A Global Optimization Heuristic for Portfolio Choice with VaR and Expected Shortfall

Manfred Gilli and Evis Këllezi*

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*M. Gilli: Department of Econometrics, 40 Bd du Pont d'Arve, University of Geneva, 1211 Geneva 4, Switzerland, email Manfred.Gilli@metri.unige.ch. E. Këllezi: Department of Econometrics and FAME, University of Geneva, 40 Bd du Pont d'Arve, 1211 Geneva 4, Switzerland, email Evis.Kellezi@metri.unige.ch. This paper bene£ted signi£cantly from conversations with Peter Winker; we would like to thank him for valuable suggestions and comments. Financial support from the Swiss National Science Foundation (project 12–5248.97) is gratefully acknowledged.

Abstract

Constraints on downside risk, measured by shortfall probability, expected shortfall etc., lead to optimal asset allocations which differ from the mean-variance optimum. The resulting optimization problem can become quite complex as it exhibits multiple local extrema and discontinuities, in particular if we also introduce constraints restricting the trading variables to integers, constraints on the holding size of assets or on the maximum number of different assets in the portfolio. In such situations classical optimization methods fail to work ef£ciently and heuristic optimization techniques can be the only way out. The paper shows how a particular optimization heuristic, called threshold accepting, can be successfully used to solve complex portfolio choice problems.

JEL codes: G11, C61, C63.

Keywords: Portfolio Optimization, Downside Risk Measures, Heuristic Optimization, Threshold Accepting.

1 Introduction

The fundamental goal of an investor is to optimally allocate his investments among different assets. The pioneering work of [Markowitz, 1952] introduced mean-variance optimization as a quantitative tool which carries out this allocation by considering the trade-off between risk – measured by the variance of the future asset returns – and return. The assumptions of the normality of the returns or of the quadratic investor's preferences allow the simpli£cation of the utility optimization problem in a relatively easy to solve quadratic program.

Notwithstanding its popularity, this approach has also been subject to a lot of criticism. Alternative approaches attempt to conform the fundamental assumptions to reality by dismissing the normality hypothesis in order to account for the fat-tailedness and the asymmetry of the asset returns. Consequently, other measures of risk, such as Value at Risk (VaR), expected shortfall, mean semi-absolute deviation, semi-variance and so on are used, leading to problems that cannot always be reduced to standard linear or quadratic programs. The resulting optimization problem often becomes quite complex as it exhibits multiple local extrema and discontinuities, in particular if we introduce constraints restricting the trading variables to integers, constraints on the holding size of assets, constraints on the maximum number of different assets in the portfolio, etc.

In such situations, classical optimization methods do not work ef£ciently and heuristic optimization techniques can be the only way out. They are relatively easy to implement and computationally attractive. The use of heuristic optimization techniques to portfolio selection has already been suggested by [Mansini and Speranza, 1999], [Chang *et al.*, 2000] and [Speranza, 1996]. This paper builds on work by [Dueck and Winker, 1992] who £rst applied a heuristic optimization technique, called Threshold Accepting, to portfolio choice problems. We show how this technique can be successfully employed to solve complex portfolio choice problems where risk is characterized by Value at Risk or Expected Shortfall.

In Section 2, we outline the different frameworks for portfolio choice as well as the most frequently used risk measures. Section 3 gives a general representation of the threshold accepting heuristic we use. The performance and ef£ciency of the algorithm is discussed in Section 4 by, £rst, comparing it with the quadratic programming solutions in the mean-variance framework and, second, applying the algorithm to the problem of maximizing the expected portfolio value under constraints on the portfolio expected shortfall or VaR. Section 5 concludes.

2 Portfolio choice models

In this section we describe the most frequently used risk measures as well as the different frameworks for portfolio choice they give rise to.

