OPTIMAL DYNAMIC PORTFOLIO SELECTION: MULTI-PERIOD MEAN-VARIANCE FORMULATION*

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Abstract

The mean-variance formulation by Markowitz in 1950s and its analytical solution by Merton in 1972 paved a foundation for modern portfolio selection analysis in single period. This paper considers an analytical optimal solution to the mean-variance formulation in multi-period portfolio selection. Specifically, analytical optimal portfolio policy and analytical expression of the mean-variance efficient frontier are derived in this paper for the multi-period mean-variance formulation. An efficient algorithm is also proposed in this paper in finding an optimal portfolio policy to maximize a utility function of the expected value and the variance of the terminal wealth.

Key Words: Multi-period portfolio selection, multi-period mean-variance formulation, utility function.

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1 Introduction

Portfolio selection is to seek a best allocation of wealth among a basket of securities. The mean-variance formulation by Markowitz [13] [14] provides a fundamental basis for portfolio selection in single period. An analytical expression of the mean-variance efficient frontier in single-period portfolio selection was derived by Merton [16] in 1972. The problem of multi-period portfolio selection has been studied in the literature, such as in Smith [25], Chen et al. [1], Mossin [18], Merton [15] [17], Samuelson [23], Fama [6], Hakansson [9] [10], Elton and Gruber [3][4][5], Winkler and Barry [26], Francis [7], Dumas and Luciano [2], Östermark [20], Grauer and Hakansson [8] and Pliska [21]. Enormous difficulty was reported by Chen et al. [1] in finding optimal solutions for a multi-period mean-variance formulation. The literature in multi-period portfolio selection has been dominated by the results of maximizing expected utility functions of the terminal wealth and/or multi-period consumption. Specifically, the investment situations where the utility functions are of power form, logarithm function, exponential function, or quadratic form have been extensively investigated in the literature.

To our knowledge, no analytical or efficient numerical method in finding the optimal portfolio policy for a multi-period mean-variance formulation and in determining the mean-variance efficient frontier has been reported in the literature. In this sense, the concept of the Markowitz's mean-variance formulation has not been fully utilized in the multi-period portfolio selection. This paper represents an extension of the existing literature to capture the spirit of risk management in dynamic portfolio selection. Especially, Merton's analytical result [16] has been generalized in this paper to multi-period portfolio selection. Analytical optimal portfolio policy is derived for the multi-period mean-variance formulation along with the analytical expression of the mean-variance efficient frontier.

The organization of this paper is as follows. In Section 2, the mean-variance formulation for multi-period portfolio selection is discussed. The analytical solution to the multi-period mean-variance formulation is stated in Section 3. Detailed derivation of the analytical results is provided in Section 4. The mean-variance formulation for multi-period portfolio selection is investigated in Section 5 for investment situations where there is a riskless asset. The multi-period mean-variance formulation is then generalized in Section 6 to investment situations where a utility function of the expected terminal wealth and the risk is maximized. Three exemplary cases are studied in Section 7. This paper concludes in Section 8 with a suggestion for further study.

2 Mean-Variance Formulation for Multi-Period Portfolio Selection

We consider a capital market with $(n+1)$ risky securities, with random rates of returns. An investor joins the market at time 0 with an initial wealth x_0. The investor can allocate his wealth among the $(n+1)$ assets. The wealth can be reallocated among the $(n+1)$ assets at the beginning of each of the following $(T-1)$ consecutive time periods. The rates of return of the risky securities at time period t within the planning horizon are denoted by a vector $\mathbf{e_t} = [e_t^0, e_t^1, ..., e_t^n]'$, where e_t^i is the random return for security i at time period t. It is assumed in this paper that vectors $\mathbf{e_t}$, t = 0, 1, ..., T - 1, are statistically independent and return $\mathbf{e_t}$ has a known mean $E(\mathbf{e_t}) = [E(e_t^0), E(e_t^1), ..., E(e_t^n)]'$ and a known covariance

$$Cov(\mathbf{e_t}) = \begin{bmatrix} \sigma_{t,00} & \cdots & \sigma_{t,0n} \\ \vdots & \ddots & \vdots \\ \sigma_{t,0n} & \cdots & \sigma_{t,nn} \end{bmatrix}.$$

Let x_t be the wealth of the investor at the beginning of the t-th period, u_t^i, i = 1, 2, ..., n, be the amount invested in the ith risky asset at the beginning of the t-th time period. The amount investigated in the 0th risky asset at the beginning of the t-th time period is equal to x_t - $\sum_{i=1}^n u_t^i$. An investor is seeking a best investment strategy, $\mathbf{u}_t = [u_t^1, u_t^2, ..., u_t^n]'$ for t = 0, 1, 2, ..., T - 1, such that i) the expected value of the terminal wealth x_T, $E(x_T)$, is maximized subject to that the variance of the terminal wealth, $Var(x_T)$, is not greater than a preselected risk level, or ii) the variance of the terminal wealth, $Var(x_T)$, is minimized subject to that the expected terminal wealth, $E(x_T)$, is not smaller than a preselected level. Mathematically, a mean-variance formulation for multi-period portfolio selection can be posed as one of the following two forms when security 0 is taken as a reference:

$$(P1(\sigma)): \qquad \max \; E(x_T) \qquad \text{s.t. } Var(x_T) \leq \sigma \tag{1}$$

$$
\begin{aligned}
x_{t+1} &= \sum_{i=1}^{n} e_t^i u_t^i + \left(x_t - \sum_{i=1}^{n} u_t^i \right) e_t^0 \\
&= e_t^0 x_t + \mathbf{P_t}'\mathbf{u_t} \quad\quad t = 0, 1, 2, ..., T-1
\end{aligned}
$$

and

$$
\begin{aligned}
(P2(\epsilon)): \quad & \min \; Var(x_T) \\
& \text{s.t. } E(x_T) \geq \epsilon \\
& x_{t+1} = \sum_{i=1}^{n} e_t^i u_t^i + \left(x_t - \sum_{i=1}^{n} u_t^i \right) e_t^0 \\
& \quad\;\; = e_t^0 x_t + \mathbf{P_t}'\mathbf{u}_t \quad\quad t = 0, 1, 2, ..., T-1
\end{aligned}
\tag{2}
$$

where

$$
\mathbf{P_t} = \left[P_t^1, P_t^2, ..., P_t^n \right]' = \left[(e_t^1 - e_t^0), (e_t^2 - e_t^0), ..., (e_t^n - e_t^0) \right]'. \tag{3}
$$

Notice that $E(\mathbf{e_t}(\mathbf{e_t})') = Cov(\mathbf{e_t}) + E(\mathbf{e_t})E(\mathbf{e_t}')$. It is reasonable to assume in this paper that $E(\mathbf{e_t}(\mathbf{e_t})')$ is positive definite for all time periods, i.e.,

$$
\begin{aligned}
E(\mathbf{e_t}(\mathbf{e_t})') &= \begin{bmatrix} E((e_t^0)^2) & E(e_t^0 e_t^1) & \dots & E(e_t^0 e_t^n) \\ E(e_t^1 e_t^0) & E((e_t^1)^2) & \dots & E(e_t^1 e_t^n) \\ \dots & \dots & \dots & \dots \\ E(e_t^n e_t^0) & E(e_t^n e_t^1) & \dots & E((e_t^n)^2) \end{bmatrix} > 0 \\
& \quad\quad \forall\; t \;=\; 0, 1, \dots, T-1
\end{aligned}
\tag{4}
$$

