A Test of the Use of the Implied Volatility Function Model to Price Exotic Options*

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Abstract

Researchers such as Derman and Kani (1994), Dupire (1994), and Rubinstein (1994) have proposed a one-factor model for asset prices that is exactly consistent with all European option prices. In this model, which we refer to as the implied volatility function (IVF) model, the asset price volatility is a function of both time and the asset price. Practitioners often use the IVF model to price exotic options. This paper explores the validity of this. It does so by assuming a two-factor stochastic volatility model for the asset price and examining the way the IVF model prices compound options and barrier options. We find the model works well for compound options, but sometimes gives rise to large pricing errors for barrier options.

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1 Introduction

The Black–Scholes (1973) model and its extensions are widely used by the market to price options. The usual approach is to imply volatilities from the prices of options that trade actively and use them to price other options. This approach is possible because volatility is the only unobservable parameter in the Black–Scholes pricing formula.

This use of implied volatilities by the market makes sense only if all options (or at least all options with the same expiration date) yield similar implied volatilities at any given time. As Rubinstein (1994) points out, two implied volatilities can be considered approximately the same if the economic consequences of using one rather than the other are relatively benign in the sense of yielding small percentage errors in option values.

Since the market crash of 1987, the implied volatilities calculated from different options on the same stock or stock index have been systematically dependent on the strike price. As the strike price increases, the implied volatility decreases. This phenomenon is referred to as a *volatility skew*. Foreign exchange markets exhibit a different pattern from equity markets. For a given maturity, the implied volatility is a U-shaped function of the strike price. The implied volatility is lowest for an option that is at or close to the money. It becomes progressively higher as an option moves either in or out of the money. This is referred to as a *volatility smile*.

When implied volatilities for European options with a particular maturity are dependent on the strike price, the risk–neutral probability distribution for the future value of the asset is non-lognormal and the assumptions underlying the Black–Scholes model do not hold. This makes it difficult for traders to price exotic options consistently with standard options.

A number of authors have proposed extensions of the Black-Scholes model. For example, Merton (1976) and Bates (1996) have proposed jump-diffusion models. Heston (1993), Hull and White (1987, 1988), and Stein and Stein (1995) have proposed models where volatility follows a stochastic process. When parameters are chosen appropriately these models produce Black–Scholes implied volatilities that have a similar pattern to those observed in the market. Although the models are popular among academic researchers, they are not widely used by practitioners. When valuing exotic options, most practitioners like to use a model that exactly matches all the observed market prices of options written on the same underlying asset. Research by Derman and Kani (1994), Dupire (1994), and Rubinstein (1994) shows that we can construct a one-factor model with this property by making volatility a function of the asset price and time. Following Rosenberg (2000), we will refer to this model as the *implied volatility function* (IVF) model.

Dumas, Fleming and Whaley (1998) test the stationarity of the IVF model. They show that there are significant errors when the IVF model is fitted to the market at a particular time and then used to price options one week later. They also find that the difference between the observed and predicted option prices is larger for complex parameterizations of the volatility functions than for a constant volatility specification. They conclude that the implied volatility model is not an improvement over Black–Scholes as a description of how asset prices evolve. Recently Rosenberg (2000) has proposed a model where the at-the-money implied volatility follows a process dependent on asset returns and other volatilities are a function of the at-the-money volatility. He carries out a similar test to Dumas, Fleming, and Whaley and shows that his model performs well when compared to particular cases of the IVF model.

We subject the IVF model to a different test — one that is more closely related to the way it is used in practice. We test whether the model is useful as a tool for relating the price of an exotic option to the prices of standard European options at one particular time. We assume that the true model describing the evolution of asset prices is a stochastic volatility model and that the market prices of all European options are consistent with this model. We fit the IVF model to European option prices and compare the prices it gives for particular exotic options with the prices given by the stochastic volatility model.

We choose this somewhat artificial test of the model because there are very little data on market prices of exotic options. The test is designed to determine whether the IVF model gives good prices for a simple, relatively well-behaved, two-factor model. If it does, we can be optimistic that it will work reasonably well for the complicated processes driving asset prices in the real world. If it does not, we can reasonably assume that it will not perform adequately as a pricing tool in the real world.

2 The Implied Volatility Function Model

In the Black-Scholes setting, an asset price, S, is assumed to follow a geometric Brownian motion,

$$\frac{dS}{S} = \mu dt + \sigma dz, \tag{1}$$

The expected return on the asset, μ, can be a function of the asset price and time, the asset price volatility, σ, is constant, and z follows a Wiener process. The market is assumed to be frictionless with no arbitrage opportunities. The spot interest rate, r, and yield on the asset, q, are assumed to be constant.

