Approximate Option Pricing *

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Abstract

As increasingly large volumes of sophisticated options are traded in world financial markets, determining a "fair" price for these options has become an important and difficult computational problem. Many valuation codes use the binomial pricing model, in which the stock price is driven by a random walk. In this model, the value of an n-period option on a stock is the expected time-discounted value of the future cash flow on an n-period stock price path. Path-dependent options are particularly difficult to value since the future cash flow depends on the entire stock price path rather than on just the final stock price. Currently such options are approximately priced by Monte Carlo methods with error bounds that hold only with high probability and which are reduced by increasing the number of simulation runs.

In this article we show that pricing an arbitrary path-dependent option is #-P hard. We show that certain types of path-dependent options can be valued exactly in polynomial time. Asian options are path-dependent options that are particularly hard to price, and for these we design deterministic polynomial-time approximate algorithms. We show that the value of a perpetual American put option (which can be computed in constant time) is in many cases a good approximation to the value of an otherwise identical n-period American put option. In contrast to Monte Carlo methods, our algorithms have guaranteed error bounds that are polynomially small (and in some cases exponentially small) in the maturity n. For the error analysis we derive large-deviation results for random walks that may be of independent interest.

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1 Introduction

Over the last decade or so, sophisticated financial instruments called **derivative securities** [10, 17] have become increasingly important in world financial markets. These are securities whose value depends on the values of more basic underlying variables. For instance a stock option is a derivative security whose value is contingent on the price of a stock. The daily trading volume in options, futures and other types of derivative securities is in the *trillions* of dollars, often exceeding the trading volume of the underlying asset [29]. In addition, corporations and financial institutions trade in complicated contingent claims (called over-the-counter options) that are tailored to their own needs and are not traded on exchanges.

Hedgers find it advantageous to trade in a derivative security on an asset rather than in the asset itself, to reduce the risk associated with the price of the asset. Also, speculators trade in options on stocks to get extra leverage from a favorable movement of the stock price. For instance, suppose the current price of a certain stock is \$20, and an investor feels that it will rise. If she buys the stock now and the price rises to say \$25 in 60 days, she can at that time realize a profit of \$5, so the return on her investment would be 5/20, or 25%. Now suppose that instead of buying the stock itself, she buys, for \$1, a *call option* that gives her the right (but *not* the obligation) to buy one share of the stock at \$20 in 60 days. If the stock price in 60 days is less than \$20, she will choose not to exercise her option, losing only her initial investment of \$1. On the other hand, if the stock price in 60 days is say \$25, she can realize a profit of \$(5−1) by exercising her option contract and buying the stock at \$20. The return on her investment is now (5−1)/1 or 400%.

As in the above example, a price must be paid to own a derivative security, and a central problem is the one of determining a "fair" price. An option is priced, or "valued", by assuming (a) some model of the price behavior of the underlying asset (e.g., a stock), and (b) a pricing theory. In a landmark article, Black and Scholes [2] introduced a continuous-time model for option valuation that underlies most pricing methods in use today. Their model is based on *Arbitrage Pricing Theory* [10, 17]. The model assumes that the asset price is driven by a Brownian motion, and specifies a stochastic differential equation that the option value must satisfy.

For many complex options, such as *Asian Options* and *(American) Lookback options*, the Black-Scholes differential equation has no known closed form solution, so numerical approximations are used. In Monte Carlo methods [4, 22, 24] one runs several continuous-time simulations of the Black-Scholes model to estimate the option price – which is the time-discounted expectation of the future cash flow. This approach is justified by the law of large numbers. In finite difference methods [7, 18, 29] the underlying stochastic differential equation is discretized and solved iteratively.

The error bound typically guaranteed by Monte Carlo methods is $O(\sigma/\sqrt{N})$, where N is the number of simulation runs, and σ is the standard deviation of the future cash flows [22]. It should be noted that this bound only holds with "high" probability, is expressed in terms of the *extrinsic* parameter N, and depends on the underlying dynamic only through σ. On the other hand, approximations based on finite-difference methods usually lack a precise quantification of the error term (see [26]).

In contrast to the above methods, the widely-used *binomial pricing model* [9, 17] is based on a simpler discrete-time process. The mathematical justification of this model is that the standard symmetric random walk, appropriately scaled, converges to Brownian motion. As in the continuous models, the price of an n-period option is the time-discounted expected value of the future cash-flows over n periods. Even under this model, *path-dependent* options [19] such as Asians and Lookbacks are particularly difficult to value: for such options, the future cash flows depend on the entire stock price path rather than on just the final stock price, and there are 2^n possible paths.

In this article, we study the option pricing problem from the rigorous perspective of computational complexity and approximation algorithms. We assume the binomial model throughout. We show that the problem of pricing arbitrary path-dependent options is #-P hard. For certain path-dependent options we show polynomial-time exact pricing algorithms. For the notoriously hard Asian option pricing problem, we design *deterministic polynomial-time* (in n) approximation algorithms. In contrast to the Monte Carlo methods, our error bounds are expressed in terms of *intrinsic* parameters such as the maturity n of the option: in fact they are polynomially and in some cases exponentially small in n. In some cases our algorithms run in time independent of n. We also show that in some cases the price of an American option can be approximated well by that of an otherwise equivalent *perpetual* option, whose value is $O(1)$-time computable. For the error analysis we prove several large-deviation results on random walks. We thus hope to demonstrate that the field of derivative securities is a rich source of opportunities for computer science research.

For more details on option pricing and Arbitrage Pricing Theory, the interested reader is referred to Hull's [17] excellent introductory text. However the present article defines all the concepts needed, and will suffice to understand the computational problems involved. Section 1.1 describes the binomial model for stock prices. Section 1.2 defines the options considered in this article, and Section 1.3 describes the pricing formulas and the specific results in the article. The remaining sections contain our results.

1.1 The binomial model for stock prices

To keep the wording simple, we only consider options on stocks. The notation described in this section will be used throughout the article. For easy reference, a summary of notation is included at the end of the article. The **binomial model** for the price of the stock underlying an n-time-period ($n \geq 1$) option is the following. The model is parametrized by the constants p, u, r. In actual applications these parameters are chosen so that the discrete-time stock-price process (to be described below) approximates a continuous-time stochastic process driven by a Brownian motion. In this paper we will simply assume these parameters are given. It will be convenient to write q for $1 - p$. In this model, n is the (possibly infinite) number of time periods up to the expiration of the option, where time 0 is the initial time, i.e., the time at which one wants to price the option. The trading dates are times $0, 1, \dots, n$. The stock price at time k is denoted S_k. The initial stock price S_0 is assumed to be non-random. u is the **up-factor**, p is the **up-tick probability,** r is the **risk-free interest rate.** At each time step, with probability

p the stock price goes up by a factor u, and with probability $q = 1 - p$ the price goes down by a factor $1/u$. The parameters u, p, q, r satisfy (see [17]):

$$\begin{aligned} u > 1, &\quad 1/u < 1 + r < u, \\ 0 < p < 1, &\quad p + q = 1, \end{aligned} \tag{1}$$

$$p = \frac{1 + r - 1/u}{u - 1/u}, \qquad \text{or equivalently,} \quad pu + q/u = 1 + r. \tag{2}$$