The following is true from (4),

$$
\begin{aligned}
& \begin{bmatrix} E((e_t^0)^2) & E(e_t^0 \mathbf{P_t'}) \\ E(e_t^0 \mathbf{P_t}) & E(\mathbf{P_t P_t'}) \end{bmatrix} \\
= & \begin{bmatrix} 1 & 0 & \dots & 0 \\ -1 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ -1 & 0 & \dots & 1 \end{bmatrix} E(\mathbf{e_t}(\mathbf{e_t})') \begin{bmatrix} 1 & -1 & \dots & -1 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{bmatrix} > 0 \\
& \quad\quad \forall\; t \;=\; 0, 1, \dots, T-1
\end{aligned}
\tag{5}
$$

Further, we have the following from (5),

$$
E(\mathbf{P_t P_t'}) > 0 \quad\quad \forall\; t \;=\; 0, 1, \dots, T-1 \tag{6}
$$

and

$$E((e_t^0)^2) - E(e_t^0\mathbf{P_t'})E^{-1}(\mathbf{P_t}\mathbf{P_t'})E(e_t^0\mathbf{P_t}) > 0 \qquad \forall\, t \;=\; 0, 1, \ldots, T-1 \quad (7)$$

One of the advantages of adopting problem formulation $(P1(\sigma))$ or $(P2(\epsilon))$ in multi-period portfolio selection over the expected utility approach is that formulation $(P1(\sigma)$ or $(P2(\epsilon))$ enables an investor to specify a risk level which he can afford when he is seeking to maximize his expected terminal wealth or to specify an expected terminal wealth he would like to achieve when he is seeking to minimize the corresponding risk. It is easier and more direct for investors to provide this kind of subjective information than for them to construct a utility function in term of the terminal wealth.

A multi-period portfolio policy is an investment sequence,

$$\begin{aligned} \pi &= \{\mu_0, \mu_1, \mu_2, ..., \mu_{T-1}\} \\ &= \left\{ \begin{bmatrix} \mu_0^1 \\ \mu_0^2 \\ \vdots \\ \mu_0^n \end{bmatrix}, \begin{bmatrix} \mu_1^1 \\ \mu_1^2 \\ \vdots \\ \mu_1^n \end{bmatrix}, \begin{bmatrix} \mu_2^1 \\ \mu_2^2 \\ \vdots \\ \mu_2^n \end{bmatrix}, ..., \begin{bmatrix} \mu_{T-1}^1 \\ \mu_{T-1}^2 \\ \vdots \\ \mu_{T-1}^n \end{bmatrix} \right\} \end{aligned} \qquad (8)$$

More specifically, π is a feedback policy and μ_t maps the wealth at the beginning of the t-th period, x_t, into a portfolio decision in the t-th period,

$$\begin{bmatrix} u_t^1 \\ u_t^2 \\ \vdots \\ u_t^n \end{bmatrix} = \begin{bmatrix} \mu_t^1(x_t) \\ \mu_t^2(x_t) \\ \vdots \\ \mu_t^n(x_t) \end{bmatrix} \qquad (9)$$

A multi-period portfolio policy, π^*, is said to be efficient if there exists no other multi-period portfolio policy, π, such that $E(x_T)|_\pi \geq E(x_T)|_{\pi^*}$ and $Var(x_T)|_\pi \leq Var(x_T)|_{\pi^*}$ with at least one equality strictly. By varying the value of σ in $(P1(\sigma))$ or the value of ϵ in $(P2(\epsilon))$, the set of efficient multi-period portfolio policies can be generated.

An equivalent formulation to either $(P1(\sigma))$ or $(P2(\epsilon))$ in generating efficient multi-period portfolio policies is,

$$\begin{aligned} (E(w)): \qquad & \max E(x_T) - wVar(x_T) \qquad (10) \\ & \text{s.t. } x_{t+1} = e_t^0 x_t + \mathbf{P_t}'\mathbf{u}_t \quad t = 0, 1, 2, ..., T-1 \end{aligned}$$

where $w \in [0, \infty)$. It is well known that if π^* solves $(E(w))$, then π^* solves $(P1(\sigma))$ with $\sigma = Var\,(x_T)\,|_{\pi^*}$ and π^* solves $(P2(\epsilon))$ with $\epsilon = E\,(x_T)\,|_{\pi^*}$. Note the relationship $w = \frac{\partial E(x_T)}{\partial Var(x_T)}$ at the optimal solution of $(E(w))$. Problem formulation $(E(w))$ is preferable to be adopted in investment situations where an investor is able to specify his desirable trade-off between the expected terminal wealth and the associated risk.

3 Analytical Solution to the Multi-period Mean-Variance Formulation

Analytical solutions to all three problems $P1(\sigma)$, $P2(\epsilon)$ and $(E(w))$ are derived in this paper. The major results of the analytical optimal multi-period portfolio policy and the analytical expression of the mean-variance efficient frontier will be stated in this section while the detailed derivation of these results will be given in the next section.

Define

$$B_t = E\left(\mathbf{P_t}'\right) E^{-1}\left(\mathbf{P_t}\mathbf{P_t'}\right) E\left(\mathbf{P_t}\right) \quad t = 0, 1, ..., T-1 \tag{11}$$

$$A_t^1 = E\left(e_t^0\right) - E\left(\mathbf{P_t'}\right) E^{-1}\left(\mathbf{P_t}\mathbf{P_t'}\right) E\left(e_t^0\mathbf{P_t}\right) \quad t = 0, 1, ..., T-1 \tag{12}$$

$$A_t^2 = E\left((e_t^0)^2\right) - E\left(e_t^0\mathbf{P_t'}\right) E^{-1}\left(\mathbf{P_t}\mathbf{P_t'}\right) E\left(e_t^0\mathbf{P_t}\right) \quad t = 0, 1, ..., T-1 \tag{13}$$

$$B_t^1 = B_t \frac{\prod_{k=t+1}^{T-1} A_k^1}{2\prod_{k=t+1}^{T-1} A_k^2} \quad t = 0, 1, ..., T-1 \tag{14}$$

$$B_t^2 = B_t \left(\frac{\prod_{k=t+1}^{T-1} A_k^1}{2\prod_{k=t+1}^{T-1} A_k^2}\right)^2 \quad t = 0, 1, ..., T-1 \tag{15}$$

where in (14) and (15) $\prod_{k=T}^{T-1} A_k^i$, $i = 1, 2$, are defined to equal to one. Define further