2.1 The mean-variance formulation

Mean-variance optimization is certainly the most popular approach to portfolio choice. In this framework, the investor is faced with a trade-off between the pro£tability of his portfolio, characterized by the expected return, and the risk, measured by the variance of the portfolio returns. The £rst two moments of the portfolio future return are suf£cient to de£ne a complete ordering of the investors preferences. This strong result is due to the simplistic hypothesis that the investors' preferences are quadratic or that the returns are normally distributed.

Denoting by x_i, $i = 1, \ldots, n_A$, the amount invested in asset i out of an initial capital v^0 and by r_i, $i = 1, \ldots, n_A$, the assets log-returns over the planning period, the expected return on the portfolio de£ned by the vector $x = (x_1, x_2, \ldots, x_{n_A})'$ is given as

$$\mu(x) = \frac{1}{v^0}\, x' E(r)\,.$$

The variance of the portfolio return is

$$\sigma^2(x) = x'\, Q\, x\,,$$

where Q is the matrix of variances and covariances of the vector of returns r.

Thus the mean-variance ef£cient portfolios, de£ned as having the highest expected return for a given variance and the minimum variance for a given expected return, are obtained by solving the following quadratic program

$$\begin{array}{rcll} \min\limits_{x} & x'\, Q\, x & & \\ \sum_j x_j r_j & \geq & \rho\, v^0 & \\ \sum_j x_j & = & v^0 & \\ x_j^\ell \leq x_j \leq x_j^u & & & j \in P\,. \end{array} \tag{1}$$

for different values of ρ, where ρ is the required return on the portfolio and P is the set of assets in the portfolio. The vectors x_j^ℓ, x_j^u, $j \in P$, represent constraints on the minimum and maximum holding size of the individual assets in the optimal portfolio.

The implementation of the Markowitz model with n_A assets requires n_A estimates of expected returns, n_A estimates of variances and $n_A(n_A-1)/2$ correlation coef£cients.

There exist several ef£cient algorithms for mean-variance optimization. Early successful parametric quadratic programming methods include the critical-line algorithm and the simplex method. For more recent developments see work by [Perold, 1984] as well as reviews in [Pardalos, Sandström and Zopounidis, 1994] and [Mansini and Speranza, 1999] and references therein.

2.2 Mean downside-risk framework

In practice investors are more concerned about the risk that their portfolio value falls below a certain level. That is the reason why different measures of the downside-risk are considered in the asset allocation problem. If we denote by v the future portfolio value, i.e. the value of the portfolio by the end of the planning period, then the probability

$$P(v < \mathtt{VaR}) \tag{2}$$

that the portfolio value falls below the VaR level is called the *shortfall probability*.

The conditional mean value of the portfolio given that the portfolio value has fallen below VaR, called the *expected shortfall*, is de£ned as

$$E(v \mid v < \mathtt{VaR})\,. \tag{3}$$

Other risk measures used in practice are the *mean semi-absolute deviation*

$$E(|v - Ev| \mid v < Ev)$$

and the *semi-variance*

$$E((v - Ev)^2 \mid v < Ev)\,,$$

where we consider only the negative deviations from the mean.

Maximizing the expected value of the portfolio for a certain level of risk characterized by one of the measures de£ned above leads to alternative ways of describing the investor's problem (e.g. [Leibowitz and Kogelman, 1991], [Lucas and Klaassen, 1998] and [Rockafellar and Uryasev, 2000]). Earlier related work had suggested a safety-£rst approach (see e.g. [Arzac and Bawa, 1977] and [Roy, 1952]) and [Rustem, Becker and Marty 2000] discusses a worst-case approach.

For example, if the risk pro£le of the investor is determined in terms of VaR, a mean-VaR ef£cient portfolio would be the solution of the following optimization problem:

$$\begin{array}{rcll} \multicolumn{3}{c}{\max\limits_{x} \; Ev} & \\ P(v < \texttt{VaR}) & \leq & \beta & \\ \sum_j x_j & = & v^0 & \\ x_j^\ell \leq x_j \leq x_j^u & & & j \in P\,. \end{array} \tag{4}$$

In other words, such an investor is trying to maximize the future value of his portfolio, requiring the probability that the future value of his portfolio falls below VaR not to be greater than β.