We now formalize the model. It will be convenient to visualize a sequence of n independent coin-tosses $\omega = (\omega_1, \omega_2, \ldots \omega_n)$, where each $\omega_i \in \{H, T\}$; an H corresponds to an "up-tick" of the stock price, and a T corresponds to a "down-tick". A particular sequence of coin-tosses ω will be referred to as a **path.** The sample space Ω is the set of all possible coin-toss sequences ω. We define the random variables $X_1, X_2, \ldots, X_n$ where for any $\omega \in \Omega$,

$$X_i(\omega) = \begin{cases} +1 & \text{if } \omega_i = H, \\ -1 & \text{otherwise.} \end{cases}$$

We define the probability measure **P** on Ω to be the unique measure for which the random variables X_i, $i = 1, 2, \ldots, n$, are independent, identically distributed (i.i.d.) with

$$\mathbf{P}[X_i = 1] = p \quad \text{and} \quad \mathbf{P}[X_i = -1] = q = 1 - p.$$

We will refer to the sequence of random variables $\{X_i\}_{i=1}^n$ with the above distribution as the **random walk with drift** p. Then the stock price S_k, $k \geq 0$, is a random variable that satisfies

$$S_{k+1} = S_k u^{X_{k+1}}.$$

We also define

$$\begin{aligned} Y_0 &= 0, \\ Y_k &= \sum_{i=1}^{k} X_i, \qquad k \geq 1, \\ T_k &= \sum_{i=1}^{k} S_i, \qquad k \geq 1, \end{aligned}$$

Thus we can write for $k \geq 0$,

$$S_k = S_0 u^{Y_k}$$

For any integer $k \geq 0$, for any random variable Z, the conditional expectation

$$\mathbf{E}[Z | X_1, X_2, \ldots, X_k]$$

of Z given the first k coin tosses will be denoted compactly by $\mathbf{E}[Z|\mathcal{F}_k]$. ($\mathcal{F}_k$ is the σ-algebra determined by the first k tosses.) In particular $\mathbf{E}[Z|\mathcal{F}_0] \triangleq \mathbf{E}[Z]$. For any integer $k \geq 1$, a random variable Z is **$\mathcal{F}_k$-measurable** if it depends only on the first k coin tosses, i.e., on $X_1, X_2, \ldots, X_k$. An $\mathcal{F}_0$-measurable random variable is non-random.

It is common to refer to a sequence of random variables as a **process.** In particular, $\{S_k\}_{k=0}^n$ is the **stock price process.** A process $\{Z_k\}_{k=0}^n$ such that each Z_k is $\mathcal{F}_k$-measurable, is said to be **adapted.** Thus the stock price process is adapted. An adapted process $\{Z_k\}_{k=0}^n$ is said to be a **martingale** if for $k = 0, 1, \ldots, n-1$,

$$\mathbf{E}(Z_{k+1}|\mathcal{F}_k) = Z_k.$$

The process is called a **supermartingale** if for $k = 0, 1, \ldots, n-1$, the above relation holds with $\leq$ replacing the equality.

An important fact in the binomial model just described is that the discounted stock price process $\{(1+r)^{-k} S_k\}_{k=0}^n$ is a martingale. This is easy to check, since for $k = 0, 1, \ldots, n-1$

$$\mathbf{E}(S_{k+1}|\mathcal{F}_k) = S_k \mathbf{E} u^{X_{k+1}} = S_k(pu + q/u) = S_k(1+r).$$

and this implies

$$\mathbf{E}[(1+r)^{-(k+1)} S_{k+1}|\mathcal{F}_k] = (1+r)^{-k} S_k, \quad k = 0, 1, \ldots n-1.$$

The martingale property also implies that

$$\mathbf{E} S_k = S_0 (1+r)^k.$$

For any process $\{Z_k\}_{k=0}^n$ we define:

$$\overline{Z}_k \triangleq \max_{0 \leq i \leq k} Z_i.$$

$$\underline{Z}_k \triangleq \min_{0 \leq i \leq k} Z_i.$$

1.2 Options

There are two basic types of options. A **call option** on a stock is a contract that gives the holder the right to *buy* the underlying stock by a certain date, for a certain price. A **put option** gives the holder the right to *sell* the underlying stock by a certain date for a certain price. The price in the contract is known as the **strike price,** and is denoted by L. The date in the contract is known as the **exercise date,** or **expiration date.** Recall that n denotes the number of time periods until the expiration of the option. The holder of the option must pay a certain price, called the **option price** to the issuer of the option. The **option pricing problem** is to determine the "fair" price to pay for an option. This will become clearer later. An **American-style option** can be exercised at any time up to the expiration date. **European-style options** can only be exercised on the expiration date itself. It is important to note that an option contract merely gives the holder the *right* to exercise; the holder need not exercise it.

The **payoff** G_k from an option (for the holder) at time k is 0 if it cannot be exercised at time k. Otherwise G_k is the maximum of 0 and the profit that can be realized by exercising the option at time k. This profit ignores the price paid by the buyer for the option. For instance consider an American Call option. If $S_k > L$, the holder can exercise the option at time k by buying the stock at L and realize a profit of $S_k - L$ by selling the stock in the market at S_k. If $S_k \leq L$, no positive profit can be made by exercising. Thus, for an American call, the payoff is the random variable

$$G_k = (S_k - L)^+, \quad k = 0, 1, 2, \ldots, n, \qquad \text{(Call)}$$

where for any $x \in \mathbb{R}$, $x^+ = \max\{x, 0\}$. Similarly for an American put,

$$G_k = (L - S_k)^+, \quad k = 0, 1, 2, \ldots, n. \qquad \text{(Put)}$$

The payoff functions for the European options are the same as for their American counterparts, except that exercise is only allowed at time $k = n$, so $G_k = 0$ for all $k < n$.

In the case of simple calls and puts, the payoff at any time depends only on the prevailing stock price, i.e., $G_k = g(S_k)$ for some function g. Such options are said to be **Markovian,** or path-independent. However there are many options that are **path-dependent** [14, 17, 19]. One class of such options we consider in this article are Asian options. An (European-style) **Asian call** option is one that can be exercised only at time n, and whose payoff G_n is given by

$$G_n = (A_n - L)^+, \qquad \text{(Asian call)}$$

where A_n is the *average* stock price *from time 1* to time n: $A_n = T_n/n$. We do not include S_0 in the computation of the average only for notational convenience; since S_0 is a fixed constant, this does not affect our results.

Similarly, a (European-style) **Asian put** has payoff

$$G_n = (L - A_n)^+. \qquad \text{(Asian put)}$$

Asian options are of obvious appeal to a company which must buy a commodity at a fixed time each year, yet has to sell it regularly throughout the year [29]. These options allow investors to eliminate losses from movements in an underlying asset without the need for continuous rehedging. Such options are commonly used for currencies [29], interest-rates and commodities such as crude oil [15].