$$\mu = \prod_{t=0}^{T-1} A_t^1 \tag{16}$$

$$\nu = \sum_{t=0}^{T-1}\left(\prod_{k=t+1}^{T-1} A_k^1\right) B_t^1 \tag{17}$$

$$\tau = \prod_{t=0}^{T-1} A_t^2 \tag{18}$$

$$a = \frac{\nu}{2} - \nu^2 \tag{19}$$

$$b = \frac{\mu\nu}{a} \tag{20}$$

$$c = \tau - \mu^2 - ab^2 \tag{21}$$

The optimal multi-period portfolio policy for problem $(E(w))$ is specified by the following analytical form:

$$\begin{aligned} \mathbf{u}_t^* &= -E^{-1}\left(\mathbf{P_t}\mathbf{P_t}'\right)E\left(e_t^0\mathbf{P_t}\right)x_t \\ &\quad +\frac{1}{2}(bx_0 + \frac{\nu}{2wa})\left(\prod_{k=t+1}^{T-1}\frac{A_k^1}{A_k^2}\right)E^{-1}\left(\mathbf{P_t}\mathbf{P_t}'\right)E\left(\mathbf{P_t}\right) \\ &\qquad \forall\ t = 0, 1, \ldots, T-2 \end{aligned} \tag{22}$$

$$\begin{aligned} \mathbf{u}_{T-1}^* &= -E^{-1}\left(\mathbf{P}_{T-1}\mathbf{P}_{T-1}'\right)E\left(e_{T-1}^0\mathbf{P}_{T-1}\right)x_{T-1} \\ &\quad +\frac{1}{2}(bx_0 + \frac{\nu}{2wa})E^{-1}\left(\mathbf{P}_{T-1}\mathbf{P}_{T-1}'\right)E\left(\mathbf{P}_{T-1}\right) \end{aligned} \tag{23}$$

The optimal multi-period portfolio policy for problems $(P1(\sigma))$ and $(P2(\epsilon))$ is specified by the following analytical form:

$$\begin{aligned} \mathbf{u}_t^* &= -E^{-1}\left(\mathbf{P_t}\mathbf{P_t}'\right)E\left(e_t^0\mathbf{P_t}\right)x_t \\ &\quad +\frac{1}{2}(bx_0 + \frac{\nu}{2w^*a})\left(\prod_{k=t+1}^{T-1}\frac{A_k^1}{A_k^2}\right)E^{-1}\left(\mathbf{P_t}\mathbf{P_t}'\right)E\left(\mathbf{P_t}\right) \\ &\qquad \forall\ t = 0, 1, \ldots, T-2 \end{aligned} \tag{24}$$

$$\begin{aligned} \mathbf{u}_{T-1}^* &= -E^{-1}\left(\mathbf{P}_{T-1}\mathbf{P}_{T-1}'\right)E\left(e_{T-1}^0\mathbf{P}_{T-1}\right)x_{T-1} \\ &\quad +\frac{1}{2}(bx_0 + \frac{\nu}{2w^*a})E^{-1}\left(\mathbf{P}_{T-1}\mathbf{P}_{T-1}'\right)E\left(\mathbf{P}_{T-1}\right) \end{aligned} \tag{25}$$

where

$$w^* = \begin{cases} \frac{\nu}{2\sqrt{a(\sigma - cx_0^2)}} & \text{when } (P1(\sigma)) \text{ is solved} \\ \frac{\nu^2}{2a[\epsilon - (\mu + b\nu)x_0]} & \text{when } (P2(\epsilon)) \text{ is solved} \end{cases} \tag{26}$$

The mean-variance efficient frontier for problems $(P1(\sigma))$, $(P2(\epsilon))$ and $(E(w))$ is specified by the following analytical form:

$$\begin{aligned} Var\left(x_T\right) &= \frac{a}{\nu^2}[E\left(x_T\right) - (\mu + b\nu)x_0]^2 + cx_0^2 \\ &\qquad \text{for } E\left(x_T\right) \geq (\mu + b\nu)x_0 \end{aligned} \tag{27}$$

With the analytical solution, the implementation of optimal multi-period portfolio policy for problem $(P1(\sigma))$, $(P2(\epsilon))$, or $(E(w))$ is straightforward. The optimal multi-period portfolio policy consists of two terms. The second term in u_t^* is dependent on the investor's risk attitude and is independent of his current wealth. It can be calculated off-line before the real investment process starts. The first term in u_t^* is dependent on the current wealth and is independent of the investor's risk attitude. It is calculated on-line at every time period when the current wealth is observed.

4 Derivation of the Analytical Solution

All three problems $(P1(\sigma))$, $(P2(\epsilon))$ and $(E(w))$ are difficult to be solved directly due to their nonseparability in the sense of dynamic programming. Variance minimization has been a notorious problem in stochastic control. Let I^t be an information set available at time t and $I^{t-1} \subset I^t$, $\forall\ t$. A key observation is that while the expectation operator satisfies the smoothing property: $E[E(\cdot \mid I^j) \mid I^k] = E(\cdot \mid I^k)$, $\forall\ j > k$, the variance operation does not: $Var[Var(\cdot \mid I^j) \mid I^k] \neq Var(\cdot \mid I^k)$, $\forall\ j > k$.

The optimal multi-period portfolio policy for problem $(E(w))$ will be first derived in this section. The solutions to problems $(P1(\sigma))$ and $(P2(\epsilon))$ will be then obtained based on the relationships between $(P1(\sigma))$, $(P2(\epsilon))$ and $(E(w))$.

A solution scheme adopted in this paper is to embed problem $(E(w))$ into a tractable auxiliary problem that is separable, investigate the relationship between the solution sets of problem $(E(w))$ and the auxiliary problem, and search for the solution of the auxiliary problem that attains the optimum point of problem $(E(w))$.

Define $\prod_E(w)$ to be the set of optimal solutions of problem $(E(w))$ with given w, i.e.

$$\Pi_E(w) = \{\pi | \pi \text{ is a maximizer of } (E(w))\} \tag{28}$$

Define

$$\begin{aligned} & \tilde{U}\left(E\left(x_T^2\right), E\left(x_T\right)\right) \\ = \ & E\left(x_T\right) - wVar\left(x_T\right) \\ = \ & -wE\left(x_T^2\right) + \left[wE^2\left(x_T\right) + E\left(x_T\right)\right] \end{aligned} \tag{29}$$

It is obvious that $\widetilde{U}$ is a convex function of $E\left(x_T^2\right)$ and $E\left(x_T\right)$. The following auxiliary problem is now constructed for $(E\left(w\right))$,

$$\begin{aligned} \left(A\left(\lambda, w\right)\right) \quad : \quad & \max E\left\{-w x_T^2 + \lambda x_T\right\} \\ & \text{s.t. } x_{t+1} = e_t^0 x_t + \mathbf{P_t}' \mathbf{u}_t \quad t = 0, 1, 2, ..., T-1 \end{aligned} \tag{30}$$

Prominent features of problem $\left(A\left(\lambda, w\right)\right)$ are that $\left(A\left(\lambda, w\right)\right)$ is of a separable structure in the sense of dynamic programming and the objective function of $\left(A\left(\lambda, w\right)\right)$ is of a quadratic form while the system dynamic is of a linear form. Define $\Pi_A\left(\lambda, w\right)$ to be the set of the optimal solutions of problem $\left(A\left(\lambda, w\right)\right)$ for given λ and w, i.e.,

$$\Pi_A\left(\lambda, w\right) = \left\{\pi | \pi \text{ is a maximizer of } \left(A\left(\lambda, w\right)\right)\right\} \tag{31}$$