Under these and other technical assumptions, it can be shown that the price $c(S,t;K,T,\sigma)$ at time t of a European call option on the asset with strike price K and maturity T is given by

$$c(S,t;K,T,\sigma) = e^{-q(T-t)}SN(d_1) - e^{-r(T-t)}KN(d_2), \tag{2}$$

where

$$d_1 = \frac{\ln(S/K) + (r - q + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \quad d_2 = d_1 - \sigma\sqrt{T-t},$$

and $N(x)$ is the cumulative probability that variable with a standardized normal distribution is less than x. Equation (2) implies that there is a one-to-one correspondence between call option prices and their volatilities. If the market price, c_{mkt}, for an European call option with a strike price K and maturity T is known, then there is a unique volatility σ_{imp} such that

$$c_{\mathrm{mkt}} = c(S, t; K, T, \sigma_{\mathrm{imp}}). \tag{3}$$

This is known as the implied volatility. The value of σ_{imp} as a function of K and T is referred to as the *volatility matrix* or *volatility surface.*

The Black–Scholes assumptions imply that σ_{imp} is independent of K and T. In practice, as already mentioned, σ_{imp} is found to vary systematically with the strike price. In the foreign exchange market there is a volatility smile. The implied volatilities for at-the-money options are typically lower than the implied volatilities of deep out-of-the-money or deep in-the-money options on the same asset. In equity markets there is a volatility skew. The Black-Scholes implied volatilities for options with the same maturity date tend to decrease as the strike price increases. This phenomenon, discussed in Jackwerth and Rubinstein (1996), has been a feature of equity markets since the market crash in 1987.

The IVF model was proposed by Derman and Kani (1994), Dupire (1994), and Rubinstein (1994) and extended by Andersen and Brotherton–Ratcliffe (1998). The volatility of the underlying asset is assumed to be a deterministic function of both time and underlying asset level. The risk-free interest rate, r, and the asset's yield, q, are assumed to be functions of time so that the risk-neutral process followed by the asset price is

$$\frac{dS}{S} = [r(t) - q(t)]dt + \sigma(S, t)dz.$$

Derivatives dependent on the asset price satisfy the differential equation

$$\frac{\partial f}{\partial t} + [r(t) - q(t)]S\frac{\partial f}{\partial S} + \frac{1}{2}\sigma(S, t)^2 S^2 \frac{\partial^2 f}{\partial S^2} = r(t)f. \tag{4}$$

As shown by Dupire (1994) and Andersen and Brotherton–Ratcliffe (1998), there is an analytic relationship between the volatility function and the prices of European options with different strike prices and times to maturity. The relationship is:

$$\sigma(K, T)^2 = 2\frac{\partial c_{\mathrm{mkt}}/\partial T + q(t)c_{\mathrm{mkt}} + [r(t) - q(t)]K\partial c_{\mathrm{mkt}}/\partial K}{K^2 \partial^2 c_{\mathrm{mkt}}/\partial K^2}. \tag{5}$$

Once the volatility function in equation (5) has been determined, exotic option prices can be obtained by solving differential equation (4) subject to appropriate boundary conditions.

3 Potential Errors in the IVF Model

The IVF model is designed so that it values all European options correctly. This means that the risk-neutral probability distribution of the asset price at all future times, conditional on the asset price at time zero, is consistent with the market. This in turn means that the IVF model correctly prices all derivatives that provide a single payoff at a time T when the payoff is contingent only on the asset price at time T.

There is no guarantee that the IVF model prices other derivatives correctly. Consider, for example, a compound option where the holder has the right at time T_1 to pay a prespecified amount of money, K_1, to enter into a European option with strike price, K_2, maturing at time T_2. The decision to exercise at time T_1 depends on the asset price at time T_1 and, possibly, other state variables. The payoff at time T_2 depends on the asset price at time T_2. The value of the option therefore depends on the joint probability distribution of the asset price at times T_1 and T_2. Because the joint probability distribution of two variables is not uniquely determined from their marginal distributions, the IVF model may be assuming a different joint probability from the market.

To express this point more formally, define $\phi_n[t_1, t_2, \ldots, t_n]$ as the joint probability distribution of the asset price at times t_1, t_2, ..., t_n. The IVF model is designed so that $\phi_1(t)$ is correct for all t, but this does not ensure that $\phi_n[t_1, t_2, \ldots, t_n]$ is correct for $n > 1$. In the case of the compound option just considered $\phi_1(T_1)$ and $\phi_1(T_2)$ are correct, but this does not guarantee that $\phi_2(T_1, T_2)$ is correct.