We consider two other path-dependent option payoffs in the article: (Let $\mathbf{1}_A$ denote the indicator function for any subset $A \subseteq \Omega$.)

$$G_k = \begin{cases} (\overline{S}_k - L)^+, \ k \geq 0 & \textbf{(Lookback)} \\ \mathbf{1}_{\overline{S}_k \geq B}(S_k - L)^+, \ k \geq 0 & \textbf{(Knock-in barrier)}. \end{cases}$$

We also consider the **American perpetual put** (APP) option, which has an associated strike price L just like an ordinary American put, except that there is no expiration date. The payoff G_k for an APP is therefore given by

$$G_k = (L - S_k)^+, \quad k \geq 0. \qquad \text{(perpetual put)}$$

1.3 Pricing formulas, and results in the article

Since a European option can be viewed as an American option with payoff $G_k = 0$ for all $k < n$, pricing formulas for American options apply equally well to European options. However, the formulas for European options are somewhat simpler and we describe them first.

For European-style options with payoff G_n, the **value** of the option at time k is defined by

$$V_k = (1+r)^k \mathbf{E}[(1+r)^{-n} G_n | \mathcal{F}_k], \qquad k = 0, 1, \ldots, n, \tag{3}$$

which is the expected payoff at expiration, discounted by the risk-free interest rate over $n - k$ periods. In particular we have $V_n = G_n$. We refer to the time-0 value V_0 as simply "the value" of the option, and denote it by V:

$$V \stackrel{\triangle}{=} V_0 = (1+r)^{-n} \mathbf{E} G_n. \tag{E}$$

The **pricing problem,** which this article deals with, consists of evaluating the formula (E) for the value V of an option. We show in Section 2 that this problem is #-P hard for an arbitrary (polynomially-specified) path-dependent European option. It is easy to see that ordinary European calls and puts can easily be valued in $O(n)$ time: there are only $n+1$ possible values of S_n, and G_n depends only on S_n. However, the valuation of Asian calls and puts is a well-known hard problem in finance and much research has been directed at this problem [3, 12, 23, 27, 29, 30]. All known valuation methods for these options either use some form of Monte Carlo estimation or use analytic approximations with no error analysis. For instance, Turnbull and Wakeman [27] have proposed an analytic approximation for Asian options, but provide no error analysis; they only experimentally test the accuracy of their approximation against Monte Carlo estimates. In Section 4 we develop *deterministic polynomial-time* approximation algorithms for the value V of Asian options, along with error bounds. For the error analysis we show several large-deviation results for random walks that may be of independent interest.

To define the value of an American option, we need to use the notion of a stopping time[28]. Let Ω be the sample space of all possible coin-toss paths ω defined in Section 1.1. A **stopping time** is a random variable

$$\tau : \Omega \rightarrow \{0, 1, 2, \ldots, n\} \cup \{\infty\}$$

with the property that for each $k = 0, 1, \ldots, n, \infty$, the set $\{\tau = k\}$ belongs to the σ-algebra $\mathcal{F}_k$. This means that membership in the set $\{\tau = k\}$ depends only on the first k coin tosses of ω. Informally, a stopping time can be thought of as a "decision rule" of when to "stop" the coin-toss sequence (or the random walk).

For an American option with payoff functions $\{G_k\}_{k=0}^n$ (where n can be infinite), the value at time k is given by

$$V_k = (1+r)^k \max_{\tau \in \mathcal{T}_k} \mathbf{E}[(1+r)^{-\tau} G_\tau | \mathcal{F}_k], \tag{4}$$

where $\mathcal{T}_k$ is the class of stopping times τ satisfying $k \le \tau \le n$ almost surely. In particular, the value of the option at time 0 (which we simply refer to as "the value" V) is

$$V = V_0 = \max_{\tau \in \mathcal{T}_0} \mathbf{E}[(1+r)^{-\tau} G_\tau]. \tag{A}$$

It turns out that the discounted value process $\{(1+r)^{-k} V_k\}_{k=0}^{n}$ is a supermartingale, i.e.,

$$\mathbf{E}[V_{k+1} | \mathcal{F}_k] \le (1+r) V_k, \quad k = 0, 1, 2, \ldots .$$

The value V of an American perpetual put (APP) does not involve n, and it can be computed in $O(1)$ time in closed form. It is natural therefore to use this value to estimate the value of an otherwise identical n-period American put. In Section 5 we investigate the error of this estimate.

For a Markovian option with payoff $G_k = g(S_k)$, the definition (4) implies that $V_k = v_k(S_k)$ for some function v_k, where v_k satisfies:

$$\begin{aligned} v_n(S_n) &= g(S_n) \qquad \text{(only for options with finite } n\text{)} \\ v_k(S_k) &= \max\left\{ g(S_k),\ \frac{1}{1+r}\left(p v_{k+1}(uS_k) + q v_{k+1}(\frac{S_k}{u}) \right) \right\}, \qquad k = 0, 1, \ldots \end{aligned} \tag{5}$$

The backward-recursion equation (5) allows V to be computed by dynamic programming in $O(n^2)$ time, since there are only $k+1$ possible different values for S_k. In Section 3 we extend this approach to certain path-dependent options (such as the Lookback and Knock-in barrier options) whose payoff can be expressed as a function of a Markov process different from the stock price process $\{S_k\}$.

2 Pricing an arbitrary European option is #P-hard

Consider a European option with an arbitrary path-dependent payoff function G_n. We will restrict our attention to payoff functions G_n that can be specified in space polynomial in n. We then wish to evaluate $V \triangleq V_0$. We show that evaluating V is #P-hard.

Theorem 1 *The problem of pricing a European option with polynomially-specified payoff function G_n, is #P-hard.*

Proof: It is well-known that the following counting problem is #P-complete: Given a graph J with edge-set $E = \{e_1, e_2, \ldots, e_n\}$, count the number $M(J)$ of perfect matchings in J. We reduce this problem to the pricing problem. We define a (path-dependent) European option with expiration time n whose payoff G_n is given by:

$$G_n(\omega) = \begin{cases} (5/2)^n & \text{if } \{e_i : \omega_i = H\} \text{ is a perfect matching of } J \\ 0 & \text{otherwise.} \end{cases}$$

Note that the quantity $(5/2)^n$ can be computed in time polynomial in n. Next we choose $u = 2$ and $r = 0.25$ so that from (2), $p = q = \frac{1}{2}$, and $1 + r = 5/4$. Thus every path ω has probability $\mathbf{P}(\omega) = (\frac{1}{2})^n$. Clearly, from Eq. (E) the value of this option is

$$\begin{aligned} V &= (1+r)^{-n} \sum_{\omega \in \{H,T\}^n} \mathbf{P}(\omega) G_n(\omega) \\ &= (4/5)^n (\tfrac{1}{2})^n \sum_{\omega \in \{H,T\}^n} G_n(\omega) \\ &= (2/5)^n (5/2)^n M(J) \\ &= M(J) \end{aligned}$$