Denote

$$\begin{aligned} d(\pi, w) &= \frac{\partial \widetilde{U}\left(E\left(x_T^2\right), E\left(x_T\right)\right)}{\partial E\left(x_T\right)}|_{\pi} \\ &= 1 + 2wE\left(x_T\right)|_{\pi} \end{aligned} \tag{32}$$

Theorem 1 *For any $\pi^* \in \Pi_E\left(w\right)$, $\pi^* \in \Pi_A\left(d\left(\pi^*, w\right), w\right)$.*

Proof By contradiction, assume that $\pi^* \notin \Pi_A\left(d\left(\pi^*, w\right), w\right)$. Then, there exists a π such that

$$\left[-w, d\left(\pi^*, w\right)\right] \left[\begin{array}{c} E\left(x_T^2\right) \\ E\left(x_T\right) \end{array}\right]\Bigg|_{\pi} > \left[-w, d\left(\pi^*, w\right)\right] \left[\begin{array}{c} E\left(x_T^2\right) \\ E\left(x_T\right) \end{array}\right]\Bigg|_{\pi^*} \tag{33}$$

Notice (32) and

$$\frac{\partial \widetilde{U}\left(E\left(x_T^2\right), E\left(x_T\right)\right)}{\partial E\left(x_T^2\right)} = -w \tag{34}$$

Since $\widetilde{U}$ is a convex function of $E\left(x_T^2\right)$ and $E\left(x_T\right)$, the following property is satisfied,

$$\begin{aligned} & \widetilde{U}\left(E\left(x_T^2\right), E\left(x_T\right)\right)|_{\pi} \geq \widetilde{U}\left(E\left(x_T^2\right), E\left(x_T\right)\right)|_{\pi^*} \\ & + \left[-w, d\left(\pi^*, w\right)\right] \left\{\left[\begin{array}{c} E\left(x_T^2\right) \\ E\left(x_T\right) \end{array}\right]\Bigg|_{\pi} - \left[\begin{array}{c} E\left(x_T^2\right) \\ E\left(x_T\right) \end{array}\right]\Bigg|_{\pi^*}\right\} \end{aligned} \tag{35}$$

Combining (33) and (35) yields

$$\widetilde{U}\left(E\left(x_T^2\right), E\left(x_T\right)\right)|_{\pi} > \widetilde{U}\left(E\left(x_T^2\right), E\left(x_T\right)\right)|_{\pi^*} \tag{36}$$

that contradicts the assumption of $\pi^* \in \Pi_E(w)$. Q.E.D.

The implication of Theorem 1 is that the solution set for problem $(E(w))$ is a subset of the solution set for problem $(A(\lambda, w))$. We can embed the non-tractable primal problem $(E(w))$ into a tractable auxiliary problem $(A(\lambda, w))$ with a quadratic utility function. The following theorem provides a necessary condition under which a solution of $(A(\lambda, w))$ constitutes an optimal multi-period portfolio policy of $(E(w))$.

Theorem 2 *Assume* $\pi^* \in \Pi_A(\lambda^*, w)$. *A necessary condition for* $\pi^* \in \Pi_E(w)$ *is* $\lambda^* = 1 + 2wE(x_T)|_{\pi^*}$.

Proof For a given w, the solution set of $(A(\lambda, w))$ can be parameterized by λ. In other words, each point in $\cup_\lambda \Pi_A(\lambda, w)$ can be expressed in terms of λ as $\{E(x_T^2(\lambda, w)), E(x_T(\lambda, w))\}$. Since $\Pi_E(w) \subseteq \cup_\lambda \Pi_A(\lambda, w)$, problem $(E(w))$ can be reduced in abstract to the following equivalent form

$$\begin{aligned} & \max_\lambda \widetilde{U}\left(E(x_T^2(\lambda, w)), E(x_T(\lambda, w))\right) \\ = \; & \max_\lambda -wE\left(x_T^2(\lambda, w)\right) + \left[wE^2(x_T(\lambda, w)) + E(x_T(\lambda, w))\right] \end{aligned} \quad (37)$$

A first-order necessary optimality condition for optimal λ^* is

$$-w\frac{\partial E(x_T^2(\lambda^*, w))}{\partial \lambda} + [1 + 2wE(x_T)|_{\pi^*}]\frac{\partial E(x_T(\lambda^*, w))}{\partial \lambda} = 0 \quad (38)$$

On the other side, when $\pi^* \in \Pi_A(\lambda^*, w)$, we have the following from [22],

$$-w\frac{\partial E(x_T^2(\lambda^*, w))}{\partial \lambda} + \lambda^*\frac{\partial E(x_T(\lambda^*, w))}{\partial \lambda} = 0 \quad (39)$$

Hence, the vector $[-w, (1 + 2wE(x_T)|_{\pi^*})]$ is proportional to $[-w, \lambda^*]$. We must have $\lambda^* = 1 + 2wE(x_T)|_{\pi^*}$. Q.E.D.

The optimal solution of the auxiliary problem $(A(\lambda, w))$ can be derived analytically using dynamic programming [12]. The optimal portfolio policy for auxiliary problem $(A(\lambda, w))$ at each time period t is of the following form,

$$\mathbf{u}_t^*(x_t; \gamma) = -\mathbf{K}_t x_t + \mathbf{v}_t(\gamma) \qquad t = 0, 1, \ldots, T-1 \quad (40)$$

where

$$\gamma = \frac{\lambda}{w} \quad (41)$$

$$\mathbf{K}_t = E^{-1}\left(\mathbf{P_t}'\mathbf{P}_t\right) E\left(e_t^0 \mathbf{P_t}\right) \tag{42}$$

$$\mathbf{v}_t(\gamma) = \frac{\gamma}{2}\left(\prod_{k=t+1}^{T-1}\frac{A_k^1}{A_k^2}\right) E^{-1}\left(\mathbf{P_t}\mathbf{P}_t'\right) E\left(\mathbf{P_t}\right) \quad t = 0, 1, 2, .., T-2 \tag{43}$$

with the following boundary condition,

$$\mathbf{v}_{T-1}(\gamma) = \frac{\gamma}{2} E^{-1}\left(\mathbf{P}_{T-1}\mathbf{P}_{T-1}'\right)^{-1} E\left(\mathbf{P}_{T-1}\right) \tag{44}$$

where A_t^1 and A_t^2 are defined in (12) and (13), respectively. Substituting (40) into the equation of wealth dynamics yields the dynamics of the wealth under portfolio policy $\mathbf{u}_t^*(x_t; \gamma)$,

$$x_{t+1}(\gamma) = \left(e_t^0 - \mathbf{P_t}'\mathbf{K}_t\right) x_t(\gamma) + \mathbf{P_t}'\mathbf{v}_t(\gamma) \tag{45}$$

