Pricing Interest Rate Derivatives: A General Approach[1]

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Abstract

Pricing Interest Rate Derivatives: A General Approach

The relationship between affine stochastic processes and bond pricing equations in exponential term structure models has been well-established (see Duffie and Kan [42]). We extend this linkage to the pricing of interest rate derivatives. This paper shows that, if the term structure model is exponential-affine, then there is a simple linkage between the bond pricing solution and the prices of many widely traded interest rate derivative securities. Our results are general, and apply to m-factor processes with n diffusions and l jump processes. Regardless of the number of shocks, the pricing solutions require at most a single numerical integral, making the model easy to implement. We provide many examples of options that yield solutions using the methods of the paper. Fast estimation of these models is possible by vectorizing the equations for the pricing solutions. A range of numerical solutions illustrates the use of the models.

1 Introduction

The literature on term structure modelling has evolved from one-factor diffusion models such as Cox, Ingersoll, & Ross [32] and Vasicek [83] to multifactor models such as Brennan & Schwartz [17], Longstaff & Schwartz [67], Balduzzi, Das, & Foresi [7], and Duffie & Singleton [43] as well as jump-diffusion models such as Ahn & Thompson [1], Das & Foresi [36], and Das [35]. The motivation for this evolution in term structure models has come from empirical papers such as Ait-Sahalia [2], Chan, Karolyi, Longstaff, & Sanders [26].[1]. However, as work proceeds on better matching the dynamics of the short rate to the observed term structure, the area of fixed income derivative pricing, the main application for modelling the short rate process, has lagged behind. In this paper, we attempt to bridge the gap between the multi-factor, jump-diffusion models of the short rate that are so commonly used now and the pricing of fixed income derivatives. Specifically, we show that *any* interest rate process (with any number of factors including stochastic volatility, stochastic central tendency, etc., or utilizing diffusion or jump-diffusion processes) that leads to an exponential term structure model, also lends itself to analytic solutions for three large classes of fixed income securities. These methods support numerical techniques which allow for easy implementation in the context of a no-arbitrage approach. It is our hope that the results of this paper will allow both researchers and practitioners to focus on the appropriate stochastic process for the short rate and its factors, and obviate concerns as to whether specific forms of the short rate lead to tractable solutions for popular fixed income securities.

The benchmark paper of Duffie and Kan [42] established the link between affine stochastic processes and exponential-affine term structure models.[2] They showed that the factor coefficients of these term structure models are solutions to a system of simultaneous Riccati equations and that these coefficients are functions of the time to maturity. The kernel of our technique resides in the fact that the solution for different types of interest rate options solves an almost identical systems of equations. The only difference between the two sets of equations is in the constant terms underlying the equations. By manipulating the Riccati equations and varying the constant terms, we develop a procedure to price options using the known components of the original term structure model. Thus, we essentially show that once the exponential-affine term structure model is derived, the pricing formulas for a wide range of popular fixed-income derivatives can be written by inspection from the components of the term structure model.[3] Specifically, we show that this approach is feasible for three large classes of fixed income derivatives: those with (1) payoffs that are linear in the short rate and factors; (2) payoffs that are exponential-affine in the short rate and factors; and (3) payoffs that are an integral over time of a linear combination of the short rate and factors. These three payoff structures encompass most of the popular fixed-income derivatives.

[1] Many other papers including those by Brown & Dybvig (1989), Litterman & Scheinkman (1991), and Stambaugh (1988) have to similar conclusions.

[2] Dai & Singleton (1997) provides a characterization of the exponential-affine class of term structure models as they unify and generalize this class.

[3] Subsequent to the original version of this paper, Bakshi and Madan [9], and Duffie, Pan, and Singleton [44] have indenpendently developed results that parallel some of those derived in this paper.

Our technique is general in that it applies to any multi-factor, *exponential-affine* term structure model with multiple Wiener and jump processes. No matter how many jump-diffusion stochastic processes are used, for standard derivatives, our approach involves evaluation of at most two one-dimensional integrals, resulting in easy computation. Furthermore, in the final section of this paper we show that the techniques of the paper can be easily extended beyond the exponential-affine class to the class of term structure models we call "exponential-separable" models, such a those of Constantinides [30] and Longstaff [65]. In addition, we also show in this section how to utilize the results of the paper in the context of no-arbitrage models, such as those of [56] and [13], which allow for exact calibration with observed data.

To demonstrate the technique, we provide closed form solutions for options under a jump-diffusion model of the type used in Das and Foresi [36]. We price options on bonds, futures, and interest rate caps and floors, since these are the most common forms of term structure derivatives. We also price options on average interest rates, in order to demonstrate a parsimonious approach based on expansion of the state space.[4] An important tool in our approach is the use of Fourier inversion methods as in Heston [53][5]. Though the results of this paper pertain to term structure models, the techniques provided extend to several other market settings.

The plan of the paper is as follows. In Section 2 we specify the interest rate process and the term structure model. We then introduce the pricing technology for fixed income securities that have general payoff functions of the interest rate process. We proceed in Sections 3, 4 and 5 to develop analytic solutions, in terms of the components of the term structure model in Section 2, for the three categories of derivative payoff functions considered in the paper: linear payoffs in the state variables are handled in Section 3, exponential-affine payoffs in Section 4, and integro-linear (or a payoff function that is an integral over time of a linear combination of the factors) payoffs are dealt with in Section 5. The details of these derivations are described in the Appendix, which contains many analytical features of interest. Section 6 then introduces an estimation procedure for the model parameters based on fitting a cross-section of bond prices. Section 7 then provides numerical examples of the procedures laid out in the paper. Section 8 shows how the techniques in the paper can be extended to the class of exponential-separable models and concludes.

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[4]The idea, very simply, is to expand the state space from that of a traditional Black-Scholes/Merton setup with m state variables to $m+1$ state variables where the additional variable is the average (i.e. arithmetic integral) of the underlying. Bakshi and Madan [9] provide a spanning analysis of this idea.

[5]Fourier methods have been used subsequently in many papers in finance including Bakshi, Cao and Chen [6], Bakshi and Madan [8], [9], Bates [10], Chacko [23], [24], Das and Foresi [36], Davydov and Linetsky [37], Duffie, Pan and Singleton [44], Eydeland and Geman [46], , Heston and Nandi [54], Leblanc and Scaillet [62], Scott [75], Singleton [78] and Steenkiste and Foresi [82]. Bakshi and Madan [8], [9] link Fourier transform methods to a state-price framework, while Duffie, Pan and Singleton [44] describe the application of these techniques to problems in the area of equity, interest rate and default risk options. Van Steenkiste and Foresi [82] show how to derive state prices in the same general framework, and apply Fast-Fourier methods to price American options.

2 Generalized Option Valuation

In this section we present the setup for the general valuation principles in the paper. We specify the general interest rate process and the term structure model for which we will able to derive general option valuation formulae. The restrictions on the interest rate dynamics imposed here will be the same as those specified in Duffie and Kan [42] for jump-diffusion processes. These restrictions lead to an exponential-affine term structure model. With the aid of the Feynman-Kac theorem, which is stated below, we derive a general valuation equation for fixed income securities. Our task in subsequent sections will be to solve this equation for large classes of fixed-income securities using only the components of the term structure model presented in this section.

2.1 Interest Rate Dynamics

The economy is a continuous-trading economy with a trading interval $[0, T]$ for a fixed $T > 0$. The uncertainty in the economy is characterized by a probability space $(\Omega, \mathcal{F}, Q)$ which represents the risk-neutral probability measure in the economy.[6]

Let $\mathbf{N}_t$ represent a vector l of orthogonal Poisson processes, and let $\mathbf{W}_t$ represent a vector of n Wiener processes. Each Poisson, or jump process, can be thought of as a counter. When a jump occurs, the jump process increments upward by 1 unit. The jump frequencies of the Poisson processes are given by λ_i, $i = 1 \ldots l$, and are constant over $[0, T]$.

The term structure of zero-coupon bond prices is formed from the instantaneous interest rate and a set of m factors in the economy. The risk-neutral processes governing the interest rate and the factors are given by a vector of strong Markov processes:

$$dr_t = \mu(r_t, \mathbf{x}_t)dt + \sigma'(r_t, \mathbf{x}_t)d\mathbf{W} + \mathbf{J}_r'\, d\mathbf{N} \tag{1}$$

$$d\mathbf{x}_t = \alpha(\mathbf{x}_t)dt + \delta(\mathbf{x}_t)d\mathbf{W} + \mathbf{J}_x\, d\mathbf{N} \tag{2}$$

The $m \times 1$ vector, $\mathbf{x}_t$, represents a set of Markov factors which influence the marginal productivity of capital, and thus the interest rate, in the economy. The magnitudes of the Poisson processes are defined by the $l \times 1$ vector $\mathbf{J}_r$ and the $m \times l$ matrix $\mathbf{J}_x$ of correlated random variables. It is assumed that the distributions governing the jump magnitudes are not functions of the state variables or the instantaneous interest rate.

We assume that the instantaneous diffusion covariance matrix of the state variables is given by $\Lambda(\mathbf{x}_t)$, while the vector of instantaneous diffusion covariances between the state variables $x_{i,t}$, $i = 1 ... m$, and r_t is given by $\rho(\mathbf{x}_t)$.

While the structure above for the interest rate process may appear to be restrictive (since the state variables are not allowed to be functions of the interest rate), this is not the case. We impose this restriction only for notational simplicity later in the paper. If the state variables are functions of the interest rate, one can get to the formulation given in (1) and

[6] Unless indicated otherwise, all computations reported in the paper are with respect to the risk-neutral probability measure and not the objective probability measure.

(2) by expanding the space of state variables. This technique can be employed, for example, to rewrite the popular multifactor CIR model.

2.2 Bond Prices

With the risk-neutral interest rate process known, we can write down an expression for the price of any traded security in the economy. Specifically, let $P_t(r, \mathbf{x}; \tau)$ represent the price at time t of a security that matures after a period of time τ. Then, we have the following partial differential difference equation (PDDE) for the price of a bond (see Black and Scholes [14], Merton [68], Cox, Ingersoll and Ross [32], and Courtadon [31]):

$$0 = \mathcal{D}P_t + \mathbf{d}\ \mathbf{s}'P_t \tag{3}$$

where $\mathbf{d}$ is a row vector of constants and $\mathcal{D}$ represents the usual differential operator.[7] $\mathbf{s} = [r_t, \mathbf{x}_t, 1]$ is a row vector comprising the current levels of the short rate and the factors, and an additional parameter required for special forms of payoff functions.

In the case of traded securities, $\mathbf{d} = \mathbf{d}^* \equiv [-1, \mathbf{0}, 0]$, but we will assume for now that $\mathbf{d}$ is an arbitrary vector of constants. We do this because in subsequent sections, we will utilize transformations to (3) where partial differential equations with $\mathbf{d} \neq \mathbf{d}^*$ will result. For a zero-coupon bond that pays off \$1 at maturity, the boundary condition that is satisfied by (3) is $P(\tau = 0) = 1$.

A solution for equations of the form given by (3) can be written in expectations form using the Feynman-Kac theorem, which we now state for reference.[8]

Axiom 1 *Feynman-Kac - For any variable X determined by a stochastic differential equations of the form*

$$dX_t = \alpha_x(X_t)dt + \delta_x(X_t)dW + J(X_t)dN(\lambda)$$

the solution, $F(X_t)$, to the expression

$$E_t\left[e^{-\int_t^T g(X_v)dv} f(X_T)\right]$$

[7]The differential operator applied to a function P_t is defined as

$$\begin{aligned}
\mathcal{D}P_t \quad = \quad & \frac{1}{2}\sum_{i=1}^{n}\sigma_i^2(r_t, \mathbf{x}_t)\frac{\partial^2 P_t}{\partial r^2} + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\Lambda_{ij}(\mathbf{x}_t)\frac{\partial^2 P_t}{\partial x_i \partial x_j} + \sum_{i=1}^{m}\rho_i(\mathbf{x}_t)\frac{\partial^2 P_t}{\partial r \partial x_i} \\
& +\mu(r, \mathbf{x}_t)\frac{\partial P_t}{\partial r} + \sum_{i=1}^{m}\alpha_i(\mathbf{x}_t)\frac{\partial P_t}{\partial x_i} + \frac{\partial P_t}{\partial t} \\
& +\sum_{i=1}^{l}\lambda_i E\left[P(r + J_{r,i}, \mathbf{x}_t + \mathbf{J}_x^{(i)};\tau) - P(r, \mathbf{x};\tau)\right]
\end{aligned}$$

where the expression $\mathbf{J}_x^{(i)}$ represents a modified version of the matrix $\mathbf{J}_x$ of jump magnitudes. $\mathbf{J}_x$ is modified so that all but the ith column of $\mathbf{J}_x$ is zero. This is a direct consequence of Ito's Lemma.

[8]See Duffie [40] for more details regarding the Feynman-Kac relation.

where $f, g \in \mathcal{C}^{2,1}$, is determined by the equation

$$\mathcal{D}F(X_t) = g(X_t)F$$

where $\mathcal{D}$ is the differential operator defined by

$$\mathcal{D}F(X_t) = \frac{1}{2}\delta_x^2 \frac{\partial^2 F}{\partial X_t^2} + \alpha_x \frac{\partial F}{\partial X_t} - \frac{\partial F}{\partial \tau} + \lambda E_{t-}[F(X_t + J) - F(X_t)]$$

The boundary condition for this partial differential-difference equation is given by $F(X_T) = f(X_T)$.

This is simply the univariate version of the Feynman-Kac theorem. We will utilize multivariate versions of this theorem throughout the paper. As an example, we can use the Feynman-Kac theorem to write the solution to (3) as

$$P(\tau) = E_t \left[e^{-\int_t^T r_v dv} 1 \right]$$

which is the standard discounted form for a discount bond price.

Since we are interested in exponential-affine models of bond pricing, we need to impose restrictions on the drift and diffusion terms in (1) and (2) in order to ensure that the solution to (3) is exponential affine. From Duffie & Kan [42], we know that the term structures of zero-coupon bond prices are of the exponential affine class, i.e., those of the form

$$P_t(\tau) = \exp[A(\tau)r_t + \sum_{i=1}^{m} B_i(\tau)x_i + C(\tau)] \tag{4}$$

where $A(\tau), B_i(\tau)$, $i = 1...m$, and $C(\tau)$ are functions of time-to-maturity only if the drift terms and the square of each diffusion term of (1) and (2) are linear in the interest rate and the factors, and if the jump magnitudes of (1) and (2) have linear (in the interest rate and the factors) moment generating functions.

The solutions to these functions are each determined by a separate ordinary differential equation (ODE). Associated with each ODE is also a unique boundary condition. For the remainder of the paper, we will need to be concerned only with the cases where the boundary conditions for the (3) are such that resulting boundary conditions on $A(\tau)$, $B_i(\tau)$, $i = 1 \ldots m$, and $C(\tau)$ are given by

$$\begin{aligned} A(0) &= a \\ B_1(0) &= b_1 \\ &\vdots \\ B_m(0) &= b_m \\ C(0) &= c \end{aligned} \tag{5}$$

where a, $b_1, \ldots, b_m$, c are constants. In the case of zero-coupon bond prices, the exponential affine form of these prices allows us to break up (3) using the well-used technique of separation of variables into a set of Riccati equations for $A(\tau)$, $B_i(\tau)$, $i = 1 \ldots m$ and $C(\tau)$. If the boundary conditions for the system of Riccati equations are given by (5), then the solutions to $A(\tau)$, $B_1(\tau)$,...,$B_m(\tau)$, $C(\tau)$ depend on the specific structure of the drifts, variances, and covariances of the interest rate and the factors. We denote the solutions to the differential equations governing these functions specifically as $A^*(\theta;\tau,\mathbf{b},\mathbf{d})$, $B_1^*(\theta;\tau,\mathbf{b},\mathbf{d})$,...,$B_m^*(\theta;\tau,\mathbf{b},\mathbf{d})$, $C^*(\theta;\tau,\mathbf{b},\mathbf{d})$, respectively, where θ denotes the vector of parameters governing the stochastic processes, and $\mathbf{b} = [a, b_1 \ldots b_m, c]$ denotes the vector of constants that make up the boundary conditions in (5).