Thus if we can compute V exactly in polynomial time, this immediately gives the value of $M(J)$. ■

3 Exact pricing of some path-dependent options

We saw in Section 1.3 that the value V of a Markovian option can be computed in $O(n^2)$ time by dynamic programming, using the backward recursion formula (5). We generalize this dynamic programming approach to certain path-dependent options, such as the Lookback option, and the Knock-in barrier option. The main observation is that the backward-recursion formula (5) depends only on the fact that the stock price process $\{S_k\}$ is a **Markov process,** i.e., for $k \geq 0$, if h is any (Borel-measurable) function, then

$$\mathbf{E}[h(S_{k+1}, \ldots, S_n) | \mathcal{F}_k] = \mathbf{E}[h(S_{k+1}, \ldots, S_n) | S_k].$$

Therefore, we have the following theorem:

Theorem 2 *Consider an American option with payoff process $\{G_k\}_{k=0}^n$ where $G_k = g(C_k)$ where C_k is an adapted Markov process such that for each k, the set of different possible values of C_k is computable in time polynomial in n. Then the value V of this option can be computed in time polynomial in n using dynamic programming.*

For instance, it is not hard to show that the process $C_k \stackrel{\triangle}{=} (S_k, \overline{S}_k), k = 0, 1, \ldots, n$ is a Markov process. Moreover, for each k, there are at most $(k+1)^2$ possible combinations of values $(S_k, \overline{S}_k)$. Both the Lookback option and the Knock-in barrier option (see Section 1.2) have payoff functions G_k expressible as functions of $(S_k, \overline{S}_k)$, so they can be priced in $O(n^3)$ time by dynamic programming.

4 Approximate Pricing of Asian Options

We wish to approximate the value V for Asian calls and puts given by the formulae in Section 1.3. Since $(1+r)^{-n}$ is a known multiplicative factor, we will focus on approximating the *undiscounted value* $V' = (1+r)^n V$. Computing $\mathbf{E}(A_n - L)^+$ (or $\mathbf{E}(L - A_n)^+$) exactly is known to be a hard problem in finance. The exact computational complexity of this problem is not known. One approach is to approximate $\mathbf{E}(A_n - L)^+$ by $(\mathbf{E}A_n - L)^+$, which by Jensen's inequality is no larger than $\mathbf{E}(A_n - L)^+$. Note that it is easy to compute $\mathbf{E}A_n$ in closed form: since the discounted stock price process is a martingale, $\mathbf{E}S_k = S_0(1+r)^k$, so

$$\mathbf{E}A_n = \frac{S_0(1+r)\left[(1+r)^n - 1\right]}{nr}. \tag{6}$$

It is not hard to see that the quantities $[\mathbf{E}A_n - L]^+$ and $[L - \mathbf{E}A_n]^+$ respectively approximate the V' for an Asian call and an Asian put to within L.

Note that since

$$(A_n - L) = (A_n - L)^+ - (L - A_n)^+,$$

we have the following relation between the V' for an Asian call and an Asian put:

$$\mathbf{E}(A_n - L) = \mathbf{E}(A_n - L)^+ - \mathbf{E}(L - A_n)^+.$$

This relation is analogous to the **put-call parity** relation for simple calls and puts (see Hull [17]). Since $\mathbf{E}(A_n - L)$ is easily computed in closed form, it follows that if we have an approximation algorithm for the Asian put (i.e. for $\mathbf{E}(L - A_n)^+$) with a certain *additive* error, we immediately have an approximation algorithm for the Asian call with the same time complexity and additive error. In what follows we therefore confine ourselves to approximating $\mathbf{E}(L - A_n)^+$.

We now describe polynomial-time approximation algorithms for $\mathbf{E}(L - A_n)^+$ that are significantly better than the crude approximation $(L - \mathbf{E}A_n)^+$. The error analysis of these algorithms is based on certain large-deviation results on random-walks that we derive in Section 4.1. We use the notation $\beta = |2p - 1|$ since this value appears frequently in the error bounds. In the following description, we use the symbols $\mathbf{P}_\ell(c, n), \mathbf{P}_e(n), \mathbf{P}_g(n)$ (corresponding to the cases p less than, equal to and greater than $\frac{1}{2}$ respectively) to stand for different probabilities that will be determined in the next section. In most cases we can express the asymptotic difference between the exact value and our approximation in the form $L\ O(f(n))$, where in using the O notation we are treating the parameters L, S_0, u, β, r as constants. In particular, we should mention that asymptotic error bounds only come into play when n exceeds certain thresholds (specified in Corollaries 6 and 10) that depend on these parameters. We prove the following theorem:

Theorem 3 *The undiscounted value $V' = \mathbf{E}(L - A_n)^+$ of an Asian put is at most $L\ O(e^{-\beta^2 n/4})$ if $p > \frac{1}{2}$, and is at most $L\ O(\frac{\log n}{\sqrt{n}})$ if $p = \frac{1}{2}$. (We can therefore use half of these upper bounds as approximations for V')*

For $p < \frac{1}{2}$, there is a polynomial-time algorithm to approximate V' to within $2S_0/n$.

Proof:

Upward drift: $p > \frac{1}{2}$. We show (Theorem 5, Corollary 6) that with probability at least $1 - \mathbf{P}_g(n)$, all stock prices after $S_{n/2}$ are at least $2L$, so that $A_n \geq L$. This means that with probability at least $1 - \mathbf{P}_g(n)$, $(L - A_n)^+ = 0$. Since $(L - A_n)^+ \leq L$ and $(L - A_n)^+ > 0$ with probability at most $\mathbf{P}_g(n)$, we can upper bound

$$\mathbf{E}(L - A_n)^+ \leq L\ \mathbf{P}_g(n) = L\ O(e^{-\beta^2 n/4}),$$

since, as we will show, $\mathbf{P}_g(n) = O(e^{-\beta^2 n/4})$.

Undrifted: $p = \frac{1}{2}$**.** We show in Theorem 9 and Corollary 10 that with probability at least $1 - \mathbf{P}_e(n)$, some stock price before time n is at least nL, so that the average stock price A_n is at least L. Therefore by the same reasoning as above,

$$\mathbf{E}(L - A_n)^+ \leq L\ \mathbf{P}_e(n) = L\ O\left(\frac{\log n}{\sqrt{n}}\right),$$

since, as we will show, $\mathbf{P}_e(n) = O\left(\frac{\log n}{\sqrt{n}}\right)$.