Taking expectation on both sides of (45) and noticing the statistical independence between $(e_t^0, \mathbf{P_t})$ and x_t, we have the following recursive expression for the expected wealth between successive time periods under portfolio policy $\mathbf{u}_t^*(x_t; \gamma)$,

$$E\left(x_{t+1}(\gamma)\right) = A_t^1 E\left(x_t(\gamma)\right) + \frac{\gamma}{2}\left(\prod_{k=t+1}^{T-1}\frac{A_k^1}{A_k^2}\right) B_t \tag{46}$$

where B_t is defined in (11). Taking square on both sides of (45) yields,

$$\begin{aligned} x_{t+1}^2(\gamma) &= \left[\left(e_t^0\right)^2 - 2e_t^0\mathbf{P_t}'\mathbf{K}_t + \mathbf{K}_t'\mathbf{P_t}\mathbf{P}_t'\mathbf{K}_t\right] x_t^2(\gamma) \\ &\quad + 2\left(e_t^0 - \mathbf{P_t}'\mathbf{K}_t\right) x_t(\gamma)\,\mathbf{P_t}'\mathbf{v}_t(\gamma) + \mathbf{v}_t(\gamma)'\,\mathbf{P_t}\mathbf{P_t}'\mathbf{v}_t(\gamma) \\ &\qquad t = 0, 1, \ldots, T-1 \end{aligned} \tag{47}$$

Taking expectation on both sides of the above equation and simplifying the resulted expression lead to the following recursive expression for the expected value of the square wealth between successive time periods under portfolio policy $\mathbf{u}_t^*(x_t; \gamma)$,

$$E\left(x_{t+1}^2(\gamma)\right) = A_t^2 E\left(x_t^2(\gamma)\right) + \frac{\gamma^2}{4}\left(\prod_{k=t+1}^{T-1}\frac{A_k^1}{A_k^2}\right)^2 B_t \tag{48}$$

Solving two recursive equations (46) and (48) yields explicit expressions for the expected values of the terminal wealth and the square of the terminal wealth under portfolio policy $\mathbf{u}_t^*(x_t;\gamma)$,

$$E(x_T(\gamma)) = \mu x_0 + \nu\gamma \tag{49}$$

$$E\left(x_T^2(\gamma)\right) = \tau x_0^2 + \frac{\nu}{2}\gamma^2 \tag{50}$$

where μ, ν and τ are defined in (16), (17) and (18), respectively.

The variance of the terminal wealth under portfolio policy $\mathbf{u}_t^*(x_t;\gamma)$ can be expressed in terms of γ using (49) and (50),

$$\begin{aligned} Var(x_T(\gamma)) &= E\left(x_T^2(\gamma)\right) - E^2(x_T(\gamma)) \\ &= a(\gamma - bx_0)^2 + cx_0^2 \end{aligned} \tag{51}$$

where a, b and c are defined in (19), (20) and (21), respectively.

It can be seen that the expected terminal wealth $E(x_T(\gamma))$ is an increasing linear function of γ whereas the variance of the terminal wealth, $Var(x_T(\gamma))$, is a quadratic function of γ. From (49) and (51), we can express $\widetilde{U}(E(x_T^2), E(x_T))$ as a function of γ,

$$\begin{aligned} &\widetilde{U}\left(E\left(x_T^2\right), E(x_T)\right) \\ = \; &\mu x_0 + \nu\gamma - w[a(\gamma - bx_0)^2 + cx_0^2] \end{aligned} \tag{52}$$

Clearly, $\widetilde{U}$ is a concave function of γ. Differentiating (52) with respect to γ yields,

$$\frac{d\widetilde{U}}{d\gamma} = \nu - 2wa(\gamma - bx_0) \tag{53}$$

The optimal γ must satisfy the optimality condition of $\frac{d\widetilde{U}}{d\gamma} = 0$, i.e.,

$$\gamma^* = bx_0 + \frac{\nu}{2wa} \tag{54}$$

Substituting the optimal γ^* in (54) into (40) yields the optimal multi-period portfolio policy for $(E(w))$ specified in (22) and (23).

Substituting (54) into (49) and (51) yields the expression for the expected value and the variance of the terminal wealth on the efficient frontier in terms of w,

$$E(x_T(w)) = (\mu + b\nu)x_0 + \frac{\nu^2}{2wa} \tag{55}$$

$$Var\left(x_T\left(w\right)\right) = \frac{\nu^2}{4aw^2} + cx_0^2 \tag{56}$$

Given a problem $(P1(\sigma))$ or $(P2(\epsilon))$, we can first calculate the associated w in terms of σ or ϵ using (55) or (56) and then compute the corresponding optimal γ^* using (54). Substituting the optimal γ^* into (40) yields the optimal multi-period portfolio policy for $(P1(\sigma))$ or $(P2(\epsilon))$ specified in (24), (25) and (26).

The mean-variance efficient frontier given in (27) can be obtained by eliminating the parameter w in (55) and (56).

5 Investment Situations with One Risk-less Asset

Investment situations where there exists a riskless asset can be regarded as a special case in the general multi-period mean-variance formulation discussed above. Let the 0th security be riskless. In other words, we consider now a capital market with n risky assets and a riskless asset offering a sure rate of return. In this case e_t^0 equals to a constant s_t and $cov\left(e_t^0, e_t^i\right) = 0$, $i = 0, 1, ..., n$, $\forall\ t = 0, 1, \ldots, T-1$. The parameters defined in (11) - (15) now take the following forms,

$$\begin{aligned} B_t &= E\left(\mathbf{P_t'}\right) E^{-1}\left(\mathbf{P_t}\mathbf{P_t'}\right) E\left(\mathbf{P_t}\right) \quad t = 0, 1, ..., T-1 && (57) \\ A_t^1 &= s_t\left(1 - B_t\right) \quad t = 0, 1, ..., T-1 && (58) \\ A_t^2 &= s_t^2\left(1 - B_t\right) \quad t = 0, 1, ..., T-1 && (59) \\ B_t^1 &= \frac{B_t}{2\prod_{k=t+1}^{T-1} s_k} \quad t = 0, 1, ..., T-1 && (60) \\ B_t^2 &= \frac{B_t}{4(\prod_{k=t+1}^{T-1} s_k)^2} \quad t = 0, 1, ..., T-1 && (61) \end{aligned}$$

where in (60) and (61) $\prod_{k=T}^{T-1} s_k$ is defined to equal to one. The expressions for μ, ν, τ, a, b, and c in (16) - (21) can be then simplified to the following using (57) to (61),

$$\mu = \prod_{t=0}^{T-1} s_t\left(1 - B_t\right) \tag{62}$$

$$\nu = \frac{1}{2}[1 - \prod_{t=0}^{T-1}\left(1 - B_t\right)] \tag{63}$$

$$\tau = \prod_{t=0}^{T-1} s_t^2 (1 - B_t) \tag{64}$$

$$a = \frac{1}{4} \prod_{t=0}^{T-1} (1 - B_t) [1 - \prod_{t=0}^{T-1} (1 - B_t)] \tag{65}$$

$$b = 2 \prod_{t=0}^{T-1} s_t \tag{66}$$

$$c = 0 \tag{67}$$

Notice that the relationship of $\prod_{t=0}^{T-1}(1 - B_t) = 1 - \sum_{t=0}^{T-1} \prod_{k=t+1}^{T-1}(1 - B_k)B_t$ is used in the above derivation.