The dependence of the price of some derivatives on the joint probability distribution of the underlying asset price at different times is quite complex. Consider a barrier option maturing at time T_N where the asset price is observed at times T_1, T_2, ..., T_N for the purposes of determining whether the barrier has been hit. The price of the option depends on $\phi_N[T_1, T_2, \ldots, T_N]$. The IVF model is designed so that $\phi_1(T_i)$ is correct for $1 \leq i \leq N$, but this does not mean $\phi_N[T_1, T_2, \ldots, T_N]$ is even approximately correct.

4 Tests

To test the IVF model we assumed that the asset price is follows a two-factor stochastic volatility model similar to the one developed by Heston (1993) so that

$$\frac{dS}{S} = (r - q)dt + v dz_S \tag{6}$$

with

$$dv = \kappa(\theta - v)dt + \xi dz_v, \tag{7}$$

where z_S and z_v are Wiener processes with an instantaneous correlation ρ. The parameters κ, θ, and ξ are the mean-reversion rate, long-run average volatility, and volatility of volatility, respectively, and are assumed to be constants. The spot rate, r and the yield on the asset q are also assumed to be constant.

A valuation formula for the European call option price, $c_h(S, v, t; K, T)$, in this model can be found through the inversion of characteristic functions of random variables. It takes the form:

$$c_h(S, v, t; K, T) = e^{-q(T-t)} S(t) F_1 - e^{-r(T-t)} K F_2. \tag{8}$$

where F_1 and F_2 are integrals that can be evaluated efficiently using numerical procedures such as quadrature. More details on the model can be found in Schöbel and Zhu (1998).

Our test of the IVF model consists of the following steps:

1. Price an exotic option using the stochastic volatility model in equations (6) and (7). We denote this price by f_{true}.
2. Fit the IVF model to the market prices of European call options that are given by the model in equations (6) and (7).
3. Use the IVF model to price the exotic option. We denote this price by f_{ivf}.
4. Use the Black–Scholes model in equation (1) to price the exotic option. We denote this price by f_{bs}.
5. Compare f_{true}, f_{ivf}, and f_{bs}.

We considered the following two sets of parameters for the stochastic volatility model.

Parameter Set I: $r = 5.9\%$, $q = 1.4\%$, $v(0) = 0.25$, $\kappa = 0.16$, $\theta = 0.3$, $\xi = 0.09$, and $\rho = -0.79$.
Parameter Set II: $r = 5.9\%$, $q = 3.5\%$, $v(0) = 0.1285$, $\kappa = 0.109$, $\theta = 0.1$, $\xi = 0.0376$, and $\rho = 0.1548$.

To choose Parameter Set I, we used a least squares procedure to provide as close a fit as possible to the volatility matrix for the S&P 500 reported in Andersen and Brotherton–Ratcliffe (1998). To choose Parameter Set II, we use the same procedure to provide as close a fit as possible to a volatility matrix for the U.S. dollar-Swiss franc exchange rate provided to us by a large U.S. investment bank. The parameter sets are, therefore, designed to give volatility matrices representative of those encountered in practice for an equity index and a currency, respectively. The implied Black-Scholes volatility matrices from the stochastic volatility model of (6) and (7) with Parameter Sets I and II are shown in Tables 1 and 2.

We calculated the market prices of European call options, c_{mkt}, using equation (8). We fitted the IVF model to these prices by calculating $\partial c_{\text{mkt}}/\partial t$, $\partial c_{\text{mkt}}/\partial K$, and $\partial^2 c_{\text{mkt}}/\partial K^2$ from equation (8) and then using equation (5).

We considered two types of exotic options: a call-on-call compound option and a knock-out barrier option. We used Monte Carlo simulation with 300 time steps and 100,000 trials to estimate the prices of these options for the stochastic volatility model.[1] For this purpose, equations (6) and (7) were discretized to

$$\ln \frac{S_{i+1}}{S_i} = \left(r - q - \frac{v_i^2}{2}\right) \Delta t + v_i \epsilon_1 \sqrt{\Delta t}, \tag{9}$$

$$v_{i+1} - v_i = \kappa(\theta - v_i)\Delta t + \xi \epsilon_2 \sqrt{\Delta t}, \tag{10}$$

where Δt is the length of the Monte Carlo simulation time step, S_i and v_i are the asset price and its volatility at time $i\Delta t$, and ϵ_1 and ϵ_2 are random samples from two unit normal distributions with correlation, ρ.