Remark 1 *The main result of this paper is that the prices of a wide range of common fixed-income derivatives can be characterized solely in terms of the functions* $A^*(\theta;\tau,\mathbf{b},\mathbf{d})$, $B_1^*(\theta;\tau,\mathbf{b},\mathbf{d})$, ..., $B_m^*(\theta;\tau,\mathbf{b},\mathbf{d})$, $C^*(\theta;\tau,\mathbf{b},\mathbf{d})$. *Therefore, we will show that if the interest rate model, regardless of how complicated it is, leads to exponential bond prices, then the prices of many interest rate-dependent claims can be easily calculated in terms of these functions.*

In the case of a discount bond, the holder receives a dollar at maturity, and the boundary condition for the bond can be stated as

$$P_T(0) = 1 = \exp\left[0r_t + \sum_{i=1}^{m} 0x_{i,t} + 0\right] \tag{6}$$

Therefore the specific boundary conditions for each Riccati equation are all zero, i.e., $\mathbf{b} = \mathbf{0}$, Hence, the price of the bond is given by

$$P_t(\tau) = \exp\left[A^*(\theta;\tau,\mathbf{0},\mathbf{d}^*)r_t + \sum_{i=1}^{m} B_i^*(\theta;\tau,\mathbf{0},\mathbf{d}^*)x_{i,t} + C^*(\theta;\tau,\mathbf{0},\mathbf{d}^*)\right] \tag{7}$$

Virtually all of the term structure models developed in the literature to date begin with interest rate/factor processes that lead to bond prices of the exponential affine form given by (7). Therefore, our goal in this paper is to derive pricing implications for derivatives written on this specific class of interest-rate processes.

2.3 Example

We now present an example of the term structure model discussed in the previous sections. The example is a one-factor jump-diffusion model. This example will be continued and extended in subsequent sections of the paper in order to illustrate the use of the theoretical results of the paper and, hopefully, to make them more concrete.

Consider the risk-neutral interest process given by the dynamics

$$dr = \kappa(\theta - r)dt + \sigma dW + J_u dN_u(\lambda_u) - J_d dN_d(\lambda_d)$$

where κ, θ, σ, λ_u, and λ_d are constants, while J_u and J_d are exponentially distributed random variables with positive means η_u and η_d, respectively. The interest rate in this specification displays persistence as well as skewness and excess kurtosis. The one-jump version of this process was first considered in Das & Foresi [36]. A two-jump model was first considered in Duffie, Pan and Singleton [44]. The version of (3) that holds for this process is given by

$$\frac{1}{2}\sigma^2 P_{rr} + \kappa(\theta - r)P_r - P_\tau + \lambda_u E[P(r + J_u) - P(r)] + \lambda_d E[P(r - J_d) - P(r)] = -drP$$

where the subscripts on $P(r)$ denote partial derivatives. The general boundary condition on $P(r)$ that leads to (5) is given by

$$P(r, \tau = 0) = \exp[ar + c]$$

Under this boundary condition, the solution to the PDDE is of the form given by (4):

$$P(r) = \exp[A(\tau)r + C(\tau)] \tag{8}$$

where $A(\tau)$ and $C(\tau)$ satisfy ordinary differential equations:

$$\begin{aligned} \frac{dA}{d\tau} &= -\kappa A + d \\ \frac{dC}{d\tau} &= \frac{1}{2}\sigma^2 A^2 + \kappa\theta A + \lambda_u E\left[e^{AJ_u} - 1\right] + \lambda_d E\left[e^{-AJ_d} - 1\right] \\ &= \frac{1}{2}\sigma^2 A^2 + \kappa\theta A + \lambda_u \left(\frac{\eta_u A}{1 - \eta_u A}\right) - \lambda_d \left(\frac{\eta_d A}{1 + \eta_d A}\right) \end{aligned}$$

with boundary conditions $A(\tau = 0) = a$ and $C(\tau = 0) = c$. Following the convention of the previous section, we label the vector $[a, c] = \mathbf{b}$. The solutions, labeled $A^*(\tau, \mathbf{b}, d)$ and $C^*(\tau, \mathbf{b}, d)$, are given by

$$\begin{aligned} A^*(\tau, \mathbf{b}, d) &= u_1 e^{-\kappa\tau} + u_2 \\ C^*(\tau, \mathbf{b}, d) &= \frac{u_1^2\sigma^2}{4\kappa}(1 - e^{-2\kappa\tau}) + \left[\frac{u_1 u_2 \sigma^2 + \kappa\theta u_1}{\kappa}\right](1 - e^{-\kappa\tau}) \\ &\quad + \left[\frac{u_2^2\sigma^2}{2} + \kappa\theta u_2 - \lambda_u - \lambda_d\right]\tau + \frac{\lambda_u}{\kappa - d\eta_u}\log\left|\frac{(1 - \eta_u u_2)e^{\kappa\tau} - \eta_u u_1}{1 - \eta_u u_1 - \eta_u u_2}\right| \\ &\quad + \frac{\lambda_d}{\kappa + d\eta_d}\log\left|\frac{(1 + \eta_d u_2)e^{\kappa\tau} + \eta_d u_1}{1 + \eta_d u_1 + \eta_d u_2}\right| + c \\ u_1 &= a - u_2 \\ u_2 &= \frac{d}{\kappa} \end{aligned}$$

As mentioned in the previous section, in the special case that $b = [0, 0]$ and $d = -1$, the solution to (8) is the price of a zero-coupon bond with maturity τ. Henceforth throughout the paper, we will utilize this example to illustrate the theoretical pricing relationships and numerical methods derived in the paper, in the hope of making these results more concrete and accessible. To begin, we specify a base set of parameters and price the bond using the equation above:

k	θ	σ	λ_u	λ_d	η_u	η_d	τ	r
0.2	0.1	0.1	5	5	0.005	0.005	0.5	0.1

The jump intensities for both jumps are set at 5 jumps a year. The jumps are symmetric, in that the mean jump size for the positive and negative jumps is 50 basis points. The discount bond price is computed to be 0.9514. We now show bond prices as we vary the jump intensities from 3 to 12 jumps per annum.

		λ_d			
		3	6	9	12
	3	0.9514	0.9531	0.9549	0.9566
λ_u	6	0.9497	0.9514	0.9532	0.9549
	9	0.9480	0.9497	0.9514	0.9532
	12	0.9463	0.9480	0.9497	0.9514

Increasing the upward jump frequency of the short rate causes the bond price to fall, while the opposite happens as we increase the downward jump frequency. Intuitively, as the upward jump frequency increases, the likelihood of higher future rates increases, and since the bond price is a discounted value of these rates, bond prices drop. As both upward and downward jump frequency increase the bond price increases slightly. For example, as the both jump frequencies rise from 3 to 6 jumps per annum, the bond price rises from 0.951419 to 0.951424 (not displayed). This is due to the convexity of bond prices with respect to the short rate, which is very mild. Therefore, an equal magnitude upward jump in the short rate has less effect on bond prices than an equal magnitude downward jump.

In the following sections, we shall use the example presented in this section to illustrate how to utilize the solutions to different options.

2.4 Option Prices

In this section, we write down a general equation for the pricing of European options where the option payoff may be a general function of the interest rate r. We denote the payoff function at time T as $f_T(\mathbf{r}, \mathbf{x}, \hat{\tau})$, where $\mathbf{r}$ represents the sample path of interest rates and state variables, $\mathbf{x}$, up to time T. In addition, $\hat{\tau}$ is a "terminal time period" parameter, which allows the payoff function to also depend on a period of time of length $\hat{\tau}$ beyond time T. Therefore, the payoff of the option on its expiration date can be expressed as

$$F_T(0; \hat{\tau}) = \max[f_T(\mathbf{r}, \mathbf{x}, \hat{\tau}) - K, 0]$$

where we use the notation $F_t(\tau; \hat{\tau})$ to represent the price of an option at time t with a period of time τ to expiration written on an underlying security with remaining maturity $\hat{\tau}$. As a result $T = t + \tau$. We can use the Feynman-Kac relation[9] to write the price of the option as

[9]See Duffie [40] for a exposition of the use of these methods.

the discounted expected value of the terminal payoff:

$$F_t(\tau;\hat{\tau}) = E_t\left[e^{-\int_t^T r_v dv}\max[f_T(\mathbf{r},\mathbf{x},\hat{\tau}) - K, 0]\right] \tag{9}$$

where the expectation is taken under the risk-neutral measure.[10] To simplify notation, we introduce the variable Z_t, defined as

$$Z_T(\tau) = \int_t^T r_v dv$$

We now decompose the price of the option into two expressions as follows.

$$\begin{aligned} F_t(\tau;\hat{\tau}) &= E_t\left[e^{-Z_t} f_T(\mathbf{r},\mathbf{x},\hat{\tau}) 1_{\{f_T(\mathbf{r},\mathbf{x},\hat{\tau})\geq K\}}\right] - E_t\left[e^{-Z_t} K 1_{\{f_T(\mathbf{r},\mathbf{x},\hat{\tau})\geq K\}}\right] \\ &= E_t[e^{-Z_t} f_T(\mathbf{r},\mathbf{x},\hat{\tau})] E_t\left[\frac{e^{-Z_t} f_T(\mathbf{r},\mathbf{x},\hat{\tau}) 1_{\{f_T(\mathbf{r},\mathbf{x},\hat{\tau})\geq K\}}}{E_t[e^{-Z_t} f_T(\mathbf{r},\mathbf{x},\hat{\tau})]}\right] \qquad (10) \\ &\quad -K E_t\left[e^{-Z_t}\right] E_t\left[\frac{e^{-Z_t} 1_{\{f_T(\mathbf{r},\mathbf{x},\hat{\tau})\geq K\}}}{E_t\left[e^{-Z_t}\right]}\right] \qquad (11) \end{aligned}$$

where $1_{\{f_T(\mathbf{r},\mathbf{x},\hat{\tau})\geq K\}}$ is an indicator function for when the option finishes up in-the-money. However, $E_t\left[e^{-Z_t}\right]$ is the price of a discount bond that matures at time $T \equiv t+\tau$. So, $E_t\left[e^{-Z_t}\right] = P_t(\tau)$. We define $\Pi_{0,t} = E_t[e^{-Z_t} f_T(\mathbf{r},\mathbf{x},\hat{\tau})]$, which is the present value of the underlying function that determines the payoff. Therefore, we can rewrite the expression above as

$$\begin{aligned} F_t(\tau;\hat{\tau}) &= \Pi_{0,t} E_t\left[\frac{e^{-Z_t} f_T(\mathbf{r},\mathbf{x},\hat{\tau}) 1_{\{f_T(\mathbf{r},\mathbf{x},\hat{\tau})\geq K\}}}{E_t[e^{-Z_t} f_T(\mathbf{r},\mathbf{x},\hat{\tau})]}\right] \\ &\quad -K P_t(T) E_t\left[\frac{e^{-Z_t} 1_{\{f_T(\mathbf{r},\mathbf{x},\hat{\tau})\geq K\}}}{E_t\left[e^{-Z_t}\right]}\right] \end{aligned}$$

It is clear that $E_t\left[\frac{e^{-Z_t} f_T(\mathbf{r},\mathbf{x},\hat{\tau}) 1_{\{f_T(\mathbf{r},\mathbf{x},\hat{\tau})\geq K\}}}{E_t[e^{-Z_t} f_T(\mathbf{r},\mathbf{x},\hat{\tau})]}\right]$ and $E_t\left[\frac{e^{-Z_t} 1_{\{f_T(\mathbf{r},\mathbf{x},\hat{\tau})\geq K\}}}{E_t\left[e^{-Z_t}\right]}\right]$ are probabilities. For convenience, we denote these two probabilities as $\Pi_{1,t}$ and $\Pi_{2,t}$ respectively. The price of the option is restated as

$$F_t(\tau;\hat{\tau}) = \Pi_{0,t}\Pi_{1,t} - K P_t(\tau)\Pi_{2,t} \tag{12}$$

The task at hand is to evaluate $\Pi_{0,t}$ and the two probabilities $\Pi_{1,t}$ and $\Pi_{2,t}$. The specific pricing equations in the paper, $\Pi_{1,t}$ and $\Pi_{2,t}$ may be calculated solely in terms of the functions $A^*(\theta;\tau,\mathbf{b},\mathbf{d})$, $B_1^*(\theta;\tau,\mathbf{b},\mathbf{d})$, ... , $B_m^*(\theta;\tau,\mathbf{b},\mathbf{d})$, $C^*(\theta;\tau,\mathbf{b},\mathbf{d})$ through the application of the Feynman-Kac theorem,.which essentially allows us to solve for any expression of the form in (9) by restating the expression as a solution to a PDDE.

In the next several sections, we utilize the Feynman-Kac theorem to tackle three different types of terminal payoff functions for the pricing of interest rate derivatives:

[10] All expectations from this point onward are under the risk-neutral measure unless indicated otherwise.

1. Payoffs that are linear functions of the state variables. These may be used to price caps, floors, yield options, and slope options.

2. Payoffs that are exponential in the state variables, used to price bond options, forwards, and futures options.

3. Payoffs that are integrals of the state variables as in the case of average rate options on the short rate, and Asian options on yields.

We now examine each one in turn.

3 Option Pricing for Linear Payoffs

In this section we consider the case when the payoff function is given by a linear function of the interest rate and state variables:

$$f_T(\mathbf{r}, \mathbf{x}, \hat{\tau}) = k_0 r_T + k_1 x_{1,T} + \ldots + k_m x_{m,T} + k_{m+1} \tag{13}$$

where $k_0, ..., k_{m+1}$ are constants. As indicated by (12), the price of a European call option for this payoff function is given by

$$F_t(\tau; \hat{\tau}) = \Pi_{0,t}\Pi_{1,t} - K P_t(\tau)\Pi_{2,t} \tag{14}$$

We now derive the function $\Pi_{0,t}$ and the two probabilities $\Pi_{1,t}$ and $\Pi_{2,t}$ for the linear payoff function. Substituting these solutions into (14) will then yield the general option pricing formula for a linear terminal payoff function.

Proposition 2 *(A) The solution for* $\Pi_{0,t}$ *is given by*

$$\Pi_{0,t} = \left\{ \Gamma_{0,t} \left[\frac{\partial A^*(\theta; \tau, \mathbf{b}_0, \mathbf{d}^*)}{\partial \phi} r_t + \sum_{i=1}^{m} \frac{\partial B_i^*(\theta; \tau, \mathbf{b}_0, \mathbf{d}^*)}{\partial \phi} x_{i,t} + \frac{\partial C^*(\theta; \tau, \mathbf{b}_0, \mathbf{d}^*)}{\partial \phi} \right] \right\}_{\phi=0}$$

where

$$\begin{aligned} \Gamma_{0,t} &= \exp\left[A^*(\theta; \tau, \mathbf{b}_0, \mathbf{d}^*) r_t + \sum_{i=1}^{m} B_i^*(\theta; \tau, \mathbf{b}_0, \mathbf{d}^*) x_{i,t} + C^*(\theta; \tau, \mathbf{b}_0, \mathbf{d}^*) \right] \\ \mathbf{b}_0 &= \begin{bmatrix} \phi k_0 \\ \phi k_1 \\ \vdots \\ \phi k_m \\ \phi k_{m+1} \end{bmatrix} \end{aligned}$$

(B) The characteristic function for $\Pi_{1,t}$ *is given by*

$$\tilde{\Pi}_{1,t} = \frac{1}{\Pi_{0,t}} \left\{ \frac{1}{i} \Gamma_{1,t} \left[\frac{\partial A^*(\theta;\tau,\mathbf{b}_1,\mathbf{d}^*)}{\partial \omega} r_t + \sum_{i=1}^{m} \frac{\partial B_i^*(\theta;\tau,\mathbf{b}_1,\mathbf{d}^*)}{\partial \omega} x_{i,t} + \frac{\partial C^*(\theta;\tau,\mathbf{b}_1,\mathbf{d}^*)}{\partial \omega} \right] \right\}$$

where

$$\begin{aligned} \Gamma_{1,t} &= \exp\left[A^*(\theta;\tau,\mathbf{b}_1,\mathbf{d}^*) r_t + \sum_{i=1}^{m} B_i^*(\theta;\tau,\mathbf{b}_1,\mathbf{d}^*) x_{i,t} + C^*(\theta;\tau,\mathbf{b}_1,\mathbf{d}^*) \right] \\ \mathbf{b}_1 &= \begin{bmatrix} i\omega k_0 \\ i\omega k_1 \\ \vdots \\ i\omega k_m \\ i\omega k_{m+1} \end{bmatrix} \end{aligned}$$

(C) The characteristic function for $\Pi_{2,t}$ *is given by*

$$\tilde{\Pi}_{2,t} = \frac{1}{P_t(\tau)} \Gamma_{1,t}$$

(D) The characteristic functions in (B) and (C), $\tilde{\Pi}_{k,t}$*,* $k = 1, 2$*, may be inverted to obtain the probabilities* $\Pi_{k,t}$ *using a version of Fourier's theorem:*[11]

$$\Pi_{k,t} = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty Re\left(\frac{1}{i\omega} e^{-i\omega K} \tilde{\Pi}_{k,t} \right) d\omega, \quad k = 1, 2$$

Proof: See the Appendix

We now price specific options that fall into this category of payoffs.