The optimal parameter γ^* for problem $(E(w))$ in the investment situations with a riskless asset can be found using (54), (63), (65) and (66),

$$\gamma^* = 2 \prod_{t=0}^{T-1} s_t x_0 + \frac{1}{w \left(\prod_{t=0}^{T-1} (1 - B_t) \right)} \tag{68}$$

The optimal portfolio policy for problem $(E(w))$ in the investment situations with a riskless asset is given as follows from (54), (22), (23) and (68),

$$\begin{aligned} & \mathbf{u}_t^* = -s_t E^{-1} (\mathbf{P_t} \mathbf{P_t}') E (\mathbf{P_t}) x_t \\ + \; & [\prod_{t=0}^{T-1} s_t x_0 + \frac{1}{2w \left(\prod_{t=0}^{T-1} (1 - B_t) \right)}] \left(\prod_{k=t+1}^{T-1} \frac{1}{s_k} \right) E^{-1} (\mathbf{P_t} \mathbf{P}_t') E (\mathbf{P_t}) \\ & \qquad t = 0, 1, \ldots, T-2 \end{aligned} \tag{69}$$

$$\begin{aligned} & \mathbf{u}_{T-1}^* = -s_{T-1} E^{-1} \left(\mathbf{P}_{T-1} \mathbf{P}_{T-1}' \right) E (\mathbf{P}_{T-1}) x_{T-1} \\ + \; & [\prod_{t=0}^{T-1} s_t x_0 + \frac{1}{2w \left(\prod_{t=0}^{T-1} (1 - B_t) \right)}] E^{-1} \left(\mathbf{P}_{T-1} \mathbf{P}_{T-1}' \right) E (\mathbf{P}_{T-1}) \end{aligned} \tag{70}$$

The expected terminal wealth and the variance of the terminal wealth under the optimal portfolio policy $\mathbf{u}_t^*$ in the investment situations with a riskless asset are given as follows using (55), (56), (62), (63), (65), (66), and (67),

$$E (x_T) = \prod_{t=0}^{T-1} s_t x_0 + \frac{\left(1 - \prod_{t=0}^{T-1} (1 - B_t) \right)}{2w \left(\prod_{t=0}^{T-1} (1 - B_t) \right)} \tag{71}$$

$$Var\left(x_T\right) = \frac{\left(1-\prod_{t=0}^{T-1}\left(1-B_t\right)\right)}{4w^2\prod_{t=0}^{T-1}\left(1-B_t\right)} \tag{72}$$

The optimal portfolio policy for $(P1(\sigma))$ and $(P2(\epsilon))$ in the investment situations with a riskless asset is given as follows from (54), (24), (25), (68), (71) and (72),

$$\mathbf{u}_t^* = -s_t E^{-1}\left(\mathbf{P_t}\mathbf{P_t}'\right) E\left(\mathbf{P_t}\right) x_t$$
$$+ \; [\prod_{t=0}^{T-1} s_t x_0 + \frac{1}{2w^*\left(\prod_{t=0}^{T-1}\left(1-B_t\right)\right)}]\left(\prod_{k=t+1}^{T-1}\frac{1}{s_k}\right) E^{-1}\left(\mathbf{P_t}\mathbf{P}_t'\right) E\left(\mathbf{P_t}\right)$$
$$t = 0, 1, \ldots, T-2 \tag{73}$$

$$\mathbf{u}_{T-1}^* = -s_{T-1} E^{-1}\left(\mathbf{P}_{T-1}\mathbf{P}_{T-1}'\right) E\left(\mathbf{P}_{T-1}\right) x_{T-1}$$
$$+ \; [\prod_{t=0}^{T-1} s_t x_0 + \frac{1}{2w^*\left(\prod_{t=0}^{T-1}\left(1-B_t\right)\right)}] E^{-1}\left(\mathbf{P}_{T-1}\mathbf{P}_{T-1}'\right) E\left(\mathbf{P}_{T-1}\right) \tag{74}$$

where

$$w^* = \begin{cases} \frac{1}{2}\sqrt{\frac{\left(1-\prod_{t=0}^{T-1}(1-B_t)\right)}{\sigma\prod_{t=0}^{T-1}(1-B_t)}} & \text{when } (P1(\sigma)) \text{ is solved} \\ \frac{\left(1-\prod_{t=0}^{T-1}(1-B_t)\right)}{2(\epsilon-\prod_{t=0}^{T-1}s_t x_0)\left(\prod_{t=0}^{T-1}(1-B_t)\right)} & \text{when } (P2(\epsilon)) \text{ is solved} \end{cases} \tag{75}$$

Finally, the analytical expression of the mean-variance efficient frontier in (27) can be reduced to the following simpler form for situations with a riskless asset using (62) - (67),

$$Var\left(x_T\right) = \frac{\prod_{t=0}^{T-1}\left(1-B_t\right)}{1-\prod_{t=0}^{T-1}\left(1-B_t\right)}\left(E\left(x_T\right) - x_0\prod_{t=0}^{T-1}s_t\right)^2$$
$$\text{for } E\left(x_T\right) \geq \prod_{t=0}^{T-1} s_t x_0 \tag{76}$$

Notice that one end point of the mean-variance efficient frontier is the point with $E(x_T) = \prod_{t=0}^{T-1} s_t x_0$ and $Var(x_T) = 0$ that is associated with the investment decision with which the investor keeps all his money in the riskless asset.

When setting T = 1 in our formulation, problems $(P1(\sigma))$ and $(P2(\epsilon))$ are reduced to the single-period mean-variance formulation [13]. It can be verified [19] that the expressions of the efficient frontier are exactly the same as given in this paper and in Eq. (35) of Merton [16] when setting T = 1. The work of multi-period mean-variance approach presented in this paper can be viewed as a generalization of the analytical work in single-period mean-variance formulation by Merton [16].

6 Multi-period Portfolio Selection via Maximizing Utility function $U(E(x_T), Var(x_T))$

We consider in this section a more general problem formulation for multi-period portfolio selection. The objective of an investor now is to maximize $U(E(x_T), Var(x_T))$, a utility that is a function of the expected value and the variance of the terminal wealth x_T. Since investors always would like to maximize their final wealth with a low risk level, utility function $U(E(x_T), Var(x_T))$ is assumed to satisfy the following,

$$\frac{\partial U(E(x_T), Var(x_T))}{\partial E(x_T)} > 0 \tag{77}$$

and

$$\frac{\partial U(E(x_T), Var(x_T))}{\partial Var(x_T)} < 0 \tag{78}$$

Knoll et al. [11] found that solutions via maximizing $U(E(x_T), Var(x_T))$ and via direct utility maximization are highly correlated to each other.