We estimated the prices given by the IVF model using the implicit Crank-Nicholson finite difference method described in Andersen and Brotherton–Ratcliffe (1998). This involves constructing a 120×70 rectangular grid of points in (x, t)-space where $x = \ln S$. The grid extends from time zero to the maturity of the exotic option, T_{mat}. Define x_{min} and x_{max} as the lowest and highest x-values considered on the grid. (We explain how these are determined later.) Boundary conditions determine the values of the exotic option on the $x = x_{\text{max}}$, $x = x_{\text{min}}$ and $T = T_{\text{mat}}$ edges of the grid. The differential equation (4) enables relationships to be established between the values of the exotic option at the nodes at the ith time point and its values at the nodes at the $(i+1)$th time point. These relationships are used in conjunction with boundary conditions to determine the value of the exotic option at all interior nodes of the grid and its value at the nodes at time zero.

4.1 Compound Options

Our first test of the IVF model used a call-on-call compound option. This is an option where the holder has the right at time T_1 to pay K_1 and obtain a call option to purchase the asset for a price K_2 at time T_2 ($T_2 > T_1$). When using Monte Carlo simulation to calculate f_{true}, each trial involved using equations (9) and (10) to calculate the asset price and its volatility at time T_1. It was not necessary to simulate beyond time T_1 because the value of a European call option with strike price K_2 and maturity T_2 can be calculated at time T_1 using equation (8). Define $S_{1,j}$ and $v_{1,j}$ as the asset price and volatility at time T_1 on the jth trial, and $w_{1,j}$ as the value at time T_1 of a call option with strike price K_2 maturing at T_2 for the jth trial. From equation (8):

$$w_{1,j} = c_h(S_{1,j}, v_{1,j}, T_1, K_2, T_2)$$

The estimate of the true value of the option given by Monte Carlo simulation is:

$$f_{\text{true}} = \frac{e^{-rT_1}}{N} \sum_{j=1}^{N} \max(w_{1,j} - K_1,\, 0)$$

[1] To reduce the variance of the estimates, we used the antithetic variable technique described in Boyle (1977).

We calculated the IVF price for the compound option by building the finite difference grid out to time T_2. Between times T_1 and T_2, we used the grid to calculate the price, w, of a European call option with strike price K_2 maturing at time T_2. This enabled the value of the compound option at the nodes at time T_1 to be calculated as $\max(w - K_1, 0)$. We then used the part of the grid between time zero and time T_1 to calculate the value of the compound option at time zero. We set $x_{\text{min}} = \ln S_{\text{min}}$ and $x_{\text{max}} = \ln S_{\text{max}}$ where S_{min} and S_{max} are very high and very low asset prices, respectively. The boundary conditions we used were:

$$
\begin{aligned}
w &= \max(e^x - K_2, 0) \text{ when } t = T_2, \\
w &= 0 \text{ when } x = x_{\text{min}} \text{ and } T_1 \le t \le T_2, \\
w &= e^x - K_2 e^{-r(T_2 - t)} \text{ when } x = x_{\text{max}} \text{ and } T_1 \le t \le T_2; \\
f_{\text{ivf}} &= 0 \text{ when } x = x_{\text{min}} \text{ and } 0 \le t \le T_1, \\
f_{\text{ivf}} &= e^x - K_2 e^{-r(T_2 - t)} - K_1 e^{-r(T_1 - t)} \text{ when } x = x_{\text{min}} \text{ and } 0 \le t \le T_1.
\end{aligned}
$$

The value of a compound option using the Black–Scholes model in equation (1) was first produced by Geske (1979). Geske shows that at time zero:

$$
f_{\text{bs}} = S(0)e^{-qT_2} M(a_1, b_1; \sqrt{T_1/T_2}) - K_2 e^{-rT_2} M(a_2, b_2; \sqrt{T_1/T_2}) - e^{-rT_1} K_1 N(a_2)
$$

where

$$
\begin{aligned}
a_1 &= \frac{\ln[S(0)/S^*] + (r - q + \sigma^2/2)T_1}{\sigma\sqrt{T_1}}, \qquad a_2 = a_1 - \sigma\sqrt{T_1}, \\
b_1 &= \frac{\ln[S(0)/K_2] + (r - q + \sigma^2/2)T_2}{\sigma\sqrt{T_2}}, \qquad b_2 = b_1 - \sigma\sqrt{T_2}
\end{aligned}
$$

and $M(a, b; \rho)$, is the cumulative probability in a standardized bivariate normal distribution that the first variable is less than a and the second variable is less than b when the coefficient of correlation between the variables is ρ. The variable S^* is the asset price at time T_1 for which the price at time T_1 of a European call option with strike price K_2 and maturity T_2 equals K_1. If the actual asset price is above S^* at time T_1, the first option will be exercised; if it is not above S^*, the compound option expires worthless. In computing f_{bs} we used the implied volatility of a European option maturing at time T_2 with a strike price of K_2.