3.1 Interest Rate Caps and Floors & Exotics

An interest rate cap is an option that pays off when the terminal interest rate exceeds the strike K.[12] These options are one of the most widely traded instruments in the fixed-income derivatives markets. Many uses are envisaged. (i) They are routinely used by corporations to cap their funding costs. (ii) Money management companies use floors to ensure a base

[11] Fourier's inversion theorem for distribution functions can be found in Kendall, Ord and Stuart [60] and Shephard [76].

[12] possible specification of the cap, making it different from a standard option on a zero coupon bond. In another popular market convention, a cap is an option on which the payoff is based on the interest rate at option maturity, but the payment takes place at the end of the period for which the underlying interest rate applies. We take this up later in this section. Interestingly, the mathematical treatment for these two conventions for caps is quite different.

level of return in their portfolios. (iii) Caps and floors bear an equivalence to swaps, which also makes them useful in managing swaps portfolios. (iv) A collar is a position containing a long cap and short floor, and one popular version of these contracts is a zero-cost collar. For example, investors with a view that interest rates will rise will buy a cap and subsidize themselves by selling a floor.

While the plain vanilla form of the interest-rate cap is now widepsread in usage, more exotic options are being traded, to which the technology of this section may be put to full use. Examples are as follows. (i) Options on credit spreads are now popular, and the modelling of the term structure of spreads lends itself easily to the pricing of derivatives. (ii) With the introduction of inflation-indexed bonds, options on inflation may be valued easily, since the term structure of inflation rates is now becoming available. These options may trade off the TIPS (Treasury Inflation-Protection Securities) market, or REALs (an older OTC version of the same security). (iii) An even more exotic application is one where options on volatility levels may be priced, provided a means of ascertaining volatility is available. Implied volatilities are now readily available, and term structures of volatility are also routinely developed, making this an envisageable product. (iv) Finally, these techniques are also useful for the commodities markets, in the pricing of options on convenience yields. Convenience yields are actively traded, and hedges against backwardation and contango risk may be easily set up using options on the term structure of convenience yields.

A short rate cap may also be viewed as an average rate option where the averaging period is the last instant before the contract expires. If the cap contract matures at time T, its payoff is given by

$$C_T = (r_T - K)1_{r_T \geq K} \tag{15}$$

¿From (15) we have the following pricing result for an interest rate cap:

$$C_0 = E_t^Q \left[e^{-Z_T} (r_T - K) 1_{r_T \geq K} \right] \tag{16}$$

This option is easily priced using the formula in Proposition 2 by setting the constants $k_0 > 0$, and $k_1 = k_2 = ... = k_{m+1} = 0$.

3.2 Yield Caps and Floors

We also consider the case when the cap payoff is made based on yields for an underlying period. We denote this period δ. We price a cap maturing at time T where the applicable interest rate (denoted R) is based on compounding over period length δ. The payoff at time $T + \delta$ is given by

$$\delta \max[0, R - K]$$

which translates into an equivalent payoff at time T of

$$P(r_T, T, T + \delta)\delta \max[0, R - K] \tag{17}$$

where $P(r_T, T, T+\delta)$ is the price of the bond with remaining maturity δ, denoted as $P(\delta)$ to simplify the notation. Noting that $1 + R\delta = P(\delta)^{-1}$, we have

$$R = \left(\frac{1}{P(\delta)} - 1 \right) \frac{1}{\delta}.$$

Using equation (17) we may write the value of the cap at time 0 as follows:

$$\begin{aligned}
\text{Cap}_{t=0} &= \delta E_0 \left[e^{-Z_T} \max \{0, P(\delta)R - P(\delta)K\} \right] \\
&= \delta E_0 \left[e^{-Z_T} \max \left\{ 0, P(\delta) \left(\frac{1}{P(\delta)} - 1 \right) \frac{1}{\delta} - P(\delta)K \right\} \right] \\
&= E_0 \left[e^{-Z_T} \max \{0, 1 - P(\delta) - \delta P(\delta)K\} \right] \\
&= E_0 \left[e^{-Z_T} \max \{0, 1 - P(\delta)(1 + \delta K)\} \right] \\
&= (1 + \delta K) E_0 \left[e^{-Z_T} \max \left\{ 0, \frac{1}{1 + \delta K} - P(\delta) \right\} \right]
\end{aligned}$$

which is straightforward to value since the expression above embeds the formula for a put option of maturity T, on a zero-coupon bond of maturity $(T + \delta)$, with a strike price of $1/(1 + \delta K)$. The formula for these options is developed in the following sections.

Yield options may be priced more generally by choosing the weights appropriately in the linear payoff function to match the coefficients of the yield equation. In order to price a cap on δ-maturity yield, Proposition 2 applies with $k_0 = A^*(\theta; \delta, \mathbf{0}, \mathbf{d}^*)$, $k_1 = B_1^*(\theta; \delta, \mathbf{0}, \mathbf{d}^*)$, ..., $k_m = B_m^*(\theta; \delta, \mathbf{0}, \mathbf{d}^*)$, $k_{m+1} = C^*(\theta; \delta, \mathbf{0}, \mathbf{d}^*)$, and recall that $\mathbf{d}^* = [-1, \mathbf{0}, 0]$.

3.3 Yield Combo Options

A "combo" option is one where the payoff depends on a basket of yields, weighted in any chosen proportions. If the payoff is determined based on n yields with weights $x_i, i = 1..n$ and maturities $\delta_i, i = 1..n$, then the option is priced using the result in Proposition 2 with the constants set as follows: $k_0 = \sum_{i=1}^{n} x_i A^*(\theta; \delta_i, \mathbf{0}, \mathbf{d}^*)$, $k_1 = \sum_{i=1}^{n} x_i B_1^*(\theta; \delta_i, \mathbf{0}, \mathbf{d}^*)$, ..., $k_m = \sum_{i=1}^{n} x_i B_m^*(\theta; \delta_i, \mathbf{0}, \mathbf{d}^*)$, $k_{m+1} = \sum_{i=1}^{n} x_i C^*(\theta; \delta_i, \mathbf{0}, \mathbf{d}^*)$.

There are many types of combo options in the market. (i) A special case of combo options are yield curve "*slope*" options, based on the difference of two yields (see Duffie, Pan and Singleton [44]). (ii) Differences in the levels of term structures in different markets may be expoited in these models. For example, "diff swaps" have been in place for quite a while - yield combo options are another way to achieve the benefits of diff swaps using packages of options. These "basis-rate" transactions are gaining in popularity as markets across the world develop much tighter interactions and linkages. (iii) In the foreign currency markets, we have "currency coupon swaps" which are options on two different LIBOR rates. These transactions have become popular with the onset of the European Monetary System. (iv) "Basket" yield options allow trading on a basket of different interest rates, usually reducing corporation hedging costs.

3.4 Example

Under the two jump example of the previous section, we price a cap on the short rate, at an exercise level of 10%. Using the equations from Proposition 2, we present the formula as:
(A) The solution for $\Pi_{0,t}$ is given by ($\mathbf{b}_0 = [\phi, 0], \mathbf{d}^* = [-1, 0]$)

$$\begin{aligned} \Pi_{0,t} &= \left\{ \Gamma_{0,t} \left[\frac{\partial A^*(\theta; \tau, \mathbf{b}_0, \mathbf{d}^*)}{\partial \phi} r_t + \frac{\partial C^*(\theta; \tau, \mathbf{b}_0, \mathbf{d}^*)}{\partial \phi} \right] \right\}_{\phi=0} \\ \Gamma_{0,t} &= \exp\left[A^*(\theta; \tau, \mathbf{b}_0, \mathbf{d}^*) r_t + C^*(\theta; \tau, \mathbf{b}_0, \mathbf{d}^*) \right] \end{aligned}$$

(B) The characteristic function for $\Pi_{1,t}$ is given by ($b_1 = [i\omega, 0]$)

$$\begin{aligned} \tilde{\Pi}_{1,t} &= \frac{1}{\Pi_{0,t}} \left\{ \frac{1}{i} \Gamma_{1,t} \left[\frac{\partial A^*(\theta; \tau, \mathbf{b}_1, \mathbf{d}^*)}{\partial \omega} r_t + \frac{\partial C^*(\theta; \tau, \mathbf{b}_1, \mathbf{d}^*)}{\partial \omega} \right] \right\} \\ \Gamma_{1,t} &= \exp\left[A^*(\theta; \tau, \mathbf{b}_1, \mathbf{d}^*) r_t + C^*(\theta; \tau, \mathbf{b}_1, \mathbf{d}^*) \right] \end{aligned}$$

(C) The characteristic function for $\Pi_{2,t}$ is given by

$$\tilde{\Pi}_{2,t} = \frac{1}{P_t(\tau)} \Gamma_{1,t}$$

For illustration, we compute prices for caps given a range of jump intensities, and the results are summarized below. The value of the option for the base case ($\lambda_u = \lambda_d = 5$) is 0.0814.

		λ_d			
		3	6	9	12
	3	0.0799	0.0774	0.0749	0.0724
λ_u	6	0.0837	0.0812	0.0787	0.0761
	9	0.0874	0.0849	0.0824	0.0799
	12	0.0911	0.0886	0.0861	0.0836

As one would expect, an increase in the downward jump frequency causes the option price to drop because the probability of the option ending in the money decreases. The opposite occurs as the upward jump frequency increases. When both upward and downward jump frequencies increase, option prices increase due to the increase in overall volatility.

4 Option Pricing for Exponential-Linear Payoffs

In this section we consider the case when the payoff function is given by an exponential-linear function of the interest rate and state variables:

$$f_T(\mathbf{r}, \mathbf{x}, \hat{\tau}) = \exp(k_0 r_T + k_1 x_{1,T} + \ldots + k_m x_{m,T} + k_{m+1}) \tag{18}$$

where $k_0, ..., k_{m+1}$ are constants. As indicated by (12), the price of a European call option for this payoff function is given by

$$F_t(\tau;\hat{\tau}) = \Pi_{0,t}\Pi_{1,t} - KP_t(\tau)\Pi_{2,t} \tag{19}$$

The following proposition develops the required option pricing formula:

Proposition 3 *(A) The solution for* $\Pi_{0,t}$ *is given by*

$$\Pi_{0,t} = \exp\left[A^*(\theta;\tau,\mathbf{b}_0,\mathbf{d}^*)r_t + \sum_{i=1}^{m} B_i^*(\theta;\tau,\mathbf{b}_0,\mathbf{d}^*)x_{i,t} + C^*(\theta;\tau,\mathbf{b}_0,\mathbf{d}^*)\right]$$

where

$$\mathbf{b}_0 = \begin{bmatrix} k_0 \\ k_1 \\ \vdots \\ k_m \\ k_{m+1} \end{bmatrix}$$

(B) The characteristic function for $\Pi_{1,t}$ *is given by*

$$\tilde{\Pi}_{1,t} = \frac{1}{\Pi_{0,t}}\exp\left[A^*(\theta;\tau,\mathbf{b}_1,\mathbf{d}^*)r_t + \sum_{i=1}^{m} B_i^*(\theta;\tau,\mathbf{b}_1,\mathbf{d}^*)x_{i,t} + C^*(\theta;\tau,\mathbf{b}_1,\mathbf{d}^*)\right] \tag{20}$$

where

$$\mathbf{b}_1 = \begin{bmatrix} (1+i\omega)k_0 \\ (1+i\omega)k_1 \\ \vdots \\ (1+i\omega)k_m \end{bmatrix}$$

(C) The characteristic function for $\Pi_{2,t}$ *is given by*

$$\tilde{\Pi}_{2,t} = \frac{1}{P_t(\tau)}\exp\left[A^*(\theta;\tau,\mathbf{b}_2,\mathbf{d}^*)r_t + \sum_{i=1}^{m} B_i^*(\theta;\tau,\mathbf{b}_2,\mathbf{d}^*)x_{i,t} + C^*(\theta;\tau,\mathbf{b}_2,\mathbf{d}^*)\right] \tag{21}$$

where

$$\mathbf{b}_2 = \begin{bmatrix} i\omega k_0 \\ i\omega k_1 \\ \vdots \\ i\omega k_m \end{bmatrix}$$

(D) We invert $\tilde{\Pi}_{k,t}$ *to obtain the probability* $\Pi_{k,t}$ *using a version of Fourier's theorem:*

$$\Pi_{k,t} = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left(\frac{1}{i\omega}e^{-i\omega K}\tilde{\Pi}_{k,t}\right)\, d\omega, \quad k = 1, 2$$

Proof: See the Appendix

The following three sections consider specific cases of this class of payoff function.

4.1 Bond Options

Bond options have been traded since the late 1970s, and are the oldest form of interest rate option contract.[13] Bond options are widely traded in pure form, and more often as embedded options in corporate and government debt, for example, in callable and puttable bonds. There are many more complicated applications of bond options, such as the delivery options on futures contracts, embedded options in mortgage-backed securities, and collateralized mortgage obligations. Options on risky debt, and options on real bonds are also traded, and this section enables easy pricing of these contracts. Valuing sinking funds may also be considered.

A European call option on a discount bond at strike K is the right but not the obligation to purchase a discount bond with remaining maturity $\hat{\tau}$ on the expiration date of the option. The option payoff is:

$$F_T(0;\hat{\tau}) = \max[P_T(\hat{\tau}) - K, 0]$$

The price of the bond option is the discounted expected value of the terminal payoff:

$$\begin{aligned} F_t(\tau;\hat{\tau}) &= E_t\left[e^{-\int_t^T r_v dv} \max[P_T(\hat{\tau}) - K, 0]\right] \\ &= E_t[e^{-Z_t(\tau)} P_T(\hat{\tau})]\Pi_{1,t} - K P_t(\tau)\Pi_{2,t} \\ &= P_t(\tau+\hat{\tau})\Pi_{1,t} - K P_t(\tau)\Pi_{2,t} \end{aligned} \tag{22}$$

This is easily priced, an examination of the results of Proposition 3 reveals that setting the constants to the following values provides the value of the bond option: $k_0 = A^*(\theta;\delta,\mathbf{0},\mathbf{d}^*)$, $k_1 = B_1^*(\theta;\delta,\mathbf{0},\mathbf{d}^*)$, ..., $k_m = B_m^*(\theta;\delta,\mathbf{0},\mathbf{d}^*)$, $k_{m+1} = C^*(\theta;\delta,\mathbf{0},\mathbf{d}^*)$.

4.2 Bond Futures and Forwards

We begin with a derivation of forward and futures prices. Let $F_{d,t}(\tau;\hat{\tau})$ denote the τ-period-ahead forward price of a $\hat{\tau}$-period bond at time t. By definition, the forward price is simply the ratio of the $(\tau+\hat{\tau})$-period bond price over the τ-period bond price:

$$F_{d,t}(\tau;\hat{\tau}) = \frac{P_t(\tau+\hat{\tau})}{P_t(\tau)} \tag{23}$$

Let $F_{u,t}(\tau;\hat{\tau})$ denote the τ-period-ahead futures price of a $\hat{\tau}$-period bond at time t. The futures price is given by a simple application of the exponential model:

$$F_{u,t}(\tau;\hat{\tau}) = \exp\left[A^*(\theta;t+\tau,\mathbf{b},\mathbf{d})r_t + \sum_{i=1}^m B_i^*(\theta;t+\tau,\mathbf{b},\mathbf{d})x_{i,t} + C^*(\theta;t+\tau,\mathbf{b},\mathbf{d})\right] \tag{24}$$

[13] Pricing formulae for bond options were available from as early as that of the the Vasicek [83] model and the Cox, Ingersoll and Ross [32] model. Since then, many other papers have dealt with bond option models: Carverhill and Clewlow [22], Courtadon [31], Das [34], Das and Foresi [36], Duffie, Pan and Singleton [44], Eydeland and Geman [46], Geman and Yor [48], Heath, Jarrow and Morton [52], Heston [53], Heston and Nandi [54], Ho and Lee [55], Jamshidian [58], Leblanc and Scaillet [62], Longstaff and Schwartz [67], Naik and Lee [69], Shirakawa [77], and Yor [85].

where

$$\mathbf{b} = \begin{bmatrix} A^*(\theta; t+\tau, \mathbf{0}, \mathbf{d}^*) \\ B_1^*(\theta; t+\tau, \mathbf{0}, \mathbf{d}^*) \\ \vdots \\ B_m^*(\theta; t+\tau, \mathbf{0}, \mathbf{d}^*) \\ C^*(\theta; t+\tau, \mathbf{0}, \mathbf{d}^*) \end{bmatrix}$$

$$\mathbf{d} = [0, \mathbf{0}, 0]$$

4.3 Bond Futures Options

Futures options are traded on exchanges and are typically liquid contracts. The T-bond futures option on the Chicago Board of Trade is the most liquid and widely traded interest rate derivatives contract. Bond Futures options usually embed two kinds of option features, that of the regular bond option, and another optionality coming from the delivery of a bond on exercise of the option. The option writer has the option to deliver the cheapest bond amongst an available set. In this paper, we abstract away from the latter option, focussing only on the optionality from pure interest rate risk.