The following multi-period portfolio selection problem is formulated,

$$\begin{aligned} (U): \quad & \max \quad U(E(x_T), Var(x_T)) \\ & \text{s.t. } x_{t+1} = e_t^0 x_t + \mathbf{P_t}'\mathbf{u}_t \quad t = 0, 1, 2, ..., T-1 \end{aligned} \tag{79}$$

Define Π_U to be the set of the optimal solutions of problem (U), i.e.

$$\Pi_U = \{\pi | \pi \text{ is the maximizer of } (U)\} \tag{80}$$

Problem formulation (U) covers a general class of multi-period portfolio selection problems. A utility function, in general, can be nonlinear with respect to $E(x_T)$ and $Var(x_T)$. The multi-period mean-variance formulation discussed in the previous sections can be seen as a special case of problem formulation (U) where the utility function is linear with respect to $E(x_T)$ and $Var(x_T)$.

Lemma 1 *If $\pi^* \in \Pi_U$, then there exists a $w > 0$ such that $\pi^* \in \Pi_E(w)$.*

Proof Since U is an increasing function of $E(x_T)$ and a decreasing function of $Var(x_T)$, the optimal solution of (U) must be on the mean-variance efficient frontier in the $\{E(x_T), Var(x_T)\}$ space. It is known from (27) that $Var(X_T)$ is a convex function of $E(x_T)$ on the efficient frontier. Thus supporting lines exist everywhere on the efficient frontier in the $\{E(x_T), Var(x_T)\}$ space. In other words, every efficient solution, including $\pi^* \in \Pi_U$, can be generated by the auxiliary problem $(E(w))$. Q.E.D.

Define the following

$$U_E(\pi) = \frac{\partial U(E(x_T), Var(x_T))}{\partial E(x_T)}|_\pi \tag{81}$$

$$U_V(\pi) = \frac{\partial U(E(x_T), Var(x_T))}{\partial Var(x_T)}|_\pi \tag{82}$$

Theorem 3 *Assume $\pi^* \in \Pi_E(w^*)$. A necessary condition for $\pi^* \in \Pi_U$ is $w^* = -\frac{U_V(\pi^*)}{U_E(\pi^*)}$.*

Proof The efficient frontier in the $\{E(x_T), Var(x_T)\}$ space can be parameterized by the coefficient w. In other words, each point on the efficient frontier can be represented by $(E(x_T(w)), Var(x_T(w)))$. Since $\Pi_U \subseteq \cup_{w\geq 0}\Pi_E(w)$, problem (U) can be reduced in abstract to the following equivalent form,

$$\max_{w\geq 0} \quad U(E(x_T(w)), Var(x_T(w))) \tag{83}$$

A first-order necessary condition for optimum $w^* > 0$ is

$$U_E(\pi^*)\frac{\partial E(x_T(w^*))}{\partial w} + U_V(\pi^*)\frac{\partial Var(x_T(w^*))}{\partial w} = 0 \tag{84}$$

On the other hand, when $\pi^* \in \Pi_E(w^*)$, we have from [22],

$$\frac{\partial E(x_T(w^*))}{\partial w} - w^*\frac{\partial Var(x_T(w^*))}{\partial w} = 0 \tag{85}$$

Thus vector $[U_E(\pi^*), U_V(\pi^*)]$ is proportional to $[1, -w^*]$. We must have $w^* = -\frac{U_V(\pi^*)}{U_E(\pi^*)}$. Q.E.D.

Lemma 1 implies that problem (U) can be embedded into problem $E(w)$. Theorem 3 gives a necessary condition for a solution of $E(w)$ to attain the optimum of (U). Problem $(E(w))$ can be further embedded into the auxiliary problem $(A(\lambda, w))$ as we know from the previous sections. Thus we can conclude that a multi-period portfolio problem of maximizing $U(E(x_T), Var(x_T))$ can be also embedded into $(A(\lambda, w))$. The following theorem gives the condition for optimal parameters with which the solution of $(A(\lambda, w))$ attains the optimal point of (U).

Theorem 4 *Assume* $\pi^* \in \Pi_A(\lambda^*, w^*)$. *Necessary conditions for* $\pi^* \in \Pi_U$ *are* $w^* = -\frac{U_V(\pi^*)}{U_E(\pi^*)}$ *and* $\lambda^* = 1 - 2\frac{U_V(\pi^*)}{U_E(\pi^*)}E(x_T)|_{\pi^*}$.

Proof The theorem can be easily proven by combining Theorems 2 and 3. Q.E.D.

The optimal solution for problem $(A(\lambda, w))$ is provided for given $\gamma = \frac{\lambda}{w}$. The computational procedure to obtain the optimal γ^* can be constructed by studying the derivative of U with respect to γ. The derivative of the utility function with respect to γ can be obtained using the following formula,

$$\frac{dU}{d\gamma} = \left(\frac{\partial U}{\partial E(x_T)} - 2E(x_T)\frac{\partial U}{\partial Var(x_T)}\right)\nu + \frac{\partial U}{\partial Var(x_T)}\nu\gamma \tag{86}$$

where $\frac{dE(x_T)}{d\gamma} = \nu$ and $\frac{dE(x_T^2)}{d\gamma} = \nu\gamma$ are used in the above equation based on (49) and (50).

By setting $\frac{dU}{d\gamma}$ in (86) equal to zero, we have the following necessary optimum condition for γ,

$$\left(\frac{\partial U}{\partial E(x_T)} - 2E(x_T)\frac{\partial U}{\partial Var(x_T)}\right) + \frac{\partial U}{\partial Var(x_T)}\gamma = 0 \tag{87}$$

i.e.

$$\gamma^* = 2E(x_T) - \frac{\partial U}{\partial E(x_T)} / \frac{\partial U}{\partial Var(x_T)} \tag{88}$$

As the derivative of $\frac{dU}{d\gamma}$ is obtainable for given γ, a numerical search method using gradient information, such as the gradient method or the false position method, can be adopted to update the value of γ in $(A(\lambda, w))$. The search process for optimal γ continues until the stopping condition (88) is satisfied. Notice that both $E(x_T)$ and $Var(x_T)$ are dependent on parameter γ. Substituting the optimal value of γ^* into (40) yields the optimal portfolio policy for problem (U). The search algorithm is straightforward and only involves a one-dimensional search.

7 Illustrative Cases

Three exemplary cases are given in this section to demonstrate the adoption of the multi-period mean-variance formulations and the efficiency of the solution methods derived in this paper.

