relationship, in our model, between the forward and futures rates of interest. in the case of the model of futures rates here, we have:

**Proposition 5** *The forward rate at time $t$ for delivery at $t+k$ of an $n$-year zero- coupon bond is*

$$G_{t,t+k} = F_{t,t+k} - cov[F_{t,t+k}, \psi], \tag{26}$$

*where*

$$\psi = B_{0,1}B_{1,2}...B_{t-1,t}/B_{0,t}$$

*and where cov refers to the covariance of the variables under the martingale measure.*

*Proof.* Applying the general result in CIR we have the time $t+k$ forward price of the zero-coupon bond for delivery at time $t$

$$P'_{t,t+k} = P_{t,t+k} + cov[P_{t,t+k}, \psi].$$

Now, defining the forward rate by the relation

$$G_{t,t+k} = (1 - P'_{t,t+k})/m$$

and given the futures rate

$$F_{t,t+k} = [1 - P_{t,t+k}]/m,$$

then, substituting in the CRR relationship we find

$$G_{t,t+k} = F_{t,t+k} - cov[F_{t,t+k}, \psi].$$

Proposition 5 allows us to compute forward rates, futures rates, and any zero-coupon bond price given the term structure of futures rates.

# 6 Conclusions

This paper has explored the relationships between models of the extended Vasicek type, such as the two-factor model of Hull and White (1994), and models of the term structure of the Brennan and Schwartz (1979) type. Basically, if we assume that the price of a zero-coupon bond or any function of the price follows a two-dimensional process, then the term structure of future prices or rates is given by a two-factor cross- sectional model. Assuming that the logarithm of the interest rate follows a two-dimensional, mean-reverting process, we find that the term structure of futures rates can be written as a log-linear function of any two rates.

The rates assumed in the lognormal model are relatively simple to compute. The cross-sectional model of futures rates can be calibrated to market estimates of futures rates and volatilities from cap-floor and swaption prices. It can be used either to value American-style or path- dependent options. Alternatively the model can be used to generate interest-rate scenarios, which can in turn be used to evaluate the risk of interest-rate dependent portfolios.

# Appendix 1: Properties of the conditional mean for two- dimensional time-series processes

Lemma 1 The variable $x_t$ follows the time series process

$$x_t = (1-c)x_{t-1} + y_{t-1} + \epsilon_t$$

where $E(x_{t-1}y_{t-1}) = 0$ and where

$$y_{t-1} = (1-\alpha)y_{t-2} + \nu_{t-1}$$

if and only if, the conditional expectation of $x_{t+k}$ is of the form

$$E_t(x_{t+k}) = a_k x_t + b_k E_t(x_{t+1})$$

where

$$b_k = \sum_{\tau=1}^{k} (1-c)^{k-\tau}(1-\alpha)^{\tau-1}$$

and

$$a_k = (1-c)^k - (1-c)b_k.$$

Proof

**Sufficiency**

Successive substitution $x_1$, $x_2$, ... ,$x_{t+k}$ and taking the conditional expectation yields

$$E_t(x_{t+k}) = x_t(1-c)^k + V_t \sum_{\tau=1}^{k} (1-c)^{k-\tau}(1-\alpha)^{\tau-1} \tag{27}$$

where

$$V_t = \sum_{\tau=0}^{t-1} \nu_{t-\tau}(1-\alpha)^{\tau}$$

Substituting the corresponding expression for $E_t(x_{t+1})$ :

$$E_t(x_{t+1}) = x_t(1-c) + V_t$$

yields

$$E_t(x_{t+k}) = a_k x_t + b_k E_t(x_{t+1}), \tag{28}$$

where

$$b_k = \sum_{\tau=1}^{k} (1-c)^{k-\tau}(1-\alpha)^{\tau-1}$$

and

$$a_k = (1-c)^k - (1-c)b_k.$$

**Necessity**

Assume

$$E_t(x_{t+k}) = a_k x_t + b_k E_t(x_{t+1})$$

where $a_k$ and $b_k$ are defined by (28 ) above, and $x_t$ and $E_t(x_{t+1})$ are not perfectly correlated. Consider the orthogonal component $z_t$ from

$$E_t(x_{t+1}) = \gamma x_t + z_t \tag{29}$$

Then

$$E_t(x_{t+1}) = (a_1 + b_1\gamma)x_t + b_1 z_t$$

and hence, since $a_1 = 0$ and $b_1 = 1$

$$x_{t+1} = \gamma x_t + z_t + \epsilon_{t+1} \tag{30}$$

where $\mathrm{E}_t(\epsilon_{t+1}) = 0$. Hence $x_t$ follows a two-dimensional process with innovations $z_t, \epsilon_{t+1}$.