A European call option on a discount bond future at strike K is the right but not the obligation to purchase a bond future with remaining maturity $\hat{\tau}$ on the expiration date of the option. Let $F_t(\tau; \hat{\tau}, \tau')$ be the price of an option with time-to-expiration τ, written on a futures contract with time-to-maturity $\hat{\tau}$ that calls for the delivery of a discount bond with time-to-maturity τ'. The option payoff is:

$$F_T(0; \hat{\tau}, \tau') = \max[F_{u,T}(\hat{\tau}; \tau') - K, 0]$$

The futures option is easily priced using the results of Proposition 3 by setting the constants to the following values provides the value of the bond option: $k_0 = A^*(\theta; \delta, \mathbf{0}, \mathbf{d})$, $k_1 = B_1^*(\theta; \delta, \mathbf{0}, \mathbf{d})$, ..., $k_m = B_m^*(\theta; \delta, \mathbf{0}, \mathbf{d})$, $k_{m+1} = C^*(\theta; \delta, \mathbf{0}, \mathbf{d})$, where $\mathbf{d} = [0, \mathbf{0}, 0]$.

4.4 Example

As an example of the models in this section, we price bond options on discount bonds of half-year remaining maturity ($\widehat{\tau} = 0.5$), where the option maturity is also a half year ($\tau = 0.5$). The representative equations from Proposition are as follows:
(A) The solution for $\Pi_{0,t}$ is given by ($\mathbf{b}_0 = [A^*(\theta; \widehat{\tau}, \mathbf{b}_0, \mathbf{d}^*), C^*(\theta; \widehat{\tau}, \mathbf{b}_0, \mathbf{d}^*)], \mathbf{d}^* = [-1, 0]$)

$$\Pi_{0,t} = \exp\left[A^*(\theta; \tau, \mathbf{b}_0, \mathbf{d}^*) r_t + C^*(\theta; \tau, \mathbf{b}_0, \mathbf{d}^*)\right]$$

(B) The characteristic function for $\Pi_{1,t}$ is given by

$$\tilde{\Pi}_{1,t} = \frac{1}{\Pi_{0,t}} \exp\left[A^*(\theta; \tau, \mathbf{b}_1, \mathbf{d}^*) r_t + C^*(\theta; \tau, \mathbf{b}_1, \mathbf{d}^*)\right]$$

$$\mathbf{b}_1 = \begin{bmatrix} (1+i\omega) A^*(\theta; \widehat{\tau}, \mathbf{b}_0, \mathbf{d}^*) \\ (1+i\omega) C^*(\theta; \widehat{\tau}, \mathbf{b}_0, \mathbf{d}^*) \end{bmatrix}$$

(C) The characteristic function for $\Pi_{2,t}$ is given by

$$\tilde{\Pi}_{2,t} = \frac{1}{P_t(\tau)} \exp\left[A^*(\theta;\tau,\mathbf{b}_2,\mathbf{d}^*)r_t + C^*(\theta;\tau,\mathbf{b}_2,\mathbf{d}^*)\right]$$
$$\mathbf{b}_2 = \begin{bmatrix} i\omega A^*(\theta;\widehat{\tau},\mathbf{b}_0,\mathbf{d}^*) \\ i\omega C^*(\theta;\widehat{\tau},\mathbf{b}_0,\mathbf{d}^*) \end{bmatrix}$$

For illustration, we compute prices for options given a range of jump intensities, and the results are summarized below. The value of the option for the base case ($\lambda_u = \lambda_d = 5$) is 0.0361.

		λ_d			
		3	6	9	12
	3	0.0362	0.0390	0.0423	0.0462
λ_u	6	0.0335	0.0357	0.0385	0.0419
	9	0.0313	0.0330	0.0353	0.0381
	12	0.0297	0.0309	0.0326	0.0348

As one would expect, an increase in the downward jump frequency causes option prices to increase. This is because an increase in the downward jump frequency in the short rate translates into an increase in the *upward* jump frequency of bond prices. Therefore, bonds of all maturities have a higher probability of being in the money. The opposite occurs with an increase in upward jump frequency.

5 Option Pricing for Integro-Linear Payoffs

In this section we consider the case when the payoff function is given by a path integral of a linear function of the interest rate and state variables:

$$f_T(\mathbf{r},\mathbf{x},\hat{\tau}) = \int_t^T (k_0 r_v + k_1 x_{1,v} + \ldots + k_m x_{m,v} + k_{m+1})dv \tag{25}$$

where $k_0, ..., k_{m+1}$ are constants. As indicated by (12), the price of a European call option for this payoff function is given by

$$F_t(\tau;\hat{\tau}) = \Pi_{0,t}\Pi_{1,t} - KP_t(\tau)\Pi_{2,t} \tag{26}$$

Proposition 4 *(A) The solution for $\Pi_{0,t}$ is given by*

$$\Pi_{0,t} = \left\{ \Phi_t \times \left[\frac{\partial A^*(\theta;\tau,\mathbf{0},\mathbf{d}_0)}{\partial\phi} r_t + \sum_{i=1}^m \frac{\partial B_i^*(\theta;\tau,\mathbf{0},\mathbf{d}_0)}{\partial\phi} x_{i,t} + \frac{\partial C^*(\theta;\tau,\mathbf{0},\mathbf{d}_0)}{\partial\phi} \right] \right\}_{\phi=0}$$

where

$$\Phi_t = \exp\left[A^*(\theta;\tau,\mathbf{0},\mathbf{d}_0)r_t + \sum_{i=1}^{m} B_i^*(\theta;\tau,\mathbf{0},\mathbf{d}_0)x_{i,t} + C^*(\theta;\tau,\mathbf{0},\mathbf{d}_0)\right]$$

$$\mathbf{d}_0' = \begin{bmatrix} \phi k_0 - 1 \\ \phi k_1 \\ \vdots \\ \phi k_m \\ \phi k_{m+1} \end{bmatrix}$$

(B) The characteristic function for $\Pi_{1,t}$ *is given by*

$$\tilde{\Pi}_{1,t} = \frac{1}{\Pi_{0,t}}\left\{\Phi_t \times \left[\frac{\partial A^*(\theta;\tau,\mathbf{0},\mathbf{d}_0)}{\partial\phi}r_t + \sum_{i=1}^{m}\frac{\partial B_i^*(\theta;\tau,\mathbf{0},\mathbf{d}_0)}{\partial\phi}x_{i,t} + \frac{\partial C^*(\theta;\tau,\mathbf{0},\mathbf{d}_0)}{\partial\phi}\right]\right\}_{\phi=i\omega}$$

(C) The characteristic function for $\Pi_{2,t}$ *is given by*

$$\tilde{\Pi}_{2,t} = \frac{1}{P_t(\tau)}\exp\left[A^*(\theta;\tau,\mathbf{0},\mathbf{d}_0)r_t + \sum_{i=1}^{m} B_i^*(\theta;\tau,\mathbf{0},\mathbf{d}_0)x_{i,t} + C^*(\theta;\tau,\mathbf{0},\mathbf{d}_0)\right]_{\phi=i\omega}$$

(D) We invert $\tilde{\Pi}_{k,t}$ *to obtain the probability* $\Pi_{k,t}$ *using a version of Fourier's theorem:*

$$\Pi_{k,t} = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left(\frac{1}{i\omega}e^{-i\omega K}\tilde{\Pi}_{k,t}\right)\,d\omega, \quad k = 1,2$$

Proof: See the Appendix

This payoff class relates to the pricing of average (Asian) options, on short rates and yields.

5.1 Asian Options

Asian options[14] have several uses: (i) Banks and corporations use them to hedge their financing costs over an extended period of time, rather than rely on more traditional contracts

[14] In the literature on Asian options, various analytical solutions have been obtained. Geman and Yor [48], provide a solution for the arithmetic average option when the underlying follows a Bessel process. Most of the work done on techniques for pricing Asian options focuses on numerical techniques such as Monte Carlo simulation or lattice-based methods. Examples of interesting numerical techniques for the Asian option problem with geometric Brownian motions include Dewynne and Wilmott [39], Yor [85], De-Schepper, Teunen, and Goovaerts [38] and Barraquand and Pudet [11] In addition, the overwhelming majority of work has focused on Asian options written on a stock price or a foreign exchange rate, where the use of geometric Brownian motion may be deemed appropriate.

such as caps, floors and collars. (ii) Corporations that have cash flows over a period of time may use an Asian option instead of a series of conventional options to hedge the risks associated with these cash flows. Asian options are often cheaper than regular options, which makes hedging cost-effective. (iii) The writers of caps and floors may use Asian options to hedge their risk on these contracts over several maturities. (iv) Interest differentials are known to follow mean-reverting processes, and Asian options written on the average interest differential of two currencies may be used to hedge risk in a portfolio of long term foreign currency options over a range of maturities. (v) Binary Asian options may be used to cover 'event risk'; such contracts pay off a fixed amount only if an event occurs. An example of such contracts is one where two parties contract on whether the EMS (European Monetary System) convergence will occur or not. In this setting, the rationale for the binary Asian option lies in the fact that interest rates will be in one of two regimes (high or low) depending on the outcome of EMS. Since regimes are often difficult to detect empirically, writing options on the average of a financial variable over a period of time is more likely to ensure that a financial variable actually resides within a regime, than when a variable is examined only at one point in time. (vi) Finally, Asian options are less susceptible to market manipulation by the option's counterparties, since it is harder to manipulate a market over an extended period of time.

Proposition 4 provides the pricing of Asian options on the short rate and yields. This complements the work of Longstaff [66] who has developed a similar single-factor model using different methods. Geman and Yor [48], use Bessel process methods to value perpetuities in both the O-U and square-root process models[15]. In this paper, alternative methods for *finite time* integrals of mean-reverting Brownian motions are developed by means of state-space expansion.[16]

Asian options on the short rate are priced using a special case of Proposition 4 where $k_0 > 0, k_1 = k_2 = ... = k_{m+1}$. However, Asian options on yields are far more widely used, such as in the case of options on the average of the 3 or 6-month yield. These are also emanable to Proposition 4 with $k_0 = A^*(\theta;\delta,\mathbf{0},\mathbf{d}^*)$, $k_1 = B_1^*(\theta;\delta,\mathbf{0},\mathbf{d}^*)$, $\ldots$, $k_m = B_m^*(\theta;\delta,\mathbf{0},\mathbf{d}^*)$, $k_{m+1} = C^*(\theta;\delta,\mathbf{0},\mathbf{d}^*)$, where the option is written on the average of the δ-maturity yield.

5.2 Example

As an example, we extend the two-jump model to the pricing of an Asian option on the short rate. The option maturity is a half year ($\tau = 0.5$) and the exercise level of the average rate is 10%. The equations from Proposition 4 are

[15] Perpetuities are also integrals of exponentials of a Brownian motion and hence are logically subsumed within the framework of Geman and Yor [48] perpetuities by Dufresne [45].

[16] In an earlier version of the paper, our method for binary Asian options on jump-diffusion processes had not been extended to standard Asian options, developed subsequently by Bakshi and Madan [8],[9] for diffusion processes. Here, we provide complete solutions to all Asian type contracts in the particular framework of this paper, which complements the results of Bakshi and Madan, and more recently, for jump-diffusions, by Duffie, Pan and Singleton [44].

(A) The solution for $\Pi_{0,t}$ is given by

$$\begin{aligned}
\Pi_{0,t} &= \left\{ \Phi_t \times \left[\frac{\partial A^*(\theta;\tau,\mathbf{0},\mathbf{d}_0)}{\partial \phi} r_t + \frac{\partial C^*(\theta;\tau,\mathbf{0},\mathbf{d}_0)}{\partial \phi} \right] \right\}_{\phi=0} \\
\Phi_t &= \exp\left[A^*(\theta;\tau,\mathbf{0},\mathbf{d}_0) r_t + C^*(\theta;\tau,\mathbf{0},\mathbf{d}_0) \right] \\
\mathbf{d}_0' &= \begin{bmatrix} \phi - 1 \\ 0 \end{bmatrix}
\end{aligned}$$

(B) The characteristic function for $\Pi_{1,t}$ is given by

$$\tilde{\Pi}_{1,t} = \frac{1}{\Pi_{0,t}} \left\{ \Phi_t \times \left[\frac{\partial A^*(\theta;\tau,\mathbf{0},\mathbf{d}_0)}{\partial \phi} r_t + \frac{\partial C^*(\theta;\tau,\mathbf{0},\mathbf{d}_0)}{\partial \phi} \right] \right\}_{\phi=i\omega}$$

(C) The characteristic function for $\Pi_{2,t}$ is given by

$$\tilde{\Pi}_{2,t} = \frac{1}{P_t(\tau)} \exp\left[A^*(\theta;\tau,\mathbf{0},\mathbf{d}_0) r_t + C^*(\theta;\tau,\mathbf{0},\mathbf{d}_0) \right]_{\phi=i\omega}$$

While the jump example is merely illustrative, an extensive array of numerical examples for the Asian option model and other models is provided in Section 7.

6 Model Implementation

The solutions provided in the previous sections provide a convenient set of results that should allow researchers to write down pricing solutions to exponential term structure models in one quick step. However, the implementation of these models for actual pricing purposes requires calibration of the base term structure model against a set of data. In this section, we extend the approach in [42] to show how calibration can readily be accomplished, and subsequently, we show how to implement option pricing using the calibrated model.

Calibration of the model using a cross-section of bond prices provides one way of obtaining the risk-neutral parameters required for derivative security pricing. In the class of models investigated in this paper, it is possible to obtain parameter estimates directly off the Riccati equations for the term structure model. We call this approach "pricing by estimation of primitives".

6.1 Calibration Methodology

We assume the existence of cross-sectional data on bond prices, i.e.there are a set of N bonds at time t: $\{P_t(\tau_k)\}_{k=1\ldots N}$. Alternatively, estimation may be undertaken using a full panel data set of T observations, in which case we have $\{P_t(\tau_k)\}_{k=1\ldots N, t=1\ldots T} \in R^{T\times N}$. These prices directly map into a set of yields: $Y_t(\tau_k), \forall k, t$. The yields are given by

$$Y(\tau) = -\frac{1}{\tau}\ln[P(\tau)] = -\frac{1}{\tau}[A(\tau)r_t + \sum_{i=1}^{m} B_i(\tau)x_i + C(\tau)] \quad (27)$$

The coefficients in the pricing equation, $A(\tau), B_1(\tau), ..., B_m(\tau), C(\tau)$, are solutions to the Riccati equation system. Cross-sectional calibration is possible using the closed form solutions for $P(\tau)$ as was undertaken in Brown and Dybvig [20].[17]

Given the set of affine processes for the term structure model, and data on the state variables, starting with the initial condition, and a guess of the parameters of the stochastic processes, we use the Riccati equations to generate values of $A(\tau_k), B_1(\tau_k), ..., B_m(\tau_k), C(\tau_k)$ for $k = 1...N$ via forward propogation in time. Using vectorization, this is done in one pass and results in fast and accurate computation for the entire set of bonds. These values and the data on the state variables $(r, x_1...x_m)$ determine the right-hand side of equation (27). Least squares minimization[18] of fitted versus actual yields allows rapid convergence of the algorithm to yield the vector of parameter estimates θ.

[17]The estimation literature for the term structure has been extended at a galloping rate. A representative sample of estimation methods is covered by the papers of Ait-Sahalia [2], Attari [3], Babbs and Webber [4], Balduzzi, Bertola, Foresi and Klapper [5], Balduzzi, Das and Foresi [7], Brandt and Santa-Clara [16], Brenner, Harjes and Kroner [18], Chacko [24], Chan, Karolyi, Longstaff and Sanders [26], Conley, Hansen, Luttmer and Scheinkman [29], Dai and Singleton [33], Das [35], Duffie and Glyn [41], Gray [49], Jagannathan and Wang [57], Koedijk, Nissen, Schottman and Wolff [61], Naik and Lee [69], Pritsker [70], Singleton [78] and Stanton [79].