Table 3 shows f_{true} and the percentage errors when the option price is approximated by f_{ivf} and f_{bs} for the case where $T_1 = 1$, $T_2 = 2$, and K_2 equals the initial asset price. It considers a wide range of values of K_1. The table shows that the IVF model works very well. For compound options where the true price is greater than 1% of the initial asset price, the IVF price is within 2% of the true price. When very high strike prices are used with Parameter Set II the error is higher, but this is because the true price of the compound option is very low. Measured as a percent of the initial asset price the absolute pricing error of the IVF model is never greater than 0.08%.

The Black–Scholes model, on the other hand, performs quite badly. For high values of the strike price, K_1, it significantly overprices the compound option in the case of Parameter Set I and significantly underprices it in the case of Parameter Set II. The reason is that, when K_1 is high, the first call option is exercised only when the asset price is very high at time T_1. Consider first Parameter Set I. As shown in Table 1, the implied volatility is a declining function of the strike price. (This is the volatility skew phenomenon for a stock index described earlier). As a result the probability distribution of the asset price at time T_1 has a fatter left tail and a thinner right tail than a lognormal distribution when the latter is calculated using the at-the-money volatility, and very high asset prices are much less likely than they are under the Black–Scholes model. This means that the first option is much more likely to be exercised in the Black–Scholes world than in the assumed true world. Consider next Parameter Set II. As shown in Table 2, the implied volatility is a U-shaped function. (This is the volatility smile phenomenon for a currency described earlier.) The results in the probability distribution of the asset price having fatter left and right tails than a lognormal distribution when the latter is calculated using the at-the-money volatility, and very high asset prices are much more likely than they are under the Black–Scholes model. This means that the first option is much less likely to be exercised in the Black–Scholes world than in the assumed true world.

Practitioners sometimes try to make the Black–Scholes model work for compound options by adjusting the volatility. Sometimes they use two different volatilities, one for the period between time zero and time T_1 and the other for the period between time T_1 and time T_2. There is of course some volatility (or pair of volatilities) that will give the correct price for any given compound option. But the price of a compound option given by the Black-Scholes model is highly sensitive to the volatility and any procedure that involves estimating the "correct" volatility is dangerous and liable to give rise to significant errors. Based on the tests reported here and other similar tests we have carried out, the IVF model provides a satisfactory approach to valuing compound options for the two-factor model we have considered. This is encouraging, but of course it provides no guarantee that the model will work well for more complicated multifactor models.

4.2 Barrier Options

The second exotic option we consider is a knock-out barrier call option. This is a European call option with strike price K and maturity T that ceases to exist if the asset price reaches a barrier level, H. When the barrier is greater than the initial asset price, the option is referred to as an up-and-out call; when the barrier is less than the initial asset price, it is referred to as a down-and-out call.

When using Monte Carlo simulation to calculate f_{true}, each trial involved using equations (9) and (10) simulate a path for the asset price between time zero and time T. For an up-and-out (down-and-out) option, if for any i, the asset price is above (below) H at time $i\Delta t$ on the jth trial the payoff from the barrier option is set equal to zero on that trial. Otherwise the payoff from the barrier option is $\max[S(T)-K,\, 0]$ at time T. The estimate of f_{true} is the arithmetic mean of the payoffs on all trials discounted from time T to time zero at rate r.[2]

We calculated the IVF price for the barrier option by building the finite difference grid out to time T. In the case of a up-and-out option, we set $x_{\max} = \ln(H)$ and $x_{\min} = \ln S_{\min}$ where $S_{\min}$ is a very low asset price; in the case of a down-and-out option, we set $x_{\min} = \ln(H)$ and $x_{\max} = \ln S_{\max}$ where $S_{\max}$ is very high asset price. For an up-and-out call option, the boundary conditions are:

$$\begin{aligned} f_{\text{ivf}} &= \max(e^x - K_2, 0) \text{ when } t = T, \\ f_{\text{ivf}} &= 0 \text{ when } x \geq \ln(H) \text{ and } 0 \leq t \leq T, \\ f_{\text{ivf}} &= 0 \text{ when } x = x_{\min} \text{ and } 0 \leq t \leq T. \end{aligned}$$