Furthermore, it would be realistic to consider an investor who cares not only for the shortfall probability, but also for the extent to which his portfolio value can fall below the VaR level. In this case, the investor's risk pro£le is de£ned via a constraint on the expected shortfall tolerated ν if the portfolio value falls below VaR. Then the mean-expected shortfall ef£cient portfolios are solutions of the following program for different values of ν:

$$\begin{array}{rcll} \multicolumn{3}{c}{\max\limits_{x} \; Ev} & \\ E[v \mid v < \texttt{VaR}] & \geq & \nu & \\ \sum_j x_j & = & v^0 & \\ x_j^\ell \leq x_j \leq x_j^u & & & j \in P\,. \end{array} \tag{5}$$

3 The threshold accepting optimization heuristic

Heuristic approaches prove useful in situations where the classical optimization methods fail to work ef£ciently. Heuristic optimization techniques like simulated annealing [Kirkpatrick *et al.*, 1983] and genetic algorithms [Holland, 1975] are used with increasing success in a variety of disciplines. The reason for their success is that they are relatively easy to implement and that the cost of computing power is no longer a matter of concern.

Threshold accepting (TA) was introduced by [Dueck and Scheuer, 1990] as a deterministic analog to simulated annealing. It is a re£ned local search procedure which escapes local minima by accepting solutions which are not worse by more than a given threshold. The algorithm is deterministic in the sense that we £x a number of iterations and explore the neighborhood with a £xed number of steps during each iteration. The threshold is decreased successively and reaches the value of zero in the last iteration.

The threshold accepting algorithm has the advantage of an easy parameterization, it is robust to changes in problem characteristics and works well for many problem instances. An extensive introduction to threshold accepting is given in [Winker, 2000].

Let us formalize our optimization problem as $f : \mathcal{X} \rightarrow \mathbb{R}$ where $\mathcal{X}$ is a discrete set and where we may have more then one optimal solution de£ned by the set

$$\mathcal{X}_{\text{min}} = \{x \in \mathcal{X} \mid f(x) = f_{\text{opt}}\} \tag{6}$$

with

$$f_{\text{opt}} = \min_{x \in \mathcal{X}} f(x)\,. \tag{7}$$

The threshold accepting heuristic described in algorithm 1 will, after completion, provide us with a solution $x \in \mathcal{X}_{\text{min}}$ or a solution close to an element in $\mathcal{X}_{\text{min}}$. The complexity of the algorithm is $\mathcal{O}(niter \times steps)$.

Algorithm 1 Pseudo-code for the threshold accepting algorithm.

Initialize $niter$ and $steps$
Initialize sequence of thresholds th_r, $r = 1, 2, \ldots, niter$
Generate starting point $x^0 \in \mathcal{X}$
for $r = 1$ to $niter$ **do**
    for $i = 1$ to $steps$ **do**
        Generate $x^1 \in \mathcal{N}_{x^0}$ (neighbor of x^0)
        if $f(x^1) < f(x^0) + th_r$ **then**
            $x^0 = x^1$
        end if
    end for
end for

The parameters of the algorithm are the number of iterations $niter$, the number of steps per iteration $steps$ and the sequence of thresholds th. In practice, we start with the de£nition of the objective function, which can be a non-trivial task if f comprises several dimensions. Second, we construct a mapping $\mathcal{N} : \mathcal{X} \rightarrow 2^{\mathcal{X}}$ which de£nes for each $x \in \mathcal{X}$ a neighborhood $\mathcal{N}(x) \subset \mathcal{X}$. Third, we de£ne the sequence of thresholds by exploring the neighborhood of randomly selected elements $x \in \mathcal{X}$.

These different steps of the implementation and parameterization of the algorithm will be illustrated with the application presented in the following section.