[18]Any other criterion may be used as well.

The algorithm may be summarized as follows:

$$
\begin{aligned}
& \min_{\theta} \sum_{t=1}^{T} \sum_{k=1}^{N} \varepsilon_t[\theta(\tau_k)]^2 \\
\text{subject to} \quad & : \\
\varepsilon_t[\theta(\tau_k)] &= Y(\tau_k) + \frac{1}{\tau}[A(\tau_k)r_t + \sum_{i=1}^{m} B_i(\tau_k)x_{it} + C(\tau_k)] \\
\frac{\partial A}{\partial \tau} &= \frac{1}{2}\sum_{i=1}^{n} \sigma_{r,i}^2 A^2 + \mu_r A + d, \quad \forall \tau_k, d = -1, A(0) = 0 \\
\frac{\partial B_1}{\partial \tau} &= \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n} \Lambda_{x_1,ij}^2 B_i B_j + \sum_{i=1}^{m} \rho_{x_1,i} A B_i \\
& + \sum_{i=1}^{n} \sigma_{x_1,i}^2 A^2 + \mu_{x_1} A + \sum_{i=1}^{m} \alpha_{x_1,i} B_i, \quad \forall \tau_k, [B_1(0) \ldots B_m(0)]' = 0 \\
& \vdots \\
\frac{\partial B_m}{\partial \tau} &= \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n} \Lambda_{x_m,ij} B_i B_j + \sum_{i=1}^{m} \rho_{x_m,i} A B_i \\
& + \sum_{i=1}^{n} \sigma_{x_m,i}^2 A^2 + \mu_{x_m} A + \sum_{i=1}^{m} \alpha_{x_m,i} B_i \quad \forall \tau_k, [B_1(0) \ldots B_m(0)]' = 0 \\
\frac{\partial C}{\partial \tau} &= \frac{1}{2}\sum_{j=1}^{n} \sigma_j A^2 + \mu A + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n} \Lambda_{ij} B_i B_j + \sum_{i=1}^{m} \alpha_i B_i + \sum_{i=1}^{m} \rho_i A B_i \\
& + \sum_{i=1}^{l} \lambda_i \left(E\left[e^{AJ_{r,i} + B_1 J_{x,1i} + \ldots B_m J_{x,mi}} \right] - 1 \right) \quad \forall \tau_k, C(0) = 0
\end{aligned} \tag{28}
$$

This approach has many useful features. First, since the Riccati equation system (28) consists entirely of first-order ordinary differential equations, generation of the value set $A(\tau_k)$, $B_1(\tau_k), \ldots, B_m(\tau_k)$, $C(\tau_k)$ for a given θ is very accurate.[19] Second, since the calibration equation is linear, and the objective function is quadratic, we obtain a well-behaved optimization problem. Third, we retain the choice of undertaking calibration either for the entire panel of data, or for a single cross-section only. Fourth, since the information used relates directly to the prices of derivative securities, all estimated parameters are with respect to the risk-neutral measure and may be used for pricing immediately.[20]

[19]Indeed, standard mathematical packages yield highly accurate results. We found the `ode45` function in Matlab to be extremely fast and accurate. This is but one example of the power of using the original Riccati equations. Other facile implementations using characteristic function estimators are considered in Chacko [24] and Singleton [78].

[20]Appendix B gives an example of the implementation for the Vasicek [83] model. Extending the estimation

6.2 The Implementation of Option Pricing

As an example, we consider the pricing of bond options, for which the equation is : $F_t(\tau;\hat{\tau}) = P_t(\tau+\hat{\tau})\Pi_{1,t} - KP_t(\tau)\Pi_{2,t}$, where K is the strike price. There are four components to this model: (i) an underlying bond of maturity $(\tau+\hat{\tau})$, (ii) a bond of the same maturity (τ) as the option, (iii) the probability of the option finishing in the money $(\Pi_{2,t})$, and (iv) the present value of a dollar conditional on the option finishing in the money $(\Pi_{1,t})$. Since the first two components are directly observable from the market, we need only compute the two probability values.

Since $\Pi_{2,t}$ is not directly computable we obtain its characteristic function, $\tilde{\Pi}_{2,t}$, which is the solution to the Riccati differential equation system (28). We propogate the differential equation system forward to time T, beginning with the appropriate initial conditions, which are (see Proposition 3):

$$\mathbf{b} = \begin{bmatrix} i\omega A^*(\theta;\hat{\tau},\mathbf{0},\mathbf{d}^*) \\ i\omega B_1^*(\theta;\hat{\tau},\mathbf{0},\mathbf{d}^*) \\ \vdots \\ i\omega B_m^*(\theta;\hat{\tau},\mathbf{0},\mathbf{d}^*) \\ i\omega C^*(\theta;\hat{\tau},\mathbf{0},\mathbf{d}^*) \end{bmatrix} \tag{29}$$

The values $A^*(\theta;\hat{\tau},\mathbf{0},\mathbf{d}^*)$, $B_1^*(\theta;\hat{\tau},\mathbf{0},\mathbf{d}^*),...,B_m^*(\theta;\hat{\tau},\mathbf{0},\mathbf{d}^*)$, $C^*(\theta;\hat{\tau},\mathbf{0},\mathbf{d}^*)$ are the coefficients from boundary conditions that have been computed from the calibration step and are therefore already available. Hence, the vector $\mathbf{b}$ is completely known, and forms an observable initial condition for forward propogation via the Riccati system. For implementation purposes we discretize the state space on which the Fourier inversion parameter ω resides, i.e. generate a finite support set $\omega \in \{0,\omega_1,\omega_2,...,\overline{\omega}\}$ with equal intervals $\Delta\omega$. This generates via the Riccati system a set of values of the probability $\tilde{\Pi}_{2,t}(\omega)$ for each value of ω. Discrete Fourier inversion yields the following probability:

$$\Pi_{2,t} = \frac{1}{2} + \frac{1}{\pi}\sum_{\omega=0}^{\overline{\omega}} \mathrm{Re}\left(\frac{1}{i\omega}e^{-i\omega\log K}\tilde{\Pi}_{2,t}(\omega)\right)\Delta\omega \tag{30}$$

A similar analysis results in the solution to the probability $\Pi_{1,t}$.

Employing the Vasicek model, the known closed-form solutions for the options are compared to the numerical values we obtained. The results in Tables 1 and 2 confirm that the procedure is accurate.

7 Illustrative Examples

In this section we present examples illustrating the techniques of the paper. The purpose of the section is not to develop pricing solutions for new models but instead to illustrate how

procedure to calibration off option prices involves only one extra dimension in the ODE generator for the parameter ω. See the following section.

to use the techniques developed in earlier sections of this paper. This is best done in the context of simple models. Consequently, first, we provide the results as applicable to the Cox-Ingersoll-Ross (CIR, [32]) model. Second, we discuss an implementation of the CIR and Vasicek [83] models with several numerical results. Third, we extend the analysis to some more exotic options such as range-Asians. Finally, we provide results for a version of the jump-diffusion example that has been used throughout this paper.

7.1 Pricing in the Cox-Ingersoll-Ross Model

In order to illustrate the generality of the technique developed in the preceding section, we derive the analytics for the square-root diffusion in closed form. Our objective here is to undertake a step-by-step example of the implementation of the technique. Cox, Ingersoll and Ross [32] assume that under the risk-neutral measure, interest rates follow a square-root diffusion:

$$dr_t = (k\,[\theta - r_t] - \xi r_t)\,dt + \eta\sqrt{r_t}dW_t$$

The long run mean of the interest rate is θ, and the interest rate reverts to this mean at rate k. The diffusion has square-root volatility with coefficient η, and W_t is the standard Wiener shock. ξ is the market price of interest rate risk. We define $\mu_0 = k\theta, \mu_1 = -(k+\xi)$, so that the process may be written as:

$$dr = (\mu_0 + \mu_1 r)\,dt + \eta\sqrt{r}dW_t$$

where it is clear that the drift and variance are linear in the interest rate r, so that the requirements for the affine class of models is met. The general PDE we need to solve (analogous to equation 3) to carry out all the analytics is as follows.

$$\frac{1}{2}\eta^2 r\frac{\partial^2 P}{\partial r^2} + (\mu_0 + \mu_1 r)\frac{\partial P}{\partial r} - \frac{\partial P}{\partial \tau} + drP = 0.$$

We guess and verify a solution to this PDE: $P(r,\tau) = \exp[A(\tau)r + C(\tau)]$. The specific problem we wish to solve is defined by the choice of values $\{a, c, d\}$, where $A(0) = a$, $C(0) = c$. Therefore, $P(r,0) = \exp[ar + c]$. In the special case when $a = c = 0, d = -1$ we obtain the bond price. Solving the PDE by separation of variables, we obtain two ODEs which we solve entirely in closed form. These ODEs are

$$\begin{aligned}
\frac{dA}{d\tau} &= \frac{1}{2}\eta^2 A^2 + \mu_1 A + dP, \quad A(0) = a \\
\frac{dC}{d\tau} &= \mu_0 A, \quad C(0) = c
\end{aligned}$$

The solutions are as follows:

$$
\begin{aligned}
A(\tau, a, d) &= \frac{2}{\eta^2}\left(\frac{[2u_2 + a\eta^2]u_1 e^{u_1\tau} - [2u_1 + a\eta^2]u_2 e^{u_2\tau}}{[2u_1 + a\eta^2]e^{u_2\tau} - [2u_2 + a\eta^2]e^{u_1\tau}}\right) \\
C(\tau, c, d) &= c + \frac{2\mu_0}{\eta^2}\ln\left(\frac{2[u_1 - u_2]}{[2u_1 + a\eta^2]e^{u_2\tau} - [2u_2 + a\eta^2]e^{u_1\tau}}\right) \\
u_1 &= \frac{\mu_1 + \sqrt{\mu_1^2 - 2d\eta^2}}{2} \\
u_2 &= \frac{\mu_1 - \sqrt{\mu_1^2 - 2d\eta^2}}{2}
\end{aligned}
$$

Using this general solution, we now price a few representative products, using the formulae provided in this equation and the results from Propositions 2, 3 and 4 of the paper.

7.2 Numerical Examples

In this section, we shall price options using two popular interest rate processes. These will illustrate the application of the general techniques of the previous section. We present some of the building blocks in closed form. In addition to the Cox, Ingersoll and Ross [32] model, we employ the Vasicek [83] model for which the stochastic process is:

$$
dr_t = (k\,[\theta - r_t] - \xi\eta)\,dt + \eta dW_t
$$

Expressions for the Vasicek model may be derived analogous to those presented in section 7.1. Using these two models, we price four different types of options: (i) Asian binary caps, (ii) Non-Asian binary caps, i.e. a simple digital option on the interest rate, (iii) Asian non-binary (regular) caps, i.e. a non-digital cap on the average interest rate, and (iv) Regular (non-Asian) caps on the interest rate. Thus, we consider Asian vs non-Asian options and their combinations with binary vs non-binary forms.

Tables 3 thru 6 present option prices for the square-root model. Tables 7 and 8 present option prices for the Ornstein-Uhlenbeck model. The notional values underlying all the options is a dollar. Several results emanate from the numerical analysis. Binary options are worth more than regular options since they always pay off a dollar.

The moneyness of the option affects the comparison of Asian and non-Asian options. This is illustrated in Tables 3 thru 8. When the exercise price K is below the interest rate r, the Asian binary option is worth more than the non-Asian binary option. An option which is in-the-money at the outset is more likely to sustain that value if it is an Asian option than a non-Asian option, since even if interest rates fall, the average of the interest rate will not fall as fast. Likewise, when the exercise price K is above the interest rate r, the Asian binary option is worth less than the non-Asian binary option - an option which is out of the money at the outset is more likely to remain out of the money if it is an Asian option than a simple option, since even if interest rates rise, the average of the interest rate will not rise as fast.

When volatility increases (as in Table 4 versus Table 3), the options that are out-of-the-money increase in value, since the likelihood of the options finishing in-the-money increases when volatility increases. Conversely, the in-the-money options decline in value when volatility increases. This effect holds for both the digital Asian option and the simple digital option.

Since mean reversion is a feature that distinguishes interest rate dependent securities from equity dependent securities, it is instructive to look at the parameter θ, the long-run mean of the interest rate. In the presence of mean reversion, the location of the long run mean is critical for the pricing of options. A comparison of Tables 3, 5 and 6 reveals that the location of the mean rate (θ) impacts options values quite severely. When the mean is low ($\theta = 0.05$) option values drop substantially. This is especially true for the non-Asian options and the effect is less marked for the Asian options. This is because when pricing caps, if the mean rate is low, then mean-reversion drags down the value of the terminal interest rate and the average interest rate. Likewise, when the mean is high ($\theta = 0.15$), option values are substantially higher. Hence, unlike with equity options, even if the option is deep in-the-money, the interest rate option may still not offer much value, if the interest rate quickly reverts to a mean level which is quite low.

One question of importance when pricing Asian options on interest rates, is that of model choice. Are the values of Asian options very sensitive to the specific choice of stochastic process for interest rates? To answer this question, we compare prices from the O-U model with those of the square-root diffusion, taking care to ensure that the average volatility in both models is held constant. Tables 7 and 8 provide results for the O-U process. They are analogous to Tables 3 and 4 for the square-root process. The average volatility of the O-U process has been set approximately equal to that of the square-root process by means of the following equation: $\eta_{OU} = \eta_{SQR}\sqrt{r_0}$, where η_{OU} is the O-U process volatility coefficient, η_{SQR} is the coefficient for the square-root process, and $r_0 = \theta$. Therefore, in Table 7 we use $\eta_{OU} = 0.20\sqrt{0.1} = 0.063246$. In Table 8, the value is $\eta_{OU} = 0.30\sqrt{0.1} = 0.094868$. In general, the prices from the O-U model are fairly close to, though higher than that of the square-root model. We can thus conclude that the choice of interest rate model does not substantially impact Asian option prices. Mean-reversion makes the level-dependent volatility of the square-root diffusion likely to be quite stable, and hence, especially for the Asian option, this average volatility will be quite close to the constant level of volatility in the O-U model.

¿From the analysis so far, it is clear that one of the most interesting differences between equity options and interest rate options is the feature of mean-reversion. Mean-reversion tends to reduce average volatility over time, reducing option values in most cases. However, it also affects the direction (skewness) of the interest rate depending on where the current rate of interest is relative to its long-run mean level. Depending on how strong the rate of mean reversion is, it may cause away from the money options to demonstrate interesting behavior. For example, if the rate of mean reversion (k) is high, out-of-the-money options become more likely to swing into-the-money, and vice versa for in-the-money options. Mean reversion (k) has an interesting impact on option prices as time to maturity (T) varies. As T increases, options prices increase at first and then decline as the effect of mean reversion begins to negate the effect of volatility.

7.3 Range-Asian Options

To examine mean reversion further, we decided to explore a more exotic option, the range-Asian. The range-Asian is an interesting option to analyze because it offers a good setting in which the joint effects of the mean rate θ, and the rate of mean-reversion k, may be examined. In general, a range option is one which pays off a certain amount each day if the value of the underlying variable lies within a pre-specified range. The range-Asian pays off each day that the current average up to that date remains within pre-specified limits. These options have daily pay offs which are based on whether the average interest rate up to time t lies within a prespecified range $[a(t), b(t)], a(t) < b(t), \forall t \in [0, T]$. In the paper we assume that $a(t) = a$ and $b(t) = b$, without loss of generality. The value of these options is simply

$$
\begin{aligned}
R[a(t), b(t), T] &= \frac{1}{d}\sum_{j=1}^{d}[Q(a(t), t) - Q(b(t), t)] \\
d &= \text{Flr}(T * 365) \\
t &= \frac{j}{365}
\end{aligned}
$$

where $Q(.)$ is the value of a binary Asian option, and $\text{Flr}(x)$ is a function that returns the greatest integer less than or equal to K. Our analyses utilize both, the square-root and the O-U process models.

In Table 9 we present prices of range-Asian options. These prices increase when the range widens. When the mean rate θ lies inside the range, increases in mean reversion (k) drive the price upwards. This is because, as k rises, the likelihood of the interest rate remaining within the range increases, thereby raising value. When the mean lies outside the range, option prices decrease when k increases because the interest rate is less likely to remain in the desired range. This is true of both cases, when the mean is above and below the range, i.e. $\theta = 0.15$ and $\theta = 0.05$ respectively.