Example 1 *Consider the case study in Chapter 7 of Sharpe et al. [24] by assuming a stationary multi-period process with* $T = 4$. *An investor has one unit wealth in the very beginning of the planning horizon. The investor is trying to find the best allocation of his wealth among three risky securities, A, B, and C in order to maximize* $E(x_4)$ *while keeping his risk not exceeding 2, i.e.* $\sigma = 2$. *The expected returns for risky securities, A, B, and C are* $E(e_t^A) = 1.162$, $E(e_t^B) = 1.246$, *and* $E(e_t^C) = 1.228$, t *= 0, 1, 2, 3. The covariance of* $\mathbf{e_t} = [e^A, e^B, e^C]'$ *is* $Cov(\mathbf{e_t}) = \begin{bmatrix} 0.0146 & 0.0187 & 0.0145 \\ 0.0187 & 0.0854 & 0.0104 \\ 0.0145 & 0.0104 & 0.0289 \end{bmatrix}$, t *= 0, 1, 2, 3.*

Take security A as the reference asset. Thus, $E(\mathbf{P_t}) = E[e_t^B - e_t^A, e_t^C - e_t^A]' = [0.084, 0.066]'$, t = 0, 1, 2, 3, $E(\mathbf{P_t P_t'}) = E\begin{bmatrix} (e^B)^2 - 2e^Ae^B + (e^A)^2 & e^Be^C - e^Ae^C - e^Ae^B + (e^A)^2 \\ e^Be^C - e^Ae^C - e^Ae^B + (e^A)^2 & (e^C)^2 - 2e^Ae^C + (e^A)^2 \end{bmatrix} = \begin{bmatrix} 0.0697 & -0.0027 \\ -0.0027 & 0.0189 \end{bmatrix}$, t = 0, 1, 2, 3. $E(e^A\mathbf{P_t'}) = \left[E\left(e_t^Ae_t^B\right) - E\left((e_t^A)^2\right), E\left(e_t^Ae_t^C\right) - E\left((e_t^A)^2\right)\right] = [0.1017, 0.0766]$, t = 0, 1, 2, 3. Furthermore, we have $B_t = 0.3566$, $A_t^1 = 0.7424$, $A_t^2 = 0.8711$, t = 0, 1, 2, 3, $\mu = 0.3038$, $\nu = 0.4077$, $a = 0.0376$, $b = 3.2933$, and $c = 0.0754$.

The mean-variance efficient frontier in this example problem is given as follows using (27),

$$Var(x_4) = 0.2262[E(x_4) - 1.6465]^2 + 0.0754$$

where $E(x_4) \geq 1.6465$.

From (26), the corresponding w^* in $(E(w))$ is 0.75773. The associated optimal portfolio policy is given as follows using (24) and (25),

$$u_t^* = -\mathbf{K}_t x_t + \mathbf{v}_t$$

where $\mathbf{K}_t = \begin{bmatrix} 1.6238 \\ 4.2907 \end{bmatrix}$, $t = 0, 1, 2, 3$, $\mathbf{v}_0 = \begin{bmatrix} 4.3548 \\ 11.9327 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 5.1094 \\ 14.0004 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 5.9948 \\ 16.4263 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 7.0335 \\ 19.2726 \end{bmatrix}$. The investment in the first security, asset A, at period t is equal to $(x_t - \sum u_t^i)$. The corresponding expected terminal wealth and the risk level are given by $E(x_4) = 4.5632$ and $Var(x_4) = 2$, respectively, using (55) and (56).

Example 2 *Consider now a modified version of Example 1. In addition to the three risky assets, A, B, and C, there exists a riskless asset with a sure return rate of 1.04. Suppose this time that the investor seeks an efficient portfolio policy with a desired trade-off between the expected terminal wealth and risk,* $\frac{\partial E(x_4)}{\partial Var(x_4)} = 2$. *More directly, the investor would like to maximize* $E(x_4)$ - $2Var(x_4)$.

We can calculate $E(\mathbf{P_t}) = E[e_t^A - s_t, e_t^B - s_t, e_t^C - s_t]' = [0.122, 0.206, 0.188]'$, $t = 0, 1, 2, 3$, and $E(P_t P_t') = Cov(e) + E(P_t)E(P_t') = \begin{bmatrix} 0.0295 & 0.0438 & 0.0374 \\ 0.0438 & 0.1278 & 0.0491 \\ 0.0374 & 0.0491 & 0.0642 \end{bmatrix}$, $t = 0, 1, 2, 3$. Furthermore, we have $B_t = E(P_t')E^{-1}(P_t P_t')E(P_t) = 0.593817$, $t = 0, 1, 2, 3$.

The mean-variance efficient frontier in this case is given as follows by using (76),

$$Var(x_4) = 0.02798[E(x_4) - 1.1699]^2$$

where $E(x_4) \geq 1.1699$.

The associated optimal portfolio policy is given as follows using (69) and (70),

$$u_t^* = -\mathbf{K}_t x_t + \mathbf{v}_t$$

where $\mathbf{K}_t = \begin{bmatrix} 0.4004 \\ 0.6496 \\ 2.3133 \end{bmatrix}$, $t = 0, 1, 2, 3$, $\mathbf{v}_0 = \begin{bmatrix} 3.5440 \\ 5.7494 \\ 20.4751 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 3.6858 \\ 5.9794 \\ 21.2941 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 3.8332 \\ 6.2185 \\ 22.1459 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 3.9865 \\ 6.4673 \\ 23.0317 \end{bmatrix}$. The investment in the riskless asset at period t is equal to $(x_t - \sum u_t^i)$. The corresponding expected terminal wealth and the risk level are $E(x_4) = 10.1043$ and $Var(x_4) = 2.2336$, respectively, using (71) and (72).

Example 3 *Consider Example 2 again. But this time the investor seeks an optimal portfolio policy that maximizes the following utility function*

$$U\left(E\left(x_4\right), Var\left(x_4\right)\right) = E^2\left(x_4\right) - \exp\left[Var\left(x_4\right)\right]$$

The derivative of U with respect to γ can be obtained from (86) and (63),

$$\frac{dU}{d\gamma} = 0.97278E\left(x_4\right)\left[1 + \exp\left(Var\left(x_4\right)\right)\right] - 0.48639\exp\left(Var\left(x_4\right)\right)\gamma$$

Adopting the false position method, the optimal value of γ^* is found to be equal to 25.8965 at which $\frac{dU}{d\gamma} = 0$ and U attains its maximum of 120.0707. The associated optimal portfolio policy is given by

$$u_t^* = -\mathbf{K}_t x_t + \mathbf{v}_t$$

where $\mathbf{K}_t = \begin{bmatrix} 0.4004 \\ 0.6496 \\ 2.3133 \end{bmatrix}$, $t = 0, 1, 2, 3$, $\mathbf{v}_0 = \begin{bmatrix} 4.4318 \\ 7.1897 \\ 25.6044 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 4.6091 \\ 7.4773 \\ 26.6286 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 4.7935 \\ 7.7764 \\ 27.6937 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 4.9852 \\ 8.0874 \\ 28.8015 \end{bmatrix}$. The corresponding expected terminal wealth and the risk are $E(x_4) = 12.6276$ and $Var(x_4) = 3.6734$, respectively.

8 Conclusions

The Markowitz's mean-variance approach has been extended in this paper to multi-period portfolio selection problems. With a solution scheme using embedding, analytical solution has been derived for the multi-period mean-variance formulation that is nontractable in its original setting. The derived analytical expression of the efficient frontier for the multi-period portfolio selection will definitely better the investors' understanding of the trade-off between the expected terminal wealth and the risk. At the same time, the derived analytical optimal multi-period portfolio policy provides investors the best strategy to follow in a dynamic investment environment. One of the future research subjects is to investigate an efficient solution methodology for the constrained multi-period mean-variance formulation.

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