4 Application

The working of the TA algorithm is £rst illustrated by solving the standard mean-variance optimization problem. The solution is also computed with the quadratic programming algorithm which will be used as a benchmark. Second we apply the TA algorithm to a non-convex optimization problem with integer variables and a variety of integer constraints such as holding and trading size.

For the following application we use a data set publically available from OR-Library[1] at `http://mscmga.ms.ic.ac.uk/jeb/orlib/portinfo.html`. The set contains 291 observations of weekly prices for the period from March 1992 to September 1997 for assets from the following £ve indices: Hang Seng (31 assets), DAX 100 (85 assets), FTSE 100 (89 assets), S&P 100 (98 assets) and Nikkei (225 assets). The reported results refer to the S&P 100 case.

4.1 Benchmarking the TA algorithm

The standard mean-variance optimization problem has already been de£ned in (1). The following is a reformulation of the problem where the initial capital v^0 has been normalized to one:

$$
\begin{aligned}
&\min_{\omega} \ \omega' Q \, \omega \\
&\omega' r \ \geq \ \rho \\
&\iota' \omega \ = \ 1 \\
&\omega_j^l \leq \omega_j \leq \omega_j^u \qquad j \in P \, .
\end{aligned}
$$

The composition of the portfolio is de£ned by the shares $\omega_i = x_i / v^0$.

De£nition of the objective function

The variance can now be minimized by exploring with the threshold accepting algorithm 1 the elements in the set $\mathcal{X}$ which satisfy the constraints. However, a better way is to accept solutions which violate the return constraint in the search process. This can be done by minimizing the following objective function

$$F(\omega) = V(\omega) + p \, (\rho - R(\omega))$$

[1] [Beasley, 1990], [Beasley, 1996].

where $V(\omega)$ and $R(\omega)$ denote respectively the variance and the return of a portfolio de£ned by ω and where p is a penalty function de£ned as

$$p = \begin{cases} c & \text{if} \quad R(\omega) < \rho \\ 0 & \text{otherwise.} \end{cases}$$

with c a scaling factor.

De£nition of neighborhood

To generate a point x^1 in the neighborhood $\mathcal{N}_{x^0}$ of a given point x^0 we draw with a probability $1/n_A$ two assets i and j out of all n_A assets. The amount of i and j in the portfolio is ω_i, respectively ω_j. We then sell an amount q_r of asset i and buy for the corresponding amount asset j. After this move the amount of i and j in the portfolio is $\omega_i - q_r$, respectively $\omega_j + q_r$. The amount q_r changes from iteration to iteration and the sequence q_r, $r = 1, \ldots, niter$ has to be tuned according to the problem nature.

Algorithm 2 De£nition of neighborhood.

```
Select two assets i and j with probability 1/n_A
t = q_r
if (ω_i − t) ≥ ω_i^l then
    ω_i = ω_i − t
else
    t = ω_i
    ω_i = 0
end if
if (ω_j + t) ≤ ω_j^u then
    ω_j = ω_j + t
else
    cash = cash + ω_j + t − ω_j^u
    ω_j = ω_j^u
end if
```

In order to avoid short selling and to respect the constraints on the holding size of the assets, the procedure for the selection of a neighbor solution must be re£ned. Algorithm 2 describes the procedure of the selection of a neighbor solution in detail.

De£nition of thresholds

In order to de£ne the sequence of thresholds, we compute the empirical distribution of the distances between values of the objective function at random points and its neighbors. As the distance between neighbors depends on the parameter q_r, the distribution has to be computed for each value of q_r.

In the following application we choose $niter = 3$ and $steps = 3000$. The parameter q_r decreases linearly from 0.05 to 0.005 during the 3 iterations.

Figure 1 shows the empirical distribution of the distances between the objective function values of neighbors of 5000 randomly drawn portfolios. In the left panel we represent the distribution for the points generated in the £rst iteration where $q_r = 0.05$ whereas in the right panel we have the distribution of the points generated during the second iteration with q_r set to 0.025.