7.4 The Jump-Diffusion Model

In this section, we analyze the pricing of Asian options in a jump-diffusion framework. In particular, we extend the results of Das and Foresi [36] to the pricing of Asian options on jump-diffusions. The underlying interest rate process is as follows

$$dr = k(\theta - r)dt + \sigma dz + J(\psi, \alpha)dQ(\lambda).$$

Here, k is the rate of mean reversion, θ is the long run mean of the interest rate, σ is the coefficient of diffusion volatility, and dz is the Wiener increment. The jump portion has a point process Q with jump arrival intensity λ and the jump J has a sign determined by parameter ψ, which represents the probability of a positive jump. The parameter $\alpha > 0$

defines the jump size and is the distribution parameter for an exponential distribution, such that it has mean $\frac{1}{\alpha}$, i.e. the probability density function is given by $f(J) = \alpha e^{-\alpha J}$. As an example, we price bonds and options with the following base case parameters:

k	θ	σ	λ	α	ψ	r	τ
2	0.1	0.02	5	50	1	0.1	3

Bond maturity is denoted τ. This results in a discount bond price of $P(0.1, 3) = 0.6545$. As an illustration we choose one interesting case, i.e. $\psi = 1$, which indicates that there will only be positive jumps, and this diminishes the probability of negative interest rates, but also injects a substantial quantity of positive skewness in interest rates. We shall vary the jump intensity (λ) from 0 to 10 to see how increasing skewness affects the price of the binary Asian option and the standard Asian option. Results are presented in Table 10.

As λ increases, the value of the binary Asian option first rises and then declines. The intuition for this is straightforward. Increasing positive skewness forces the binary option further into the money, making it more valuable. At the same time, the skewness increases the average discount rate for the payoff reducing the value of the option. When λ reaches the value of 3, the binary option is maximized in value, and declines thereafter as the discounting effect swamps the increasing payoff.

In Table 10, the standard Asian option of course has lower values, and since it is probability weighted, the payoff effect is always strong enough to outweigh the discounting effect, resulting in a monotonically increasing option value as skewness increases.

Thus, we have demonstrated with several numerical examples that the solutions derived in this paper are easy to implement in practice. Since the solution equations are merely a few lines, and do not contain more than a single integral, they are easy to write computer code for, and implementation with a mathematical software package is simple.

8 Extensions

In this section, we conclude the paper by showing that the pricing model can be applied in settings beyond those described thus far in the paper. The first setting is that of a no-arbitrage model where the short-rate process is allowed to have time-varying components, allowing for an exact match with the current term structure of interest rates, volatilities, etc. Examples of such a model are those of [56] and [13]. In this setting, the pricing formulas derived above continue to hold. Thus, even in settings where calibration of the model to currently observed data must be exact, the general pricing formulas derived above can still be used for pricing popular fixed income securities. The second setting is the case where the term structure model is no longer exponential-affine, but is exponential separable, i.e., where log bond prices are given by

$$\log P_t(\tau) = A_1(\tau) r_t + \sum_{i=2}^{q} A_i(\tau) f_i(r_t, \mathbf{x}_t) + B(\tau)] \tag{31}$$

where $P_t(\tau)$ represents the price of a bond maturing in τ periods, $A_i(\tau)$ and $B(\tau)$ represent functions of time-to-maturity only, and $f_i(r_t, \mathbf{x}_t)$ represent (possibly) nonlinear functions of any factors determining bond prices. The generalization here from the exponential-affine class is to allow for non-linear functions of factors, but to restrict the non-linear structure so that the time-varying component may still be separated out from the factors. Examples of this class of models are [30] and [65]. In this setting, the pricing formulas derived above continue to hold but with the factors, $\mathbf{x}_t$, in each pricing formula replaced by their respective nonlinear functions, $f_i(r_t, \mathbf{x}_t)$.

8.1 No-Arbitrage Models

Exact calibration of pricing models to currently observable data is an important requirement for most practitioners. One class of no-arbitrage models that allows for this defines the short rate process with time-varying components in the drift, volatility, and/or jump terms and uses these time-varying components as free variables to match up to observed data. Examples of this class of models include [56] and [13]. We now show via our jump-diffusion example how to use the pricing formulas derived above in the context of such models in which the bond price is also exponential-affine.

We first extend the one-factor, jump-diffusion example we have been using throughout this paper to allow for a time-varying central tendency. The interest rate process is defined by

$$dr = \kappa[\theta(t) - r]dt + \sigma dW + J_u dN_u(\lambda_u) - J_d dN_d(\lambda_d)$$

where the central tendency is now a time-varying function, $\theta(t)$, rather than a constant. In this case the generalized term structure model is given by

$$\exp[A^*(\tau, \mathbf{b}, d)r_t + C^*(\tau, \mathbf{b}, d)]$$

where

$$\begin{aligned}
A^*(\tau, \mathbf{b}, d) &= (a - \frac{d}{k})e^{-\kappa\tau} + \frac{d}{k} \\
C^*(\tau, \mathbf{b}, d) &= -\frac{u_1^2\sigma^2}{4\kappa\tau}(e^{-2\kappa\tau} - 1) + \frac{u_1 u_2 \sigma^2}{\kappa}(e^{-\kappa\tau} - 1) + \left[\frac{u_2^2\sigma^2}{2} - \lambda_u + \lambda_d\right]\tau \\
&\quad + \frac{\lambda_u}{\kappa - d\eta_u}\log|[(1 + \eta_u u_2)e^{\kappa\tau} - \eta_u u_1]/[1 - \eta_u u_1 - \eta_u u_2]| \\
&\quad + \frac{\lambda_u}{\kappa + d\eta_u}\log|[(1 + \eta_d u_2)e^{\kappa\tau} - \eta_d u_1]/[1 + \eta_d u_1 + \eta_d u_2]| \\
&\quad + c - a + \int_0^\tau \kappa\theta(v)A^*(v, \mathbf{b}, d)dv \\
u_1 &= a - u_2 \\
u_2 &= \frac{d}{\kappa}
\end{aligned}$$

Here, the bond price (formed when $a = 0$, $c = 0$, $d = -1$) is a function of the time-varying function $\theta(t)$, which appears in the expression for $C^*(\tau, \mathbf{b}, d)$. Therefore, by choosing the function for $\theta(t)$ appropriately, the current term structure of interest rates, volatilities, etc. can be matched perfectly. Furthermore, because the price of a bond is exponential-affine here, all of the pricing formulas for fixed-income derivatives derived in the paper apply with this model as well. Consequently, once calibration of this model to currently observed data is accomplished, pricing formulas for popular fixed-income securities can be written by inspection using the formulas derived earlier.

8.2 Exponential-Separable Models

Considerable research is now being focused on non-affine term structure models. While few such models have been found with closed-form solutions, we want to allow for the use of the formulas derived above for a certain class of models which seem promising: the exponential-separable class. Bond prices for this class of models have the form given in (31). Examples of this class for which closed-form solutions exist include [30] and [65].

It is easy to show that the pricing models derived in this paper can easily accomodate term structure models of this class with minor changes. Specifically, we first introduce the new variables y_i, $i = 2 \ldots q$. These variables are defined as

$$\begin{aligned} y_2 &= f_2(r_t, \mathbf{x}_t) \\ y_3 &= f_3(r_t, \mathbf{x}_t) \\ &\vdots \\ y_q &= f_q(r_t, \mathbf{x}_t) \end{aligned}$$

Notice now that the term structure model is now exponential-affine in the interest rate and these new variables. Therefore, from [42], we know that the stochastic differential equations governing must have linear drifts and instataneous variances. Thus, the interest rate and the new factors, y_i, now have linear drifts and variances in y_i. Thus, the transformed term structure model fits into the exponential-affine class of models and all of the pricing results derived in this paper apply. Consequently, the results of this paper apply to exponential-separable models as well but with the individual factors, $\mathbf{x}_t$, in the pricing formulas replaced by their non-linear function counterparts, $f_i(r_t, \mathbf{x}_t)$ from the generalized exponential-separable term structure model.

9 Concluding Comments

Duffie and Kan [42] established the relationship between affine stochastic processes and bond pricing equations in exponential term structure models. We extend the results in their paper to the pricing of interest rate derivatives. This paper shows that if an exponential affine structure is assumed for the term structure, there is a fundamental link between the components of the bond pricing solution and the prices of many widely traded interest rate derivative securities.

The intuition for our results stems from the fact that derivative prices are derived from a set of differential equations that are similar to those for bond prices upto a modification of constant terms. Our results apply to multifactor processes with multiple diffusions and jump processes. Regardless of the number of shocks, the pricing solutions require at most a single numerical integral, making the model easy to implement. In addition, we show that the results of the paper can be easily extended to no-arbitrage models of the type developed in [56], with time-varying components in the short rate or factors as well as a class of non-linear term structure models: exponential-separable term structure models, such as that in [30].

We provide many examples of options that yield solutions using the methods of the paper. While the general approach is the same, the mathematical details for each option vary, resulting in three separate option models, based on the structure of the payoff function. We show that fast estimation of these models is possible by vectorizing the Riccati equations for the pricing solutions. A range of numerical solutions illustrates the use of the models. The analytical solutions for European options are useful in speeding up the pricing of American options, and for less tractable options, such as swaptions and options on coupon bonds in multifactor settings.

A Solving for the Probability Functions

In this section, we solve for various probability functions needed for the different options priced in this paper. Each probability function varies in subtle ways from the other, and requires different techniques for their solution. The following subsections are categorized by structure of the payoff function.

A.1 Linear Payoff Functions

A.1.1 Solution for $\Pi_{0,t}$ with linear payoffs

To solve for the function $\Pi_{0,t} = E_t[e^{-Z_t(T)}(k_0 r_T + k_1 x_{1,T} + \ldots + k_m x_{m,T} + k_{m+1})]$, we first note that

$$\begin{aligned} & E_t[e^{-Z_t(T)}(k_0 r_T + k_1 x_{1,T} + \ldots + k_m x_{m,T} + k_{m+1,T})] \\ = & \left\{ \frac{\partial}{\partial \phi} E_t \left[e^{-Z_t(T)} e^{\phi(k_0 r_T + k_1 x_{1,T} + \ldots + k_m x_{m,T} + k_{m+1,T})} \right] \right\}_{\phi=0} \end{aligned}$$

Therefore, we need to solve for $E_t \left[e^{-Z_t(T)} e^{\phi(k_0 r_T + k_1 x_{1,T} + \ldots + k_m x_{m,T} + k_{m+1,T})} \right] \equiv \Gamma_{0,t}$ and simply evaluate the partial derivative of this expression with respect to ϕ at $\phi = 0$. We now apply the Feynman-Kac relation, from which we know that this equation solves the PDDE

$$0 = \mathcal{D}\Gamma_{0,t} - r_t \Gamma_{0,t} \tag{32}$$

with boundary condition

$$\Gamma_{0,T} = \exp f_T(\mathbf{r}, \mathbf{x}, \hat{\tau}) = \exp[\phi k_0 r_T + \phi k_1 x_{1,T} + \ldots + \phi k_m x_{m,T} + \phi k_{m+1,T}] \tag{33}$$

In comparing (32) and (33) with (3) and (6), we see that the PDDEs are exactly the same, while the boundary conditions differ by only a set of constant coefficients in front of the interest rate and factors. Therefore, by analogy, we can write down the solution to $\Gamma_{0,t}$ as

$$\Gamma_{0,t} = \exp \left[A^*(\theta; \tau, \mathbf{b}_0, \mathbf{d}^*) r_t + \sum_{i=1}^{m} B_i^*(\theta; \tau, \mathbf{b}_0, \mathbf{d}^*) x_{i,t} + C^*(\theta; \tau, \mathbf{b}_0, \mathbf{d}^*) \right] \tag{34}$$

where

$$\mathbf{b}_0 = \begin{bmatrix} \phi k_0 \\ \phi k_1 \\ \vdots \\ \phi k_m \\ \phi k_{m+1} \end{bmatrix}$$

Thus, the solution for $\Pi_{0,t}$ is given by

$$\begin{aligned}
\Pi_{0,t} &= \left.\frac{\partial}{\partial\phi}\Gamma_{0,t}\right|_{\phi=0} \\
&= \left\{\Gamma_{0,t}\left[\frac{\partial A^*(\theta;\tau,\mathbf{b}_1,\mathbf{d}^*)}{\partial\phi}r_t + \sum_{i=1}^{m}\frac{\partial B_i^*(\theta;\tau,\mathbf{b}_1,\mathbf{d}^*)}{\partial\phi}x_{i,t} + \frac{\partial C^*(\theta;\tau,\mathbf{b}_1,\mathbf{d}^*)}{\partial\phi}\right]\right\}_{\phi=0}
\end{aligned}$$

A.1.2 Solution for $\Pi_{1,t}$ with linear payoffs

For the linear payoff function in (13), the probability $\Pi_{1,t}$ is given by the expression

$$\begin{aligned}
\Pi_{1,t} &= E_t\left[\frac{e^{-Z_t(T)}f_T(\mathbf{r},\mathbf{x},\hat{\tau})1_{\{f_T(\mathbf{r},\mathbf{x},\hat{\tau})\geq K\}}}{E_t[e^{-Z_t}f_T(\mathbf{r},\mathbf{x},\hat{\tau})]}\right] \\
&= \frac{1}{\Pi_{0,t}}E_t\left[e^{-Z_t(T)}f_T(\mathbf{r},\mathbf{x},\hat{\tau})1_{\{f_T(\mathbf{r},\mathbf{x},\hat{\tau})\geq K\}}\right]
\end{aligned}$$

Using the Feynman-Kac relation, one can show that the probability satisfies a PDDE very similar to the bond price equation but with a discontinuous boundary condition. However, the discontinuity of the boundary condition makes this an extremely difficult equation to solve. Instead, we will first solve for the characteristic function, denoted $\tilde{\Pi}_{1,t}$, associated with this probability and then invert this characteristic function to obtain the probability. The characteristic function is defined as

$$\tilde{\Pi}_{1,t} = \frac{1}{\Pi_{0,t}}E_t\left[e^{-Z_t(T)}f_T(\mathbf{r},\mathbf{x},\hat{\tau})e^{i\omega f_T(\mathbf{r},\mathbf{x},\hat{\tau})}\right] \tag{35}$$

where $i = \sqrt{-1}$ and ω is a real-valued dummy variable. We can rewrite the characteristic function as follows:

$$\begin{aligned}
\tilde{\Pi}_{1,t} &= \frac{1}{\Pi_{0,t}}\frac{1}{i}\frac{\partial}{\partial\omega}E_t\left[e^{-Z_t}e^{i\omega f_T(\mathbf{r},\mathbf{x},\hat{\tau})}\right] \\
&= \frac{1}{\Pi_{0,t}}\frac{1}{i}\frac{\partial}{\partial\omega}\Gamma_{1,t}
\end{aligned} \tag{36}$$

where $\Gamma_{1,t} \equiv E_t\left[e^{-Z_t}e^{i\omega f_T(\mathbf{r},\mathbf{x},\hat{\tau})}\right]$. Note that $\Gamma_{1,t}$ is equivalent to $\Gamma_{0,t}$ evaluated at $\phi = i\omega$. Therefore, from the solution for $\Gamma_{0,t}$ in (34), we have the solution for $\Gamma_{1,t}$:

$$\Gamma_{1,t} = \exp\left[A^*(\theta;\tau,\mathbf{b}_1,\mathbf{d}^*)r_t + \sum_{i=1}^{m}B_i^*(\theta;\tau,\mathbf{b}_1,\mathbf{d}^*)x_{i,t} + C^*(\theta;\tau,\mathbf{b}_1,\mathbf{d}^*)\right] \tag{37}$$

where

$$\mathbf{b}_1 = \begin{bmatrix} i\omega k_0 \\ i\omega k_1 \\ \vdots \\ i\omega k_{m+1} \end{bmatrix}$$

Then, we have the solution for $\tilde{\Pi}_{1,t}$:

$$\tilde{\Pi}_{1,t} = \frac{1}{\Pi_{0,t}} \left\{ \frac{1}{i} \Gamma_{1,t} \left[\frac{\partial A^*(\theta; \tau, \mathbf{b}_1, \mathbf{d}^*)}{\partial \omega} r_t + \sum_{i=1}^{m} \frac{\partial B_i^*(\theta; \tau, \mathbf{b}_1, \mathbf{d}^*)}{\partial \omega} x_{i,t} + \frac{\partial C^*(\theta; \tau, \mathbf{b}_1, \mathbf{d}^*)}{\partial \omega} \right] \right\} \tag{38}$$