We now show that $\gamma = (1-c)$ and also that $z_t$ follows a mean reverting process with mean reversion $\alpha$. Suppose by way of contradiction, that $\gamma = (1-c')$. Also, suppose there is a shock to $x_t$ changes while the difference, $E_t(x_{t+1}) - x_t$ is constant, then $E_t(x_{t+k})$ will not be given by equation (28), since $c \neq c'$. It follows that we must have $\gamma = (1-c)$. Second, suppose that $\gamma = (1-c)$, but $z_t$ mean reverts at a rate different from $\alpha$. Then, if the difference, $E_t(x_{t+1}) - x_t$, changes, while $x_t$ is constant, then again $E_t(x_{t+k})$ will not be given by equation (28). Hence, a necessary condition is that $z_t$ mean reverts at a rate $\alpha$. $\square$

## Appendix 2: Proof of Lemma 5

**Proof that**

$$\mu(0,t,t+k) = \mu(0,t+k,t+k) + \frac{kn}{2}\sigma^2(t,t+k,t+k).$$

*Proof.* In the proof of Lemma 4, we have the no-arbitrage condition

$$F(t,t+k) = \mathrm{E}_t(i_{t+k}). \tag{31}$$

Hence, the expectation of the futures rate is given by

$$\mathrm{E}_0[F(t,t+k)] = \mathrm{E}_0(i_{t+k}), \tag{32}$$

by the law of iterated expectations.

Taking the logarithm of equation (32) and using the lognormal property, we have

$$\mu(0,t,t+k) + \frac{tn}{2}\sigma^2(0,t,t+k) = \mu(0,t+k,t+k) + \frac{(t+k)n}{2}\sigma^2(0,t+k,t+k). \tag{33}$$

From the lognormality of $i_{t+k}$,

$$(t+k)n\sigma^2(0,t+k,t+k) = \mathrm{var}_0[\mu(t,t+k,t+k)] + kn\sigma^2(t,t+k,t+k). \tag{34}$$

But, using Lemma 2,

$$\mathrm{var}_0[\mu(t,t+k,t+k)] = nt\sigma^2(0,t,t+k). \tag{35}$$

Substituting equations (35) into (34), and then (34) into (33), yields

$$\mu(0,t,t+k) = \mu(0,t+k,t+k) + \frac{kn}{2}\sigma^2(t,t+k,t+k).$$

∎

## Appendix 3: Derivation of the Process for the $k$th Futures Rate

Using Lemmas 4 and 5, the deviation of the $k$th futures rate from its expectation is related to that of the spot rate by the equation [substitue (22) in (17)]

$$f(t,t+k) - \mu(0,t,t+k) = (1-c)^k[f(t,t) - \mu(0,t,t)] + V_t[K] \tag{36}$$

where

$$\begin{aligned} V_t &= \sum_{\tau=0}^{t-1} \nu_{t-\tau}\alpha^\tau \\ K &= \sum_{\tau=1}^{k} (1-c)^{k-\tau}\alpha^{\tau-1} \end{aligned}$$

Also, by assumption, the spot rate is

$$f(t,t) - \mu(0,t,t) = (1-c)[f(t-1,t-1) - \mu(0,t-1,t-1)] + V_{t-1} + \varepsilon_t \tag{37}$$

The $k$th forward at time $t-1$ is similarly given by

$$f(t-1,t+k-1) \quad - \quad \mu(0,t-1,t+k-1)$$

$$\begin{aligned} &= (1-c)^k[f(t-1,t-1) - \mu(0,t-1,t-1)] \\ &\quad + V_{t-1}[K] \end{aligned} \tag{38}$$

Substituting (38) in (37) and (37) in (36) yields the corollary.

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