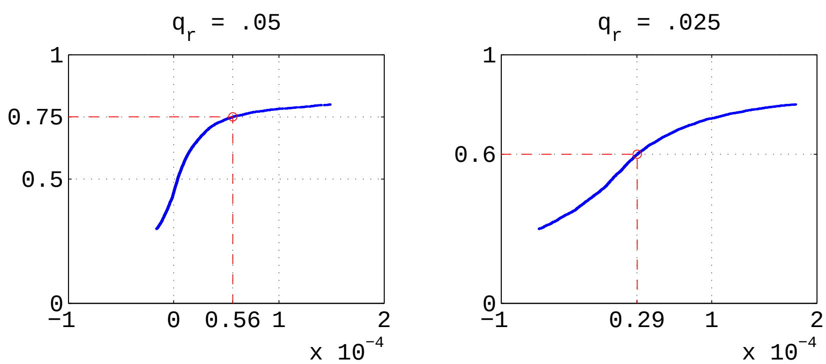

Figure 1: Empirical distribution of distances between x^0 and neighbors x^1.

If for example we want to reject 25% most distant neighbors in the £rst iteration and 40% in the second, we must choose a threshold sequence of $10^{-6}\begin{bmatrix} 56 & 29 & 0 \end{bmatrix}$. We observed that such threshold sequences are robust to different sets of random points and to different data sets.

We solve the mean-variance optimization problem for an investment opportunity set composed of the $n_A = 98$ assets in S&P 100 with the above parameter setting and we compare it with the solution given by the QP algorithm. Weights in the optimal portfolio are allowed to be any real number between 0 and 1 and the maximum number of assets is not restricted.

Figure 2 illustrates how the algorithm searches its way to the solution for $\rho = 0.0085$ lying on the ef£cient frontier computed with QP [2]. At the optimal solution the expected

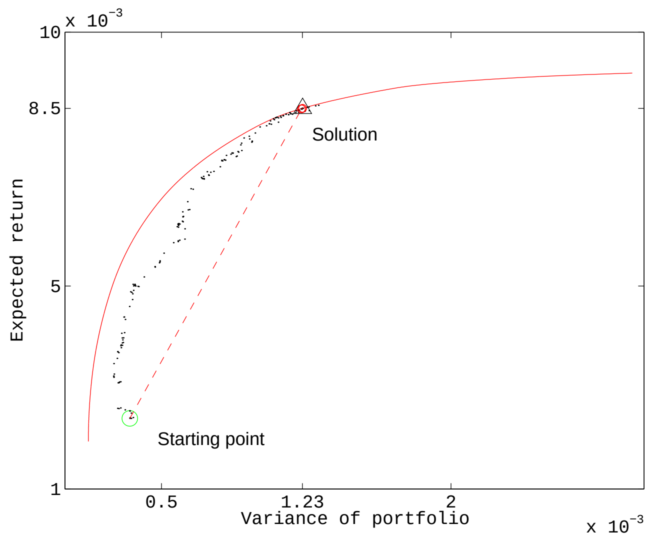

Figure 2: Working of the TA algorithm (S&P 100).

return and the variance are practically the same for the QP and TA algorithms. The optimal portfolio contains assets 34, 42, 82 and 89. Weights of the assets in the optimal portfolio for both algorithms are given in Figure 3. These results con£rm the good performance of the TA algorithms.

[2] The ef£cient frontier values are provided in the data set we used.

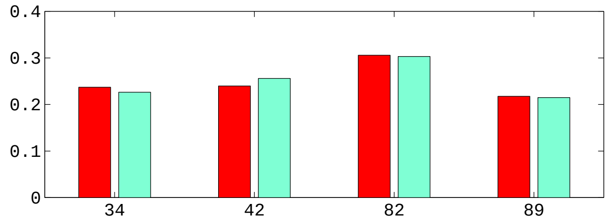

Figure 3: Composition of the optimal portfolio for QP (left bars) and TA (right bars).