A.1.3 Solution for $\Pi_{2,t}$ with linear payoffs

For the linear payoff function in (13), the probability $\Pi_{2,t}$ is given by the expression

$$\begin{aligned} \Pi_{2,t} &= E_t \left[\frac{e^{-Z_t(T)} 1_{\{f_T(\mathbf{r},\mathbf{x},\hat{\tau}) \geq K\}}}{E_t\left[e^{-Z_t}\right]} \right] \\ &= \frac{1}{P_t(\tau)} E_t \left[e^{-Z_t(T)} 1_{\{f_T(\mathbf{r},\mathbf{x},\hat{\tau}) \geq K\}} \right] \end{aligned}$$

It is easy to show that $\Pi_{2,t}$ satisfies a PDDE very similar to the bond price equation, but as with $\Pi_{1,t}$ above, this PDDE has a discontinuity in its boundary condition, which makes the equation extremely difficult to solve. Therefore, just as we solved for $\Pi_{1,t}$, we will solve for $\tilde{\Pi}_{2,t}$ by first calculating the characteristic function, $\Pi_{2,t}$, associated with $\Pi_{2,t}$ and then inverting this characteristic function to obtain $\Pi_{2,t}$. The characteristic function is defined as

$$\tilde{\Pi}_{2,t} = \frac{1}{P_t(\tau)} E_t \left[e^{-Z_t(T)} e^{i\omega f_T(\mathbf{r},\mathbf{x},\hat{\tau})} \right]$$

However, $E_t \left[e^{-Z_t(T)} e^{i\omega f_T(\mathbf{r},\mathbf{x},\hat{\tau})} \right]$ was calculated above in the derivation for $\Pi_{1,t}$.

$$E_t \left[e^{-Z_t(T)} e^{i\omega f_T(\mathbf{r},\mathbf{x},\hat{\tau})} \right] = \Gamma_{1,t}$$

Thus, we have the result

$$\begin{aligned} \tilde{\Pi}_{2,t} &= \frac{1}{P_t(\tau)} \Gamma_{1,t} \\ &= \frac{1}{P_t(\tau)} \exp \left[A^*(\theta; \tau, \mathbf{b}_1, \mathbf{d}^*) r_t + \sum_{i=1}^{m} B_i^*(\theta; \tau, \mathbf{b}_1, \mathbf{d}^*) x_{i,t} + C^*(\theta; \tau, \mathbf{b}_1, \mathbf{d}^*) \right] \end{aligned}$$

where

$$\mathbf{b}_1 = \begin{bmatrix} i\omega k_0 \\ i\omega k_1 \\ \vdots \\ i\omega k_m \\ i\omega k_{m+1} \end{bmatrix}$$

A.2 Exponential Linear Payoffs

A.2.1 Solution for $\Pi_{0,t}$ with exponential-linear payoffs

¿From (12),

$$\begin{aligned}\Pi_{0,t} &= E_t[e^{-Z_t(T)} f_T(\mathbf{r}, \mathbf{x}, \hat{\tau})] \\ &= E_t[e^{-Z_t(T)} \exp(k_0 r_T + k_1 x_{1,T} + \ldots + k_m x_{m,T} + k_{m+1,T})]\end{aligned}$$

We now apply the Feynman-Kac relation, from which we know that this equation solves the PDDE

$$0 = \mathcal{D}\Pi_{0,t} - r_t \Pi_{0,t} \tag{39}$$

with boundary condition

$$\Pi_{0,T} = \exp f_T(\mathbf{r}, \mathbf{x}, \hat{\tau}) = \exp[k_0 r_T + k_1 x_{1,T} + \ldots + k_m x_{m,T} + k_{m+1,T}] \tag{40}$$

Notice that this is the exact same PDDE and boundary condition as (32) and (33) with $\phi = 1$. Therefore, we can write the solution for $\Pi_{0,t}$ as

$$\Pi_{0,t} = \exp\left[A^*(\theta; \tau, \mathbf{b}_0, \mathbf{d}^*) r_t + \sum_{i=1}^{m} B_i^*(\theta; \tau, \mathbf{b}_0, \mathbf{d}^*) x_{i,t} + C^*(\theta; \tau, \mathbf{b}_0, \mathbf{d}^*)\right] \tag{41}$$

where

$$\mathbf{b}_0 = \begin{bmatrix} k_0 \\ k_1 \\ \vdots \\ k_m \\ k_{m+1} \end{bmatrix}$$

A.2.2 Solution for $\Pi_{1,t}$ with exponential linear payoffs

For the exponential linear payoff function in (18), the probability $\Pi_{1,t}$ is given by the expression

$$\begin{aligned}\Pi_{1,t} &= E_t\left[\frac{e^{-Z_t(T)} f_T(\mathbf{r}, \mathbf{x}, \hat{\tau}) 1_{\{f_T(\mathbf{r},\mathbf{x},\hat{\tau}) \geq K\}}}{E_t[e^{-Z_t} f_T(\mathbf{r}, \mathbf{x}, \hat{\tau})]}\right] \\ &= \frac{1}{\Pi_{0,t}} E_t\left[e^{-Z_t(T)} f_T(\mathbf{r}, \mathbf{x}, \hat{\tau}) 1_{\{f_T(\mathbf{r},\mathbf{x},\hat{\tau}) \geq K\}}\right] \\ &= \frac{1}{\Pi_{0,t}} E_t\left[e^{-Z_t(T)} f_T(\mathbf{r}, \mathbf{x}, \hat{\tau}) 1_{\{\log f_T(\mathbf{r},\mathbf{x},\hat{\tau}) \geq \log K\}}\right]\end{aligned}$$

As with the linear payoff function, $\Pi_{1,t}$ satisfies a PDDE similar to the bond price equation but with a discontinuous boundary condition. As we did with $\Pi_{1,t}$ with the linear payoff function, we will first solve for the characteristic function, denoted $\tilde{\Pi}_{1,t}$, associated with this probability and then invert this characteristic function to obtain the probability. The characteristic function is defined as

$$\begin{aligned}\tilde{\Pi}_{1,t} &= \frac{1}{\Pi_{0,t}} E_t\left[e^{-Z_t(T)} f_T(\mathbf{r},\mathbf{x},\hat{\tau}) e^{i\omega \log f_T(\mathbf{r},\mathbf{x},\hat{\tau})}\right] \\ &= \frac{1}{\Pi_{0,t}} E_t\left[e^{-Z_t(T)} \exp\left[(1+i\omega)(k_0 r_T + k_1 x_{1,T} + \ldots + k_m x_{m,T})\right]\right] \qquad (42)\end{aligned}$$

¿From the Feynman-Kac relation, it is easy to see that $\Pi_{0,t}\tilde{\Pi}_{1,t}$ satisfies

$$0 = \mathcal{D}\Pi_{0,t}\tilde{\Pi}_{1,t} - r_t \Pi_{0,t}\tilde{\Pi}_{1,t} \qquad (43)$$

with boundary condition

$$\Pi_{0,T}\tilde{\Pi}_{1,T} = \exp\left[(1+i\omega)(k_0 r_T + k_1 x_{1,T} + \ldots + k_m x_{m,T} + k_{m+1,T})\right] \qquad (44)$$

This is the same PDDE and boundary condition as (32) and (33) with $\phi = 1 + i\omega$. Therefore, we can write the solution for $\tilde{\Pi}_{1,t}$ as

$$\tilde{\Pi}_{1,t} = \frac{1}{\Pi_{0,t}} \exp\left[A^*(\theta;\tau,\mathbf{b}_0,\mathbf{d}^*) r_t + \sum_{i=1}^{m} B_i^*(\theta;\tau,\mathbf{b}_0,\mathbf{d}^*) x_{i,t} + C^*(\theta;\tau,\mathbf{b}_0,\mathbf{d}^*)\right] \qquad (45)$$

where

$$\mathbf{b}_0 = \begin{bmatrix} (1+i\omega)k_0 \\ (1+i\omega)k_1 \\ \vdots \\ (1+i\omega)k_m \\ (1+i\omega)k_{m+1} \end{bmatrix}$$

A.2.3 Solution for $\Pi_{2,t}$ with exponential-linear payoffs

For the exponential-linear payoff function in (13), the probability $\Pi_{2,t}$ is given by the expression

$$\begin{aligned}\Pi_{2,t} &= E_t\left[\frac{e^{-Z_t(T)} 1_{\{f_T(\mathbf{r},\mathbf{x},\hat{\tau}) \geq K\}}}{E_t\left[e^{-Z_t(T)}\right]}\right] \\ &= \frac{1}{P_t(\tau)} E_t\left[e^{-Z_t(T)} 1_{\{\log f_T(\mathbf{r},\mathbf{x},\hat{\tau}) \geq \log K\}}\right]\end{aligned}$$

It is easy to show that $\Pi_{2,t}$ satisfies a PDDE very similar to the bond price equation, but as with $\Pi_{1,t}$ above, this PDDE has a discontinuity in its boundary condition, which makes the equation extremely difficult to solve. Therefore, just as we solved for $\Pi_{1,t}$, we will solve for $\tilde{\Pi}_{2,t}$ by first calculating the characteristic function, $\Pi_{2,t}$, associated with $\Pi_{2,t}$ and then inverting this characteristic function to obtain $\Pi_{2,t}$. The characteristic function is defined as

$$\begin{aligned}\tilde{\Pi}_{2,t} &= \frac{1}{P_t(\tau)} E_t\left[e^{-Z_t(T)} e^{i\omega \log f_T(\mathbf{r},\mathbf{x},\hat{\tau})}\right] \\ &= \frac{1}{P_t(\tau)} E_t\left[e^{-Z_t(T)} \exp[i\omega(k_0 r_T + k_1 x_{1,T} + \ldots + k_m x_{m,T} + k_{m+1,T})]\right]\end{aligned}$$

From the Feynman-Kac relation, it is easy to see that $P_t(\tau)\tilde{\Pi}_{1,t}$ satisfies

$$0 = \mathcal{D}P_t(\tau)\tilde{\Pi}_{2,t} - r_t P_t(\tau)\tilde{\Pi}_{2,t} \tag{46}$$

with boundary condition

$$P_T(0)\tilde{\Pi}_{2,T} = \exp\left[i\omega(k_0 r_T + k_1 x_{1,T} + \ldots + k_m x_{m,T} + k_{m+1,T})\right] \tag{47}$$

This is the same PDDE and boundary condition as (32) and (33) with $\phi = i\omega$. Therefore, we can write the solution for $\tilde{\Pi}_{2,t}$ as

$$\tilde{\Pi}_{2,t} = \frac{1}{P_t(\tau)} \exp\left[A^*(\theta;\tau,\mathbf{b}_0,\mathbf{d}^*)r_t + \sum_{i=1}^{m} B_i^*(\theta;\tau,\mathbf{b}_0,\mathbf{d}^*)x_{i,t} + C^*(\theta;\tau,\mathbf{b}_0,\mathbf{d}^*)\right] \tag{48}$$

where

$$\mathbf{b}_0 = \begin{bmatrix} i\omega k_0 \\ i\omega k_1 \\ \vdots \\ i\omega k_m \\ i\omega k_{m+1} \end{bmatrix}$$

A.3 Integro-Linear Payoffs

A.3.1 Solution for $\Pi_{0,t}$ with integro-linear payoffs

For the integro-linear payoff in (25), $\Pi_{0,t}$ is defined as

$$\Pi_{0,t} = E_t[e^{-Z_t(T)} f_T(\mathbf{r},\mathbf{x},\hat{\tau})] \tag{49}$$

$$\quad = E_t[e^{-Z_t(T)} X_T] \tag{50}$$

where the new variable $X_T = \int_t^T (k_0 r_v + k_1 x_{1,v} + \ldots + k_m x_{m,v} + k_{m+1})dv$ represents an expansion of the state space. Using the Feynman-Kac relation, $\Pi_{0,t}$ satisfies the PDDE

$$0 = \mathcal{D}\Pi_{0,t} - r_t \Pi_{0,t} + (k_0 r_t + k_1 x_{1,t} + \ldots + k_m x_{m,t} + k_{m+1}) \frac{\partial \Pi_{0,t}}{\partial X_t}$$

with the boundary condition $\Pi_{0,T} = X_T$. To solve this PDDE we make the following observation:

$$\begin{aligned} E_t\left[e^{-Z_t(T)}X_T\right] &= \frac{\partial}{\partial\phi}E_t\left[e^{-Z_t(T)+\phi X_T}\right]\Big|_{\phi=0} \\ &= \frac{\partial\Phi_t}{\partial\phi}\Big|_{\phi=0} \end{aligned} \tag{51}$$

where $\Phi_t = E_t\left[e^{-Z_t(T)+\phi X_T}\right]$ and ϕ is an arbitrary constant.[21] Using the Feynman-Kac formula, Φ_t satisfies the following PDDE:

$$0 = \mathcal{D}\Phi_t + [(\phi k_0 - 1)r_t + \phi k_1 x_{1,v} + \ldots + \phi k_m x_{m,v} + \phi k_{m+1}]\Phi_t$$

The boundary condition for this equation is $\Phi_T = 1$. Since this is the same PDDE and boundary condition as (3) and 6 but with $d = \mathbf{d}_0$, where

$$\mathbf{d}_0' = \begin{bmatrix} \phi k_0 - 1 \\ \phi k_1 \\ \vdots \\ \phi k_m \\ \phi k_{m+1} \end{bmatrix}$$

we can immediately calculate the solution for Φ_t:

$$\Phi_t = \exp\left[A^*(\theta;\tau,\mathbf{0},\mathbf{d}_0)r_t + \sum_{i=1}^{m} B_i^*(\theta;\tau,\mathbf{0},\mathbf{d}_0)x_{i,t} + C^*(\theta;\tau,\mathbf{0},\mathbf{d}_0)\right] \tag{52}$$

¿From (51) we then have the solution for $\Pi_{0,t}$.

$$\Pi_{0,t} = \left\{\Phi_t \times \left[\frac{\partial A^*(\theta;\tau,\mathbf{0},\mathbf{d}_0)}{\partial\phi}r_t + \sum_{i=1}^{m}\frac{\partial B_i^*(\theta;\tau,\mathbf{0},\mathbf{d}_0)}{\partial\phi}x_{i,t} + \frac{\partial C^*(\theta;\tau,\mathbf{0},\mathbf{d}_0)}{\partial\phi}\right]\right\}_{\phi=0} \tag{53}$$

A.3.2 Solution for $\Pi_{1,t}$ with integro-linear payoffs

For the integro-linear payoff, $\Pi_{1,t}$ in (26) is defined as

$$\begin{aligned} \Pi_{1,t} &= E_t\left[\frac{e^{-Z_t(T)}Y_t(T)1_{\{Y_t(T)\geq K\}}}{E_t[e^{-Z_t(T)}Y_t(T)]}\right] & (54) \\ &= \frac{1}{\Pi_{0,t}}E_t\left[e^{-Z_t(T)}Y_t(T)1_{\{Y_t(T)\geq K\}}\right] & (55) \end{aligned}$$

[21]Given that $e^{-\phi Y_t - Z_t}$ is bounded in time and the interest rate process $(r,\mathbf{x})$ is strong Markov, the dominated convergence theorem holds. Therefore, the application of Fubini's theorem is permitted here.