Downward drift: $p < \frac{1}{2}$**.** Theorem 5 and Corollary 6 establish that for any $c > 0$, with probability at least $1 - \mathbf{P}_\ell(c, n)$, all stock prices after $m = O(c \log n)$ steps are at most S_0/n, and the difference between $A_n = T_n/n$ and T_m/n is at most S_0/n. Thus we can approximate $\mathbf{E}(L - A_n)^+$ by $\mathbf{E}(L - T_m/n)^+$, which can be computed in time $O(2^m) = n^{O(c)}$ since we need only consider coin-toss sequences of length m. When $(T_n - T_m) > S_0$ (which occurs with probability at most $\mathbf{P}_\ell(c, n)$), $(L - T_m/n)^+$ is at most $(L - A_n)^+ + L$. When $(T_n - T_m) \leq S_0$ (which occurs with probability at least $1 - \mathbf{P}_\ell(c, n)$), $(L - T_m/n)^+$ is at most $(L - A_n)^+ + S_0/n$. Combining these facts gives

$$\mathbf{E}(L - T_m/n)^+ \leq \mathbf{E}(L - A_n)^+ + L\mathbf{P}_\ell(c, n) + S_0/n.$$

It can be worked out that with $c = 2$, for n sufficiently large (as specified in Corollary 6), $\frac{S_0}{n} > L\mathbf{P}_\ell(c, n)$. Thus with an $n^{O(1)}$ running time we can achieve an error bound of $2S_0/n$.
■

In the next subsection we derive the large-deviation results that we assumed above. In subsection 4.2 we describe an algorithm that performs better in practice than the ones we described above.

4.1 Large-deviation results

We first show a fact about *drifted* random walks. We use the notation for random walks from Section 1.1. In particular recall that $Y_k = \sum_{i=1}^k X_i$ is the k'th partial sum of the random walk, and that $T_k = \sum_{i=1}^k S_i$. We will need the following bound due to Hoeffding [16]:

Theorem 4 (Hoeffding[16]) *Let $X_1, X_2, \ldots, X_n$ be independent, identically-distributed (i.i.d.) random variables whose values lie in the interval $[-1, 1]$, and let $Y_n = \sum_{i=1}^n X_i$. Then for $a > 0$ the following holds:*

$$\begin{aligned} \mathbf{P}[Y_n - \mathbf{E}Y_n < -a] &< e^{-\frac{a^2}{2n}} \\ \mathbf{P}[Y_n - \mathbf{E}Y_n > +a] &< e^{-\frac{a^2}{2n}} \end{aligned} \tag{H}$$

In particular, for the random walk with drift p, we have the i.i.d. random variables $X_1, X_2, \ldots, X_n$ where $\mathbf{P}[X_i = +1] = p$ and $\mathbf{P}[X_i = -1] = 1 - p$, and so $\mathbf{E}Y_i = (2p-1)i$.

Using this we show:

Theorem 5 (Drifted Random Walk) *Consider the random walk with drift p where $p \neq \frac{1}{2}$. Let $a \in (0, \beta n/2)$, and $m \in (2a/\beta, n)$. Then with probability at least $1 - \frac{2\sqrt{e}}{\beta^2} \exp\{\beta a - \beta^2 m/2\}$, for every integer $k \in [m, n]$:*

$$Y_k \geq a \quad \textit{if } p > \tfrac{1}{2}, \quad \textit{and} \quad Y_k \leq -a \quad \textit{if } p < \tfrac{1}{2}.$$

Proof: Suppose $p > \frac{1}{2}$, and let E_k denote the event $\{Y_k < a\}$. Then we have

$$\begin{aligned}\mathbf{P}[E_k] &= \mathbf{P}[Y_k < a] \\ &= \mathbf{P}\left[\mathbf{E}Y_k - Y_k > \mathbf{E}Y_k - a\right],\end{aligned}$$

and since $\mathbf{E}Y_k = (2p-1)k = \beta k \geq \beta m > 2a > a$, by Hoeffding bounds (eq. (H)):

$$\begin{aligned} &\leq \exp\left\{-\frac{1}{2k}\left(\beta^2 k^2 - 2k\beta a + a^2\right)\right\} \\ &= \exp\left\{\beta a - \beta^2 k/2\right\} \cdot \underbrace{e^{-\frac{1}{2}a^2/k}}_{\leq 1} \\ &\leq \exp\left\{\beta a - \beta^2 k/2\right\}. \end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{k=m}^{n} \mathbf{P}[E_k] &\leq e^{\beta a} \int_{m-1}^{\infty} e^{-\beta^2 x/2}\, dx \\ &= \frac{2}{\beta^2} \exp\left\{\beta a - \beta^2 (m-1)/2\right\} \\ &= \frac{2}{\beta^2} e^{\beta^2/2} \exp\left\{\beta a - \beta^2 m/2\right\} \\ &\leq \frac{2\sqrt{e}}{\beta^2} \exp\left\{\beta a - \beta^2 m/2\right\} \qquad \text{(since } \beta \leq 1\text{)}\end{aligned}$$

The proof for $p < \frac{1}{2}$ is exactly analogous. ■

Corollary 6 (Average Stock Price: Drifted Case) *Consider the binomial stock price process $\{S_k\}_{k=0}^{n}$ with $p \neq \frac{1}{2}$. Suppose L is the strike price of an Asian option (call or put). Let*

$$\begin{aligned}\mathbf{P}_g(n) &\triangleq \frac{2\sqrt{e}}{\beta^2}\left(\frac{2L}{S_0}\right)^{\frac{\beta}{\ln u}} e^{-\beta^2 n/4}, \\ \mathbf{P}_\ell(c, n) &\triangleq \frac{2\sqrt{e}}{\beta^2}\frac{1}{n^c} \quad \textit{(c is any positive constant)}, \\ m &\triangleq \left(\frac{2c}{\beta^2} + \frac{2}{\beta \ln u}\right) \ln n.\end{aligned}$$

Then:

1. *If $p > \frac{1}{2}$ and $n > \frac{4}{\beta}\log_u(2L/S_0)$ then with probability at least $1 - \mathbf{P}_g(n)$, every stock price S_i for $i \geq n/2$ is at least $2L$, and in particular $A_n \geq L$.*

2. *If $p < \frac{1}{2}$, then with probability at least $1 - \mathbf{P}_\ell(c, n)$, every stock price S_i for $i \geq m$ is at most S_0/n and in particular $A_n - \frac{T_m}{n} \leq \frac{S_0}{n}$.*

Proof: Let $X_1, X_2, \ldots, X_n$ denote the random walk underlying the binomial process, and let Y_i be the i'th partial sum as in Theorem 5. Thus the stock price after i coin tosses is $S_0 u^{Y_i}$.

Case 1: $p > \frac{1}{2}$**.** Applying Theorem 5 with $m = \lfloor n/2 \rfloor$ and $a = \log_u(2L/S_0)$, we see that with probability at least $1 - \mathbf{P}_g(n)$, we have that $Y_i \geq \log_u(2L/S_0)$ for every $i \geq n/2$, or in other words, the stock price S_i for $i \geq n/2$ is at least $2L$, in which case average over *all* stock prices S_i is at least L.