4.2 Mean downside risk optimization

Our second illustration of the working of the TA algorithm is a non-convex optimization problem with integer variables and a variety of constraints such as holding and trading size. We remind that this kind of problem cannot be solved with standard QP methods. The solution of the resulting mixed-integer programming model can be tackled by heuristic methods (see e.g. [Mansini and Speranza, 1999]) which provide an approximation of the exact solution.

In the following, the quantity of each asset in the portfolio is restricted to be an integer number. The generation of neighbors $x^1 \in \mathcal{N}_x^0$ to a given solution x^0 is again performed by drawing randomly two assets i and j. We then sell k_i assets i, transfer the amount to the cash and buy k_j assets j from cash. In order to make sure that each transfer is of approximatively the same amount, the number of assets k_i and k_j to be transferred are de£ned as $k_i = \lceil \frac{\max p^0}{p_i^0} \rceil$ and $k_j = \lceil \frac{\max p^0}{p_j^0} \rceil$, where p^0 is the vector of current asset prices. This procedure is summarized in algorithm 3 where we omit the details necessary to check for short selling and holding constraints.

Algorithm 3 De£nition of neighborhood in the case of integer variables.

1: Randomly select asset i to sell
2: $x_i = x_i - k_i$
3: cash $=$ cash $+ k_i\, p_i^0$
4: Randomly select asset j to buy
5: $x_j = x_j + k_j$
6: cash $=$ cash $- k_j\, p_j^0$

Using the S&P 100 data set as in the previous problem, we compute the solutions of the mean-VaR and the mean-ES problems de£ned in (4) and (5).

Dealing with integer variables and introducing cardinality constraints and constraints on the minimum and th maximum holding size, leads to the following reformulation of the mean-VaR problem:

$$\begin{array}{rcl} & \max\limits_{x} \; Ev & \\ P(v < \mathtt{VaR}) & \leq & \beta \\ x'\,p^0 & = & v^0 \\ \#\{P\} & \leq & K \\ \left\lceil \frac{\omega_j^l\, v^0}{p_j^0} \right\rceil \leq x_j \leq \left\lfloor \frac{\omega_j^u\, v^0}{p_j^0} \right\rfloor & & j \in P \end{array}$$

where x_j, $j \in P$ are the integer quantities of each asset in the portfolio and K is the maximum number of assets allowed in the portfolio. Similarly, for the mean-ES problem we have

$$\begin{array}{rcl} & \max\limits_{x} \; Ev & \\ E(v \,|\, v < \mathtt{VaR}) & \geq & \nu \\ x'\,p^0 & = & v^0 \\ \#\{P\} & \leq & K \\ \left\lceil \frac{\omega_j^l\, v^0}{p_j^0} \right\rceil \leq x_j \leq \left\lfloor \frac{\omega_j^u\, v^0}{p_j^0} \right\rfloor & & j \in P\,. \end{array}$$

The uncertainty about future returns, i.e. about future portfolio value v, is modeled through a set of possible realizations, called scenarios. Scenarios of future outcomes can be generated relying on a statistical model, past returns or experts' opinions. In our application they are randomly drawn from the empirical distribution of past returns.

In our data set we have $T = 291$ historical prices p^t, $t = 1, \ldots, T$. For each point in time, we compute the realized return vector over the previous period as

$$r_j^t = \log(p_j^t / p_j^{t-1}) \qquad j = 1, \ldots, n_A$$

from which we draw the bootstrap sample of returns of dimension n_S. The price scenarios for the end of the planning period are computed as

$$p^s = p^T r^s \qquad s = 1, \ldots, n_S\,.$$

Introducing price scenarios in the previous mean-VaR and mean-ES formulations, we obtain the following problem for the mean-VaR case

$$
\begin{aligned}
\min_x \; -\frac{1}{n_S}\sum_{s=1}^{n_S} x' p^s & \\
\#\{s \mid x' p^s < \mathtt{VaR}\} &\leq \beta\, n_S \\
x'\, p^0 &= v^0 \\
\#\{P\} &\leq K \\
\left\lceil \frac{\omega_j^l\, v^0}{p_j^0} \right\rceil \leq x_j \leq \left\lfloor \frac{\omega_j^u\, v^0}{p_j^0} \right\rfloor & \qquad j \in P
\end{aligned}
$$