For a down-and-out call, the boundary conditions are the similar except that

$$f_{\text{ivf}} = e^x - K_2 e^{-r(T-t)} \text{ when } x = x_{\max}.$$

The value of knock-out options using the Black–Scholes assumption in equation (1) was first produced by Merton (1973). He showed that at time zero, the price of a down-and-out call option is

$$\begin{aligned} f_{\text{bs}} &= S(0)N(d_1)e^{-qT} - KN(d_2)e^{-rT} - S(0)e^{-qT}[H/S(0)]^{2\lambda}N(y) \\ &\quad + Ke^{-rT}[H/S(0)]^{2\lambda-2}N(y - \sigma\sqrt{T}), \end{aligned}$$

and that the price of an up-and-out call is

$$\begin{aligned} f_{\text{bs}} &= S(0)e^{-qT}[N(d_1) - N(x_1)] - Ke^{-rT}[N(d_2) - N(x_1 - \sigma\sqrt{T})] \\ &\quad + S(0)e^{-qT}[H/S(0)]^{2\lambda}[N(-y) - N(-y_1)] \\ &\quad - Ke^{-rT}[H/S(0)]^{2\lambda-2}[N(-y + \sigma\sqrt{T}) - N(-y_1 + \sigma\sqrt{T})], \end{aligned}$$

where

$$\begin{aligned} \lambda &= \frac{r - q + \sigma^2/2}{\sigma^2}, \\ y &= \frac{\ln\{H^2/[S(0)K]\}}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T}, \\ x_1 &= \frac{\ln[S(0)/H]}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T}, \\ x_2 &= \frac{\ln[H/S(0)]}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T}, \end{aligned}$$

[2] To improve computational efficiency we applied the correction for discrete observations in Broadie, Glasserman, and Kou (1997).

and d_1 and d_2 are as in equation (2). In computing $f_{\rm bs}$ we used the implied volatility of a regular European call option with strike price K maturing at time T.

Tables 4 and 5 shows $f_{\rm true}$ and the percentage errors when the option is approximated by $f_{\rm ivf}$ and $f_{\rm bs}$ for the cases where K are 90% and 100% of the initial asset price. We consider a wide range of values for the barrier H. (When $H > 100$ the option is an up-and-out call; when $H < 100$ it is a down and out call.) A comparison of Table 3 with Tables 4 and 5 shows that the IVF model does not perform as well as for barrier options as it does for compound options. For example, when $H = 98$ for Parameter Set I and $H = 110$ for Parameter Set II, the errors are high in both absolute terms and percentage terms. The Black–Scholes model sometimes works better and sometimes works worse than the IVF model and is clearly not a satisfactory alternative. Based on these and other similar tests we have carried out, the IVF model does not always give satisfactory prices for barrier options. A more sophisticated multifactor model appears necessary to handle these types of options adequately.

5 Summary

The implied volatilities of European call options with different strike prices and maturities define the unconditional probability distribution of the underlying asset price at all future times. The IVF model matches these volatilities exactly. It, therefore, also matches the unconditional probability distribution for the asset price at all future times. An exotic option, whose payoff is contingent on the asset at just one future time is, therefore, correctly priced by the IVF model. Unfortunately, many exotic options depend on the joint probability distribution of the asset price at two or more times. There is no guarantee that the IVF model will provide a reasonably accurate representation of these joint distributions.

In this paper we test the IVF model by assuming that the the asset price follows a stochastic volatility model and then comparing the prices of compound options and barrier options with those given by the IVF model. We find that the IVF model gives reasonably good results for compound options. The results for barrier options are much less satisfactory. The IVF model does not recover enough aspects of the dynamic features of the asset price process to give reasonably accurate prices for some combinations of the strike price and barrier level. A more sophisticated multifactor model appears to be necessary to handle barrier options adequately.

Academics tend to have different views from traders on how models should be used. Academics prefer stationary models where parameters are not functions of time. Traders consider it important to exactly fit all observed market prices and are prepared to tolerate a high degree of nonstationarity to achieve this objective. This paper has tested one nonstationary model that is popular among traders. It has produced evidence to show that there is an element of data over-fitting in the model and it should be used with caution.

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Table 1: Volatility matrix for Parameter Set I. K=strike price as a percent of the initial asset price; T=time to maturity.