Case 2: $p < \frac{1}{2}$**.** Applying Theorem 5 with m as in the statement of the present theorem and $a = \log_u n$ we see that with probability at least $1 - \mathbf{P}_\ell(c, n)$, we have that $Y_i \leq -\log_u n$ for every $i \geq m$, or in other words, every stock price S_i for $i \geq m$ is at most S_0/n. In such an event, the contribution of each stock price after $S_i, i \geq m$, to A_n is no more than S_0/n^2, so that the "error" in estimating A_n by T_m/n is at most S_0/n. ■

For the case $p = \frac{1}{2}$ we would like to use an argument similar to the one in the proof of Corollary 6 to show that with high probability the stock prices S_i are all "large" (e.g., at least $2L$) after say $n/2$ steps. That argument rests on the fact (Theorem 5) that with high probability all partial sums in a random walk after a certain point are "large". However the proof of Theorem 5 does not work with $p = \frac{1}{2}$. Instead, we show that with high probability at *some* time the stock price is at least nL, so that the average is at least L. For this we use the Berry-Essen Theorem and the Reflection Principle, which we quote below:

Theorem 7 (Berry-Essen [11, 21]) Let X_i, $i = 1, 2, \ldots, n$ be i.i.d. with $\mathbf{E}X_i = 0$, $\mathbf{E}X_i^2 = \sigma^2$, and $\mathbf{E}|X_i|^3 = \xi$. If $F_n(x)$ is the distribution of $\frac{(X_1+X_2+\ldots+X_n)}{\sigma\sqrt{n}}$ and $N(x)$ is the standard normal distribution, then

$$|F_n(x) - N(x)| \leq \frac{\xi}{\sigma^3\sqrt{n}}.$$

Theorem 8 (Reflection Principle[11]) Imagine drawing the paths of the random walk on the x-y plane as follows: for each $i = 0, 1, \ldots$, draw an edge from (i, Y_i) to $(i+1, Y_{i+1})$, where Y_i is the i'th partial sum. If T, T' are positive integers, the number of paths from $(0, T)$ to (n, T') that touch or cross the x-axis is equal to the number of paths from $(0, -T)$ to (n, T').

We first show a large-deviation result for the maximum partial sum of an undrifted random walk.

Theorem 9 (Undrifted random walk) *Consider the random walk with $p = \frac{1}{2}$. Recall that $\overline{Y}_n \triangleq \max_{0 \leq i \leq n} Y_i$. Then for any $a > 0$,*

$$\mathbf{P}[\overline{Y}_n \leq \lfloor a \rfloor] \leq \sqrt{2/\pi}\frac{a}{\sqrt{n}} + \frac{2}{\sqrt{n}}.$$

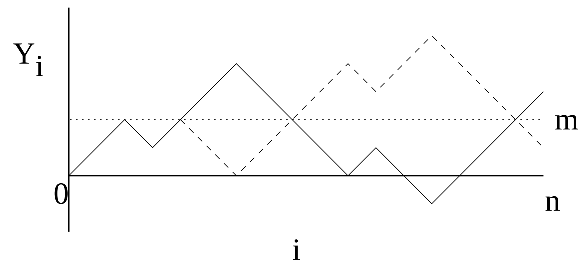

Figure 1: *Use of the Reflection Principle in Thm. 9.*

Proof:

For any integer $m > 0$ by the Reflection Principle (see Fig. 1) we have

$$\mathbf{P}(\overline{Y}_n \geq m) \geq 2\mathbf{P}(Y_n > m),$$

since with every path ω such that $Y_n(\omega) > m$, we can associate *two* paths for which $\overline{Y}_n \geq m$: one path is ω itself, and the other path ω' is identical to ω until time i, where i is the smallest k satisfying $Y_{k+1} > m$; from time $i+1$ to n, ω' is the reflection of ω through the line $y = m$. Thus,

$$\begin{aligned}
\mathbf{P}[\overline{Y}_n > \lfloor a \rfloor] \geq 2\mathbf{P}\left[Y_n > \lfloor a \rfloor\right] \quad &\geq \quad 2\mathbf{P}\left[\frac{Y_n}{\sqrt{n}} > \frac{a}{\sqrt{n}}\right] \\
&= \quad 2\left[1 - \mathbf{P}\left[\frac{Y_n}{\sqrt{n}} \leq \frac{a}{\sqrt{n}}\right]\right] \\
&\geq 2\left[1 - N\left(\frac{a}{\sqrt{n}}\right) - \frac{1}{\sqrt{n}}\right] \qquad \text{(Berry-Essen Theorem)} \\
&\geq 1 - \sqrt{\frac{2}{\pi}}\frac{a}{\sqrt{n}} - \frac{2}{\sqrt{n}} \qquad \text{(since } N(x) \leq \frac{1}{2} + x/\sqrt{2\pi} \text{ for } x > 0\text{).}
\end{aligned}$$

■

The following is a straightforward application of this theorem.

Corollary 10 (Average Stock Price: Undrifted Case.) *Consider the binomial stock price process with $p = \frac{1}{2}$, starting with price S_0 at time 0, and let L be the strike price in the Asian option. Let*

$$\mathbf{P}_e(n) \triangleq \sqrt{\frac{2}{\pi}}\frac{\log_u(nL/S_0)}{\sqrt{n}} + \frac{2}{\sqrt{n}}.$$

Then for $n > S_0/L$, with probability at least $1 - \mathbf{P}_e(n)$, the maximum stock price on a path is at least nL, and in particular $A_n \geq L$.

Proof: Let $X_1, X_2, \ldots, X_n$ be the random walk underlying the binomial model, and let Y_i be the i'th partial sum as before. Applying Theorem 9 with $a = \log_u(nL/S_0)$ we see that with probability at least $1 - \mathbf{P}_e(n)$ the highest Y_i is at least a, so that the highest stock price is at least $S_0 u^a = nL$. In this case the average stock price over the path is at least L. ■

4.2 A path-clustering approximation

We present here an $O(n^3)$-time approximation for an Asian Call for the case $p \geq \frac{1}{2}$ that is significantly better in practice than the expression $(\mathbf{E}A_n - L)^+$ presented earlier. We leave the error analysis as an open problem. As before, we consider the undiscounted value $V' = (1+r)^n V$.

Recall that $Y_n = \sum_{i=1}^n X_i$ is the position at time n of the random walk underlying our model. Now for an Asian call,

$$\begin{aligned} V' &\stackrel{\triangle}{=} \mathbf{E}(A_n - L)^+ \\ &= \mathbf{E}\left\{\mathbf{E}\left[(A_n - L)^+ \middle| Y_n\right]\right\} \\ &\geq \mathbf{E}\left\{[\mathbf{E}(A_n \mid Y_n) - L]^+\right\} \qquad \text{(Jensen's inequality)} \\ &= \sum_{k=0}^{n} p^k q^{n-k} \binom{n}{k} [\mathbf{E}(A_n \mid Y_n = k) - L]^+ \end{aligned} \tag{7}$$

We use the expression 7 as an approximation to $\mathbf{E}(A_n - L)^+$. Note that the quantity $W_k = \mathbf{E}(A_n | Y_n = k)$ is the expected value of A_n over the "cluster" of paths that have k more H's than T's. This approximation is therefore more refined than our earlier approximation $[\mathbf{E}A_n - L]^+$. Rogers and Shi [23] have considered a similar approximation based on conditional expectations, for an Asian option in the continuous setting. They show their approximation to be extremely good, both empirically and analytically. However their analysis does not appear to be adaptable to the binomial pricing model.