and

$$
\begin{aligned}
\min_x \; -\frac{1}{n_S}\sum_{s=1}^{n_S} v^s & \\
\frac{1}{\#\{s \mid v^s < \mathtt{VaR}\}} \sum_{s \mid v^s < \mathtt{VaR}} v^s &\geq \nu \\
x'\, p^0 &= v^0 \\
\#\{P\} &\leq K \\
\left\lceil \frac{\omega_j^l\, v^0}{p_j^0} \right\rceil \leq x_j \leq \left\lfloor \frac{\omega_j^u\, v^0}{p_j^0} \right\rfloor & \qquad j \in P
\end{aligned}
$$

in the mean-ES.

Assuming an initial capital of $v^0 = 8\,000\,000$ we seek the portfolio which maximises the expected return given the following constraints: shortfall probability $\beta = 0.05$ for a VaR level of $7\,700\,000$, minimum and maximum holding size for a particular asset $0.01\, v^0$ respectively $0.40\, v^0$ and a maximum of 5 assets in the portfolio. Figure 4 shows the results of the TA algorithm with the setting $niter = 4$ and $steps = 1500$.

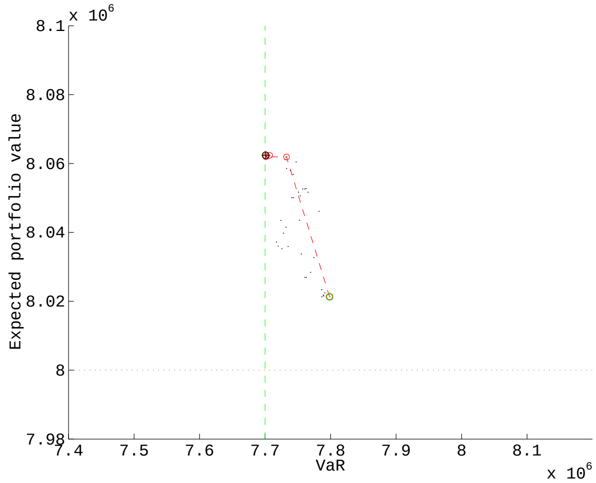

Figure 4: Search path of the TA algorithm for Mean-VaR.

The composition of the portfolio that TA found to be optimal is given in £gure 5. It contains the assets $\{14,\ 23,\ 27,\ 51\}$. The smallest position is $413\,500$ in asset 27 and the maximum position is $3\,200\,000$ in asset 23, which is the maximum allowed. Thus the constraints on the holding size and the number of assets in the portfolio are satis£ed.

This optimal portfolio has an expected return of $E(v) = 8\,062\,400$ with a shortfall probability of 0.052 for a VaR level of $7\,700\,960$, which almost fully satisfy the constraints.

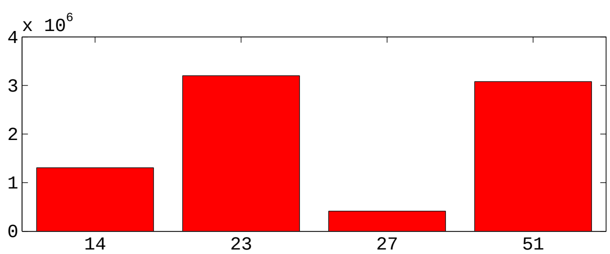

Figure 5: Optimal portfolio computed by TA for the mean-VaR problem.

For an investor it would be interesting to know what are the investment opportunities in terms of expected return and downside risk. Intuitively, the higher the capital gain one requires is, the bigger the downside risk that one accepts to run should be. We can observe this in £gures 6 and 7 where we represent the ef£cient frontiers faced by investors in stocks of the S&P 100 index. In both cases, the minimum and maximum individual weights allowed are 0.01 and 0.30, the maximum number of assets in the portfolio is restricted to 10, and the shortfall probability is required to be less than $\beta = 0.05$.