$T\backslash K$	60.00	70.00	80.00	85.00	90.00	95.00	100.00	105.00	110.00	115.00	120.00	130.00	140.00
0.175	33.48	29.67	28.06	27.31	26.58	25.87	25.16	24.47	23.79	23.13	22.48	21.29	21.35
0.425	31.35	29.71	28.20	27.47	26.76	26.07	25.39	24.71	24.05	23.40	22.77	21.56	20.49
0.695	31.40	29.82	28.34	27.64	26.95	26.27	25.61	24.95	24.31	23.68	23.06	21.88	20.81
0.940	31.46	29.91	28.47	27.78	27.11	26.45	25.8	25.17	24.54	23.93	23.33	22.17	21.11
1.000	31.48	29.94	28.50	27.82	27.15	26.49	25.85	25.22	24.60	23.99	23.39	22.24	21.18
1.500	31.59	30.11	28.74	28.08	27.45	26.82	26.21	25.61	25.02	24.45	23.88	22.78	21.76
2.000	31.69	30.27	28.95	28.32	27.71	27.12	26.54	25.97	25.41	24.86	24.32	23.28	22.30
3.000	31.85	30.53	29.31	28.73	28.18	27.63	27.10	26.58	26.08	25.58	25.09	24.15	23.26
4.000	31.97	30.74	29.61	29.08	28.56	28.06	27.57	27.10	26.63	26.18	25.74	24.89	24.07
5.000	32.07	30.91	29.86	29.36	28.88	28.42	27.97	27.53	27.11	26.69	26.29	25.51	24.76

The parameters for the stochastic volatility model used in generating this table are: $r = 5.9\%$, $q = 1.4\%$, $v_0 = 0.0114$, $\kappa = 0.16$, $\theta = 0.3$, $\xi = 0.09$, and $\rho = -0.79$. The volatility matrix is similar to that for an equity index

Table 2: Volatility matrix for Parameter Set II. K=strike price as a percent of the initial asset price; T=time to maturity.

$T\backslash K$	60.00	70.00	80.00	85.00	90.00	95.00	100.00	105.00	110.00	115.00	120.00	130.00	140.00
0.175	32.30	23.03	14.56	12.85	12.73	12.75	12.83	12.96	13.13	13.32	13.56	16.16	21.10
0.425	21.13	14.97	12.90	12.77	12.72	12.74	12.81	12.93	13.09	13.27	13.46	13.88	14.50
0.695	16.80	13.44	12.89	12.77	12.71	12.72	12.79	12.90	13.06	13.23	13.42	13.81	14.20
0.940	14.86	13.32	12.89	12.76	12.70	12.71	12.77	12.88	13.02	13.19	13.37	13.76	14.14
1.000	16.46	13.31	12.89	12.76	12.70	12.70	12.76	12.87	13.02	13.18	13.36	13.74	14.12
1.500	17.94	13.30	12.89	12.76	12.69	12.68	12.73	12.82	12.96	13.11	13.29	13.65	14.02
2.000	17.83	13.30	12.89	12.76	12.69	12.67	12.70	12.78	12.90	13.05	13.21	13.56	13.92
3.000	17.82	13.30	12.91	12.77	12.68	12.65	12.66	12.72	12.81	12.94	13.08	13.40	13.73
4.000	18.15	13.31	12.92	12.78	12.69	12.64	12.63	12.67	12.74	12.84	12.97	13.25	13.56
5.000	18.71	13.31	12.93	12.80	12.70	12.64	12.62	12.63	12.69	12.77	12.87	13.13	13.41

The parameters for the stochastic volatility model used in generating this table are: $r = 5.9\%$, $q = 3.5\%$, $v_0 = 0.1285$, $\kappa = 0.1090$, $\theta = 0.10$, $\xi = 0.0376$, and $\rho = 0.1548$. The volatility matrix is similar to that for a foreign currency

Table 3: Numerical Results for Compound Options. Parameter Set I gives a volatility matrix similar to that obtained from options on an equity index (See Table 1). Parameter Set II gives a volatility matrix similar to that obtained from options on a foreign currency (see Table 2)