We now show that W_k can be computed in $O(n^2)$ time. For $t \leq n,\ h \leq t$, we say that a path $\omega \in \{H, T\}^n$ "goes through the point (t, h)" if there are exactly h H's in the first t tosses of ω. If we write down the expression for W_k we see that the stock price at point (t, h) (which is $S_0 u^{2h-t}$) gets multiplied by a factor

$$\frac{1}{(n+1)} \frac{\binom{t}{h}\binom{n-t}{k-h}}{\binom{n}{k}},$$

so that W_k may be written

$$W_k = \frac{S_0}{\binom{n}{k}(n+1)} \sum_{t=0}^{n} \sum_{h=0}^{t} \binom{t}{h} \binom{n-t}{k-h} u^{2h-t}.$$

Figure 2 shows some examples of the use of this approximation, for an Asian call.

5 Approximating an n-period American put with a perpetual put

Recall that a Markovian American option (MAO) is one whose payoff is given by $G_k = g(S_k)$ for some function g. The dynamic programming algorithm (based on the backward recursion

n	S_0	L	u	$1+r$	p	V	V_{apx}	% Error
10	1.0	2.0	1.2	1.10	.727	0.039	0.034	12.82
10	4.0	6.0	1.1	1.05	.738	0.039	0.026	33.33
10	4.0	6.0	1.5	1.01	.412	0.683	0.575	15.81
10	4.0	3.0	1.2	1.10	.727	1.443	1.442	0.07
10	4.0	3.0	1.5	1.05	.460	1.549	1.477	4.65
15	4.0	6.0	1.3	1.20	.811	1.032	1.030	0.19
10	4.0	6.0	1.5	1.40	.880	1.034	1.034	0.00
10	4.0	6.0	2.0	1.01	.340	1.411	1.145	18.85
10	4.0	3.0	2.0	1.01	.340	2.015	1.808	10.27
7	4.0	6.0	1.2	1.10	.727	0.235	0.219	6.81
10	4.0	6.0	1.2	1.10	.727	0.478	0.441	7.74
12	4.0	6.0	1.2	1.10	.727	0.616	0.587	4.71
14	4.0	6.0	1.2	1.10	.727	0.731	0.709	3.01
16	4.0	6.0	1.2	1.10	.727	0.822	0.806	1.95

Figure 2: *Examples of the path clustering approximation, for the value V of an Asian call*

(5)) for pricing a MAO requires $O(n^2)$ time. On the other hand, the value V of some *perpetual* Markovian American options (PMAO), such as perpetual American puts, can be computed in closed form in only $O(1)$ time. It is therefore of interest to investigate how well the value of a PMAO approximates the value of an otherwise identical MAO. In this section we first show a general formula bounding the difference between a PMAO and an otherwise identical MAO, and then apply it to the case of American puts. It is not hard to show (see Hull [17]) that under the pricing model of this article, it is never optimal to exercise an American *call* before expiration. An American call is therefore equivalent to a European call and can be priced in $O(n)$ time. Thus much research has focused on devising fast pricing methods for American puts [20, 13, 6]. It is known [10] that the value of an American *perpetual* put can be computed in $O(1)$ time. In this section we investigate the difference between an American put and an otherwise equivalent American perpetual put.

Let $\mathcal{T}_0$ be the set of stopping times τ such that $\tau \geq 0$ almost surely. The value of a n-period MAO with initial (non-random) stock price S_0 is denoted by V^n. The value of a PMAO with initial stock price S_0 is denoted by V. From Section 1.3 we have the following formulas:

$$
\begin{aligned}
V^n &= \max_{\tau \in \mathcal{T}_0} \mathbf{E}[(1+r)^{-\tau \wedge n} g(S_{\tau \wedge n})] \\
V &= \max_{\tau \in \mathcal{T}_0} \mathbf{E}[(1+r)^{-\tau} g(S_\tau)],
\end{aligned}
\tag{8}
$$

where $x \wedge y \stackrel{\triangle}{=} \min\{x, y\}$.

We prove the following lemma bounding the difference between a MAO and an otherwise identical PMAO with payoff $G_k = g(S_k)$.

Lemma 11 *Let τ^* be a stopping time such that*

$$V \;=\; \mathbf{E}[(1+r)^{-\tau^*} g(S_{\tau^*})].$$

Then $V - V^n$ is at most

$$\mathbf{E}\left[\mathbf{1}_{\{\tau^*>n\}}\left((1+r)^{-\tau^*}g(S_{\tau^*}) - (1+r)^{-n}g(S_n)\right)\right].$$

Proof: We have

$$\begin{aligned} V^n &\geq \mathbf{E}\left[(1+r)^{-\tau^*\wedge n}g(S_{\tau^*\wedge n})\right] \\ &= \mathbf{E}\left[\mathbf{1}_{\tau^*>n}(1+r)^{-n}g(S_n)\right] + \mathbf{E}\left[\mathbf{1}_{\tau^*\leq n}(1+r)^{-\tau^*}g(S_{\tau^*})\right]. \\ V &= \mathbf{E}\left[\mathbf{1}_{\tau^*>n}(1+r)^{-\tau^*}g(S_{\tau^*})\right] + \mathbf{E}\left[\mathbf{1}_{\tau^*\leq n}(1+r)^{-\tau^*}g(S_{\tau^*})\right], \end{aligned}$$

and the lemma follows. ∎

Now consider an American perpetual put. The payoff function in this case is given by $g(S_k) = (L - S_k)^+$, where L is the strike price. The following Lemma is known [10].

Lemma 12 *For any integer $j \in \mathbf{Z}$, let τ_j denote the stopping time*

$$\tau_j = \min\{k : S_k \leq S_0 u^j\} = \min\{k : Y_k \leq j\}.$$

Given an American put with strike price L, there exists an integer $s \geq 0$ such that τ_{-s} achieves the $\max$ in Eq.(8):

$$\begin{aligned} V &= \mathbf{E}[(1+r)^{-\tau_{-s}}g(S_{\tau_{-s}})] \\ &= (L - S_0u^{-s})\mathbf{E}[(1+r)^{-\tau_{-s}}]. \end{aligned} \tag{9}$$

The last expression in (9) can be computed in closed form [10]. In the following we assume that s denotes the non-negative integer of Lemma 12. Let E_k^n denote the event $\{S_n = S_0u^k, \quad \tau_{-s} > n\}$, and let $P_k^n \stackrel{\triangle}{=} \mathbf{P}[E_k^n]$. We now upper bound the difference between an n-period American put and the corresponding perpetual put.