In £gure 6, we give the ef£cient frontier for the Mean-VaR problem. We observe that as the constraint on VaR becomes more conservative, the expected return on the portfolio decreases. For instance, in the case where we do not allow the VaR level to be below the initial portfolio value, the only solution returned by our algorithm is to put almost everything in cash.

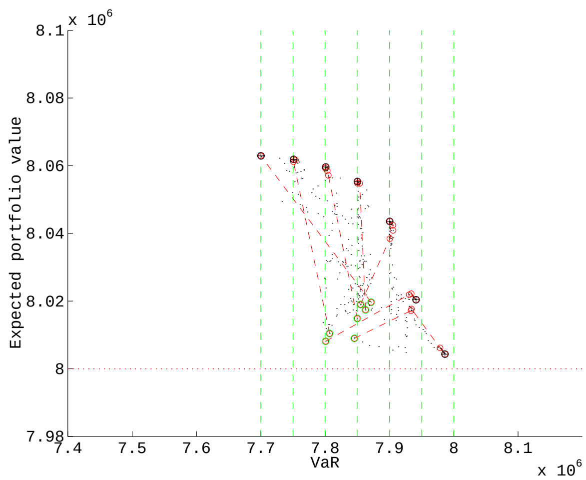

Figure 6: Mean-VaR ef£cient frontier.

In £gure 7, we give the ef£cient frontier for the Mean-ES problem. We notice that the Mean-ES frontier is smoother than the Mean-VaR one. This can be explained by the fact that the objective function itself in the former case is smoother given that ES is computed as a mean. In both cases constraints (represented in the above £gures by vertical dashed lines) may not be exactly satis£ed due to the integer nature of the variables.

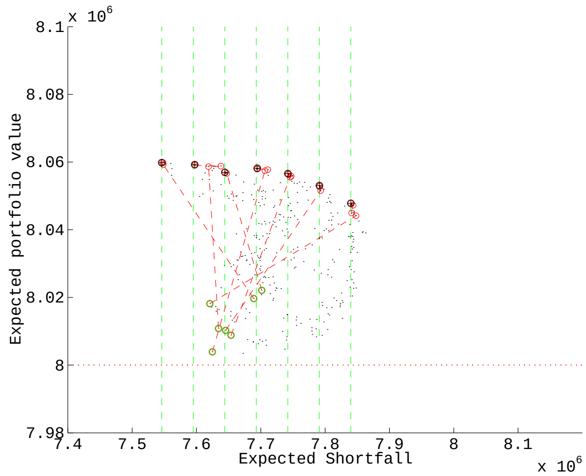

Figure 7: Mean-ES ef£cient frontier.

5 Concluding remarks

In this paper we illustrate how heuristic optimization algorithms like the threshold accepting method can be successfully applied to solve realistic non-convex portfolio optimization problems. We show that, in the cases where these problems contain non-linear and non-convex constraints, the heuristic methods are the only reasonable way out. Examples of these situations can be problems where constraints on downside risk preferences are introduced, where the solutions are required to be integers, etc.

The working of the threshold accepting algorithm is £rst illustrated by solving a standard mean-variance optimization problem. The solution is also computed with the quadratic programming algorithm which is used as a benchmark and thus provides some insight into the quality of the threshold accepting heuristic, which appears very satisfactory.

In a second example we apply the threshold accepting algorithm in order to solve non-convex optimization problems with integer variables and constraints on holding sizes and portfolio set cardinality. The future return on the portfolio is maximized under VaR and expected shortfall constraints.

From our results we conclude that the threshold accepting algorithm opens new perspectives in the practice of portfolio management as it allows to deal easily with all sorts of constraints of practical importance, it provides useful approximations of the optimal solutions, it appears to be computationally ef£cient and is relatively easy to implement. We also observed that the algorithm is robust to changes in problem characteristics.

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