Strike	Parameter Set I			Parameter Set II		
K_1	True Price	IVF % Error	BS % Error	True Price	IVF % Error	BS % Error
3.45	15.45	-0.05	0.43	6.24	-0.95	-0.72
5.90	13.60	-0.14	1.23	4.82	-1.64	-1.76
8.35	11.93	-0.25	2.47	3.70	-1.67	-3.13
10.80	10.42	-0.36	4.14	2.83	-1.73	-4.78
13.25	9.07	-0.45	6.28	2.14	-1.63	-6.77
15.70	7.85	-0.54	8.99	1.62	-1.42	-9.22
18.15	6.76	-0.53	12.31	1.21	-1.07	-12.13
20.60	5.79	-0.59	16.34	0.90	-0.77	-15.49
23.05	4.92	-0.47	21.14	0.67	-1.04	-19.49
25.50	4.16	-0.36	26.93	0.50	-0.80	-24.05
27.95	3.50	-0.26	33.77	0.37	0.81	-28.72
30.40	2.92	-0.14	41.89	0.27	1.47	-33.82
32.85	2.417	0.00	51.39	0.20	1.00	-39.00
35.30	1.99	0.25	62.58	0.15	2.74	-43.84
37.75	1.623	0.62	75.85	0.11	4.67	-49.53
40.20	1.32	1.06	91.48	0.08	5.13	-55.13
42.65	1.06	1.42	110.02	0.056	7.02	-59.65
45.10	0.84	1.68	131.75	0.04	7.14	-64.29

The table shows the true price as a percent of the initial asset price, and the percentage error when this is approximated using the IVF model and the Black–Scholes model. The maturity of the first option, T_1, is 1 year; the maturity of the second option, T_2, is 2 years; the second strike price, K_2 equals the initial asset price; the first strike price K_1 is shown in the table as a percentage of the initial asset price.

Table 4: Numerical Results for Knock-Out Barrier Options when the Strike Price is 90% of the Initial Asset Price. Parameter Set I gives a volatility matrix similar to that obtained from options on an equity index (See Table 1). Parameter Set II gives a volatility matrix similar to that obtained from options on a foreign currency (see Table 2)

Barrier	Parameter Set I			Parameter Set II		
H	True Price	IVF % Error	BS % Error	True Price	IVF % Error	BS % Error
50	24.00	0.09	0.18	14.94	0.07	0.02
60	23.77	-0.06	0.64	14.94	0.04	0.02
70	22.69	0.00	1.67	14.93	0.09	0.05
80	19.74	-0.42	0.75	14.72	0.33	0.37
90	12.97	-0.69	-3.44	12.08	-0.75	-0.57
92	10.95	-0.74	-4.67	10.58	-0.51	-0.09
94	8.72	-1.20	-6.54	8.67	-0.56	0.10
96	6.18	-1.84	-8.75	6.32	-1.32	-0.08
98	3.36	-4.08	-12.84	3.42	-1.44	0.50
102	0.01	-31.92	-19.81	0.09	-23.14	-28.20
104	0.03	-26.60	-17.49	0.26	-21.65	-26.61
106	0.06	-25.91	-20.02	0.52	-18.28	-22.92
108	0.11	-26.61	-23.54	0.87	-16.81	-20.16
110	0.18	-24.16	-24.84	1.32	-14.33	-17.43
120	1.01	-18.54	-31.40	4.42	-6.62	-4.37
130	2.90	-13.85	-36.74	7.74	-2.71	3.58
140	5.91	-10.39	-40.49	10.41	-1.37	6.40
150	9.66	-8.15	-42.35	12.19	-0.52	6.81

The table shows the true price as a percent of the initial asset price, and the percentage error when this is approximated using the IVF model and the Black–Scholes model. The barrier is shown in the table as a percent of the initial asset price.

Table 5: Same as Table 4, except that the strike price equals the initial asset price.

Barrier	Parameter Set I			Parameter S et II		
H	True Price	IVF % Error	BS % Error	True Price	IVF % Error	BS % Error
50	18.40	0.07	0.13	8.93	0.01	0.02
60	18.29	-0.05	0.54	8.93	0.01	0.02
70	17.61	0.11	2.12	8.93	0.02	0.04
80	15.61	-0.25	2.79	8.86	0.39	0.42
90	10.55	-0.54	0.52	7.72	0.48	0.37
92	8.96	-0.59	-0.37	6.91	1.14	1.10
94	7.18	-0.97	-1.91	5.84	1.11	1.10
96	5.12	-1.64	-3.91	4.39	0.30	0.64
98	2.80	-3.90	-7.96	2.45	0.04	0.88
102	0.00	n.a.	n.a.	0.00	n.a.	n.a.
104	0.00	-42.41	-9.93	0.01	-46.41	-34.04
106	0.00	-36.50	-19.52	0.03	-33.51	-29.92
108	0.01	-40.69	-28.23	0.09	-30.36	-25.62
110	0.03	-32.00	-27.10	0.19	-24.08	-21.73
120	0.34	-24.21	-33.29	1.44	-10.56	-4.18
130	1.39	-18.08	-38.61	3.41	-4.36	6.64
140	3.40	-13.40	-42.20	5.27	-2.33	10.12
150	6.20	-10.18	-43.88	6.62	-0.90	10.38