Theorem 13 *If V^n is the value of an n-period American put and V is the value of an otherwise identical American perpetual put (APP), then*

$$V - V^n \leq (1+r)^{-n}\sum_{k=-s+1}^{n} P_k^n\left((L - S_0u^{-s})\alpha^{k+s} - (L - S_0u^k)^+\right),$$

where $\alpha \stackrel{\triangle}{=} \mathbf{E}[(1+r)^{-\tau_{-1}}]$, and

$$P_k^n \;=\; p^{\frac{n+k}{2}}(1-p)^{\frac{n-k}{2}}\left(\binom{n}{\frac{n+k}{2}} - \binom{n}{\frac{n+k+2s}{2}}\right).$$

Proof: We use Lemma 11, with $\tau^* = \tau_{-s}$:

$$\begin{aligned}
&\mathbf{E}\left[\mathbf{1}_{\tau^*>n}(1+r)^{-\tau^*}g(S_{\tau^*})\right] \\
&= \sum_{k=-s+1}^{n} P_k^n \mathbf{E}\left[(1+r)^{-\tau^*}g(S_{\tau^*})\middle| E_k^n\right] \\
&= (1+r)^{-n}\sum_{k=-s+1}^{n} P_k^n g(S_{\tau_{-s}})\mathbf{E}[(1+r)^{-\tau_{-s-k}}] \\
&= (1+r)^{-n}\sum_{k=-s+1}^{n} P_k^n (L - S_0u^{-s})\alpha^{k+s}, \\
&\mathbf{E}[\mathbf{1}_{\tau^*>n}(1+r)^{-n}g(S_n)] \\
&= (1+r)^{-n}\sum_{k=-s+1}^{n} P_k^n (L - S_0u^{k})^{+}.
\end{aligned}$$

We are using the fact that

$$\mathbf{E}[(1+r)^{-\tau_{-k}}] = (\mathbf{E}[(1+r)^{-\tau_{-1}}])^k \ \ \text{(for } k \geq 1).$$

The expression for P_k^n can be derived using the Reflection Principle. ■

We now obtain an asymptotic error bound from this Theorem.

Theorem 14 *For $p \leq \frac{1}{2}$, the value of a perpetual American put with strike price L exceeds that of an otherwise identical n-period American put by at most*

$$L(1+r)^{-n}O\left(\frac{1}{\sqrt{n}}\right).$$

Proof: Let $\underline{Y}_n \triangleq \min_{0\leq i\leq n} Y_i$, where Y_i is as before the i'th partial sum in the random walk underlying the model. Noting that $(L - S_0u^{-s}) \leq L$ and $(L - S_0u^k)^+ \geq 0$, we have for $p = \frac{1}{2}$, from Theorem 13:

$$\begin{aligned}
V - V^n &\leq L(1+r)^{-n}\sum_{k=-s+1}^{n} P_k^n \\
&= L(1+r)^{-n}\mathbf{P}[\underline{Y}_n > -s] \\
&= L(1+r)^{-n}\mathbf{P}[\overline{Y}_n < s] \\
&\leq L(1+r)^{-n}\left(\sqrt{\frac{2}{\pi}}\frac{s}{\sqrt{n}} + \frac{2}{\sqrt{n}}\right) \qquad \text{(Thm. 9)} \\
&= L(1+r)^{-n}O\left(\frac{1}{\sqrt{n}}\right).
\end{aligned}$$

Since the value $\mathbf{P}[\underline{Y}_n > -s]$ is a non-decreasing function of p, the above error bound also applies for $p < \frac{1}{2}$. ■

We leave the asymptotic error analysis for $p > \frac{1}{2}$ as an open problem.

6 Further research

Some problems left open in this article are: (a) obtaining a more accurate error bound for the Asian call approximation for $p < \frac{1}{2}$ (Section 4), and for the American put for $p > \frac{1}{2}$ (Section 5); (b) establishing the hardness of pricing an (European style) Asian option.

There are plenty of research directions to pursue in option pricing. We mention a few here. One important problem is the approximate pricing of *American style* Asian options, i.e., those that can be exercised at any time up to expiration. We saw in Sections 1.3 and 3 that certain American options can be priced in polynomial-time (in the maturity n) using dynamic programming. Devising fast (say linear-time) approximate algorithms for such options would be a significant contribution to quantitative finance. Another problem is option pricing with time-varying interest rate r and time-varying up-factor u. Finally, we mention that Arbitrage Pricing Theory depends on the ability to perfectly hedge the option being priced. Soner, Shreve and Cvitanic [25] have shown for the continuous-time setting that when proportional transaction costs (such as broker commissions) are present, perfect hedging becomes impossible, and the pricing formulas of Section 1.3 no longer hold. An intriguing problem is therefore to develop a satisfactory pricing theory in the presence of transaction costs. Some initial work in this direction for simple calls and puts has been done [1, 5].

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7 Summary of Notation

For each symbol, we mention the page where it is defined, and give a brief definition.

Symbol	**Page**	**Brief definition**
β	10	$\|2p-1\|$
$\mathcal{F}_k$	4	σ-field generated by the first k coin-tosses.
G_k	6	The payoff from an option exercised at time k.
L	5	Strike price of an option.
A_n	6	Average stock price up to time n, T_n/n.
n	3	Maturity of the option. (Infinite for perpetual options)
ω	4	A coin-toss sequence $\omega_1, \omega_2, \ldots, \omega_n$ of length n.
Ω	4	The sample space of all coin-toss sequences of length n.
p	4	Up-tick probability, i.e., probability of occurrence of H.
$\mathbf{P}_\ell(c,n)$	12	Probability bound defined in Corollary 6, for the case $p < \frac{1}{2}$.
$\mathbf{P}_g(n)$	12	Probability bound defined in Corollary 6, for the case $p > \frac{1}{2}$.
$\mathbf{P}_e(n)$	14	Probability bound defined in Corollary 10, for the case $p = \frac{1}{2}$.
S_k	4	Stock price at time k, $= S_0 u^{Y_k}$
$\overline{S}_k$	5	$\max_{0 \le i \le k} S_i$.
T_k	4	$T_k = \sum_{i=1}^{k} S_i$.
τ	7	Generic stopping time.
τ_j	17	The specific stopping time $\min\{k : Y_k = j\}$.
$\mathcal{T}_k$	7	Class of stopping times τ such that $k \le \tau \le n$.
u	4	The up-factor.
V	7	Value of the option under consideration.
	16	In Section 5 this is the value of an American *perpetual* put.
V^n	16	Value of an n-period American put.
X_k	4	Random variable: $X_k(\omega) = 1$ if $\omega_k = H$ and $X_k(\omega) = -1$ otherwise.
Y_k	4	$Y_k = \sum_{i=1}^{k} X_i$; $Y_0 = 0$.
$\overline{Y}_n$	5	$\max_{0 \le i \le n} Y_i$.
$\underline{Y}_n$	5	$\min_{0 \le i \le n} Y_i$.
$x \wedge y$		$\min\{x, y\}$
x^+		$\max\{x, 0\}$