Using the Feynman-Kac relation, $\Pi_{1,t}$ satisfies a PDDE similar to the bond price equation but with a discontinuous boundary condition. Therefore we will first solve for the characteristic function, denoted $\tilde{\Pi}_{1,t}$, associated with this probability and then invert this characteristic function to obtain the probability. The characteristic function is defined as

$$\tilde{\Pi}_{1,t} = \frac{1}{\Pi_{0,t}} E_t\left[e^{-Z_t(T)} Y_t(T) e^{i\omega Y_t(T)}\right] \tag{56}$$

Notice, however, that $E_t\left[e^{-Z_t(T)+\phi Y_t(T)}\right]$ was calculated already in the derivation for $\Pi_{0,t}$ for (26) above. In this derivation $E_t\left[e^{-Z_t(T)+\phi Y_t(T)}\right]$ was defined as Φ_t and solved in (52). Therefore, we can write the solution to $\tilde{\Pi}_{1,t}$ as

$$\begin{aligned} \tilde{\Pi}_{1,t} &= \left.\frac{1}{\Pi_{0,t}} \frac{\partial \Phi}{\partial \phi}\right|_{\phi=i\omega} \qquad (57) \\ &= \frac{1}{\Pi_{0,t}} \left\{ \Phi_t \times \left[\frac{\partial A^*(\theta;\tau,\mathbf{0},\mathbf{d}_0)}{\partial \phi} r_t + \sum_{i=1}^{m} \frac{\partial B_i^*(\theta;\tau,\mathbf{0},\mathbf{d}_0)}{\partial \phi} x_{i,t} + \frac{\partial C^*(\theta;\tau,\mathbf{0},\mathbf{d}_0)}{\partial \phi} \right] \right\}_{\phi=i\omega} \end{aligned}$$

A.3.3 Solution for $\Pi_{2,t}$ with integro-linear payoffs

For the integro-linear payoff function in (25), the probability $\Pi_{2,t}$ is given by the expression

$$\begin{aligned} \Pi_{2,t} &= E_t\left[\frac{e^{-Z_t(T)} 1_{\{f_T(\mathbf{r},\mathbf{x},\hat{\tau}) \geq K\}}}{E_t\left[e^{-Z_t(T)}\right]}\right] \\ &= \frac{1}{P_t(\tau)} E_t\left[e^{-Z_t(T)} 1_{\{Y_t(T) \geq K\}}\right] \end{aligned}$$

It is easy to show that $\Pi_{2,t}$ satisfies a PDDE very similar to the bond price equation, but as with $\Pi_{1,t}$ above, this PDDE has a discontinuity in its boundary condition, which makes the equation extremely difficult to solve. Therefore, just as we solved for $\Pi_{1,t}$, we will solve for $\tilde{\Pi}_{2,t}$ by first calculating the characteristic function, $\Pi_{2,t}$, associated with $\Pi_{2,t}$ and then inverting this characteristic function to obtain $\Pi_{2,t}$. The characteristic function is defined as

$$\begin{aligned} \tilde{\Pi}_{2,t} &= \frac{1}{P_t(\tau)} E_t\left[e^{-Z_t(T)} e^{i\omega Y_t(T)}\right] \\ &= \frac{1}{P_t(\tau)} E_t\left[e^{-Z_t(T)+i\omega Y_t(T)}\right] \end{aligned}$$

However, $E_t\left[e^{-Z_t(T)+i\omega Y_t(T)}\right] \equiv \Phi_t$, which has already been calculated in (52). Therefore, we have the solution for the characteristic function:

$$\begin{aligned} \tilde{\Pi}_{2,t} &= \left.\frac{\Phi_t}{P_t(\tau)}\right|_{\phi=i\omega} \qquad (58) \\ &= \frac{1}{P_t(\tau)} \exp\left[A^*(\theta;\tau,\mathbf{0},\mathbf{d}_0) r_t + \sum_{i=1}^{m} B_i^*(\theta;\tau,\mathbf{0},\mathbf{d}_0) x_{i,t} + C^*(\theta;\tau,\mathbf{0},\mathbf{d}_0) \right]_{\phi=i\omega} \qquad (59) \end{aligned}$$

B Illustrative Callibration: The Vasicek Model

The interest rate process used is one with constant coefficients:

$$dr_t = \alpha[\beta - r_t]dt + \eta dW_t$$

where α is the coefficient of mean reversion, β is the long-run mean of the interest rate, and η is the volatility coefficient for the driving Wiener process dW_t. The estimation problem is a particular version of the system (28):

$$\begin{aligned}
& \min_{\theta} \sum_{t=1}^{T} \sum_{k=1}^{N} \varepsilon_t[\theta(\tau_k)]^2 \\
\text{subject to} \quad & : \quad \theta = \{\alpha, \beta, \eta\} \\
\varepsilon_t[\theta(\tau_k)] & = Y(\tau_k) + \frac{1}{\tau}[A(\tau_k) r_t + C(\tau_k)] \\
\frac{\partial A}{\partial \tau} & = -\alpha A - 1, \quad \forall \tau_k, A(0) = 0 \\
\frac{\partial C}{\partial \tau} & = \frac{1}{2}\eta^2 A^2 + \alpha\beta A, \quad \forall \tau_k, C(0) = 0
\end{aligned}$$

We employed a panel of monthly data from the well-known McCulloch-Kwon database. This data has zero-coupon yields for several maturities from 1 month to 20 years. We used the period 8/1985-2/1991, because the database for this period is constructed from non-callable bonds, and is not confounded with options effects. The estimation exercise took a few seconds and resulted in the following parameter estimates: $[\alpha = 1.6074, \beta = 0.0874, \eta = 0.0408]$. The estimated parameters may be used directly in the pricing of interest-rate derivatives.

Table 1: **Comparison of Numerical Option values versus Analytical solutions** This table presents call options values for a range of bond ($\tau + \hat{\tau}$) and option (τ) maturities. In each case, five strike prices are chosen centered on the at the money forward price. The two values on each side of the middle strike price are $\pm 5\%$ and $\pm 10\%$ away from the money. Parameters estimated in the previous section are used. The data was drawn from the McCulloch-Kwon dataset of monthly zero coupon yields for the period 8/1985-2/1991.

Option Maturity τ	Bond Maturity $\tau + \hat{\tau}$	Exercise Price K	Call Option Price Vasicek $F(\tau;\hat{\tau})_0$	Call Option Price Numerical $F(\tau;\hat{\tau})$
0.25	0.75	0.9288	0.0382	0.0380
0.25	0.75	0.9482	0.0191	0.0191
0.25	0.75	0.9675	0.0022	0.0022
0.25	0.75	0.9869	0.0000	0.0000
0.25	0.75	1.0062	0.0000	0.0000
0.25	1.25	0.8953	0.0368	0.0367
0.25	1.25	0.9139	0.0184	0.0184
0.25	1.25	0.9326	0.0030	0.0030
0.25	1.25	0.9512	0.0000	0.0000
0.25	1.25	0.9699	0.0000	0.0000
0.25	1.75	0.8615	0.0354	0.0353
0.25	1.75	0.8794	0.0177	0.0177
0.25	1.75	0.8974	0.0033	0.0033
0.25	1.75	0.9153	0.0001	0.0000
0.25	1.75	0.9333	0.0000	0.0000

Table 2: **Comparison of Numerical Option values versus Analytical solutions**
This table presents call options values for a range of bond $(\tau + \hat{\tau})$ and option (τ) maturities. In each case, five strike prices are chosen centered on the at the money forward price. The two values on each side of the middle strike price are ±5% and ±10% away from the money. Parameters estimated in the previous section are used. The data was drawn from the McCulloch-Kwon dataset of monthly zero coupon yields for the period 8/1985-2/1991.

Option Maturity τ	Bond Maturity $\tau + \hat{\tau}$	Exercise Price K	Call Option Price Vasicek $F(\tau;\hat{\tau})_0$	Call Option Price Numerical $F(\tau;\hat{\tau})$
1.00	1.50	0.9244	0.0361	0.0360
1.00	1.50	0.9436	0.0181	0.0180
1.00	1.50	0.9629	0.0027	0.0027
1.00	1.50	0.9822	0.0000	0.0000
1.00	1.50	1.0014	0.0000	0.0000
1.00	2.00	0.8891	0.0347	0.0346
1.00	2.00	0.9076	0.0175	0.0174
1.00	2.00	0.9261	0.0038	0.0038
1.00	2.00	0.9447	0.0001	0.0001
1.00	2.00	0.9632	0.0000	0.0000
1.00	2.50	0.8547	0.0334	0.0333
1.00	2.50	0.8725	0.0169	0.0169
1.00	2.50	0.8903	0.0041	0.0041
1.00	2.50	0.9081	0.0002	0.0002
1.00	2.50	0.9259	0.0000	0.0000

Table 3: **Options Prices for the square-root model** ($\eta = 0.20$)
This table presents the values of four options: (i) Asian binary cap, (ii) Regular binary cap, (iii) Asian cap and (iv) Regular cap. The parameters that are varied are: (a) exercise price (K), (b) time to maturity (T). The base parameters used are: volatility ($\eta = 0.2$), initial interest rate ($r_0 = 0.1$), mean reversion ($k = 1.5$), mean rate ($\theta = 0.1$), market price of risk ($\xi = 0$).

Maturity	Option Type	$K = 0.08$	$K = 0.09$	$K = 0.10$	$K = 0.11$	$K = 0.12$
$T = 0.10$	Asian Binary cap	0.9303	0.7426	0.4914	0.2420	0.0585
$T = 0.10$	Regular Binary cap	0.8528	0.6852	0.4753	0.2804	0.1403
$T = 0.10$	Asian cap	0.2093	0.1596	0.1007	0.0471	0.0108
$T = 0.10$	Regular cap	0.1918	0.1473	0.0974	0.0546	0.0259
$T = 0.50$	Asian Binary cap	0.8018	0.6377	0.4452	0.2718	0.1458
$T = 0.50$	Regular Binary cap	0.6759	0.5563	0.4360	0.3261	0.2333
$T = 0.50$	Asian cap	0.1804	0.1371	0.0912	0.0530	0.0269
$T = 0.50$	Regular cap	0.1520	0.1196	0.0893	0.0635	0.0431
$T = 1.00$	Asian Binary cap	0.7300	0.5779	0.4124	0.2661	0.1565
$T = 1.00$	Regular Binary cap	0.6162	0.5097	0.4055	0.3112	0.2311
$T = 1.00$	Asian cap	0.1642	0.1242	0.0845	0.0519	0.0289
$T = 1.00$	Regular cap	0.1386	0.1095	0.0831	0.0607	0.0427
$T = 2.00$	Asian Binary cap	0.6643	0.5193	0.3633	0.2291	0.1316
$T = 2.00$	Regular Binary cap	0.5507	0.4553	0.3630	0.2797	0.2090
$T = 2.00$	Asian cap	0.1494	0.1116	0.0744	0.0446	0.0243
$T = 2.00$	Regular cap	0.1239	0.0979	0.0744	0.0545	0.0386

Table 4: **Options Prices for the square-root model ($\eta = 0.30$)**
This table presents the values of four options: (i) Asian binary cap, (ii) Regular binary cap, (iii) Asian cap and (iv) Regular cap. The parameters that are varied are: (a) exercise price (K), (b) time to maturity (T). The base parameters used are: volatility ($\eta = 0.3$), initial interest rate ($r_0 = 0.1$), mean reversion ($k = 1.5$), mean rate ($\theta = 0.1$), market price of risk ($\xi = 0$).

Maturity	Option Type	$K = 0.08$	$K = 0.09$	$K = 0.10$	$K = 0.11$	$K = 0.12$
$T = 0.10$	Asian Binary cap	0.8662	0.6950	0.4832	0.2769	0.1189
$T = 0.10$	Regular Binary cap	0.7444	0.6087	0.4655	0.3328	0.2229
$T = 0.10$	Asian cap	0.1948	0.1494	0.0990	0.0540	0.0220
$T = 0.10$	Regular cap	0.1675	0.1308	0.0954	0.0649	0.0412
$T = 0.50$	Asian Binary cap	0.6907	0.5605	0.4299	0.3127	0.2165
$T = 0.50$	Regular Binary cap	0.5800	0.4958	0.4159	0.3431	0.2786
$T = 0.50$	Asian cap	0.1554	0.1205	0.0881	0.0609	0.0400
$T = 0.50$	Regular cap	0.1305	0.1065	0.0852	0.0669	0.0515
$T = 1.00$	Asian Binary cap	0.6204	0.5039	0.3924	0.2942	0.2132
$T = 1.00$	Regular Binary cap	0.5251	0.4510	0.3820	0.3197	0.2646
$T = 1.00$	Asian cap	0.1395	0.1083	0.0804	0.0573	0.0394
$T = 1.00$	Regular cap	0.1181	0.0969	0.0783	0.0623	0.0489
$T = 2.00$	Asian Binary cap	0.5594	0.4464	0.3408	0.2503	0.1780
$T = 2.00$	Regular Binary cap	0.4677	0.4015	0.3402	0.2851	0.2365
$T = 2.00$	Asian cap	0.1258	0.0959	0.0698	0.0488	0.0329
$T = 2.00$	Regular cap	0.1052	0.0863	0.0697	0.0555	0.0437

Table 5: **Options Prices for the square-root model** ($\eta = 0.20$, $\theta = 0.05$)
This table presents the values of four options: (i) Asian binary cap, (ii) Regular binary cap, (iii) Asian cap and (iv) Regular cap. The parameters that are varied are: (a) exercise price (K), (b) time to maturity (T). The base parameters used are: volatility ($\eta = 0.2$), initial interest rate ($r_0 = 0.1$), mean reversion ($k = 1.5$), mean rate ($\theta = 0.05$), market price of risk ($\xi = 0$).

Maturity	Option Type	$K = 0.08$	$K = 0.09$	$K = 0.10$	$K = 0.11$	$K = 0.12$
$T = 0.10$	Asian Binary cap	0.8750	0.6580	0.3983	0.1655	0.0151
$T = 0.10$	Regular Binary cap	0.7474	0.5409	0.3321	0.1720	0.0755
$T = 0.10$	Asian cap	0.1968	0.1414	0.0816	0.0322	0.0028
$T = 0.10$	Regular cap	0.1681	0.1163	0.0680	0.0335	0.0139
$T = 0.50$	Asian Binary cap	0.5523	0.3537	0.1966	0.0958	0.0414
$T = 0.50$	Regular Binary cap	0.3545	0.2484	0.1661	0.1066	0.0658
$T = 0.50$	Asian cap	0.1242	0.0760	0.0403	0.0186	0.0076
$T = 0.50$	Regular cap	0.0797	0.0534	0.0340	0.0207	0.0121
$T = 1.00$	Asian Binary cap	0.3474	0.2049	0.1097	0.0539	0.0245
$T = 1.00$	Regular Binary cap	0.2108	0.1419	0.0927	0.0590	0.0367
$T = 1.00$	Asian cap	0.0781	0.0440	0.0224	0.0105	0.0045
$T = 1.00$	Regular cap	0.0474	0.0305	0.0190	0.0115	0.0067
$T = 2.00$	Asian Binary cap	0.1692	0.0842	0.0386	0.0166	0.0067
$T = 2.00$	Regular Binary cap	0.1253	0.0794	0.0491	0.0298	0.0177
$T = 2.00$	Asian cap	0.0380	0.0181	0.0079	0.0032	0.0012
$T = 2.00$	Regular cap	0.0282	0.0170	0.0100	0.0058	0.0032

Table 6: **Options Prices for the square-root model** ($\eta = 0.20$, $\theta = 0.15$)
This table presents the values of four options: (i) Asian binary cap, (ii) Regular binary cap, (iii) Asian cap and (iv) Regular cap. The parameters that are varied are: (a) exercise price (K), (b) time to maturity (T). The base parameters used are: volatility ($\eta = 0.2$), initial interest rate ($r_0 = 0.1$), mean reversion ($k = 1.5$), mean rate ($\theta = 0.15$), market price of risk ($\xi = 0$).

Maturity	Option Type	$K = 0.08$	$K = 0.09$	$K = 0.10$	$K = 0.11$	$K = 0.12$
$T = 0.10$	Asian Binary cap	0.9728	0.8183	0.5841	0.3267	0.1145
$T = 0.10$	Regular Binary cap	0.9214	0.8041	0.6211	0.4136	0.2350
$T = 0.10$	Asian cap	0.2188	0.1759	0.1197	0.0637	0.0211
$T = 0.10$	Regular cap	0.2073	0.1728	0.1273	0.0806	0.0434
$T = 0.50$	Asian Binary cap	0.9129	0.8410	0.7082	0.5321	0.3531
$T = 0.50$	Regular Binary cap	0.8709	0.8066	0.7198	0.6166	0.5064
$T = 0.50$	Asian cap	0.2054	0.1808	0.1451	0.1037	0.0653
$T = 0.50$	Regular cap	0.1959	0.1734	0.1475	0.1202	0.0936
$T = 1.00$	Asian Binary cap	0.8672	0.8255	0.7406	0.6140	0.4660
$T = 1.00$	Regular Binary cap	0.8346	0.7911	0.7303	0.6547	0.5692
$T = 1.00$	Asian cap	0.1951	0.1774	0.1518	0.1197	0.0862
$T = 1.00$	Regular cap	0.1877	0.1700	0.1497	0.1276	0.1053
$T = 2.00$	Asian Binary cap	0.7627	0.7496	0.7107	0.6340	0.5224
$T = 2.00$	Regular Binary cap	0.7341	0.7045	0.6618	0.6065	0.5413
$T = 2.00$	Asian cap	0.1716	0.1611	0.1457	0.1236	0.0966
$T = 2.00$	Regular cap	0.1651	0.1514	0.1356	0.1182	0.1001

Table 7: **Options Prices for the O-U model** ($\eta = 0.063246$)
This table presents the values of four options: (i) Asian binary cap, (ii) Regular binary cap, (iii) Asian cap and (iv) Regular cap. The parameters that are varied are: (a) exercise price (K), (b) time to maturity (T). The base parameters used are: volatility ($\eta = 0.1$), initial interest rate ($r_0 = 0.1$), mean reversion ($k = 1.5$), mean rate ($\theta = 0.063246$), market price of risk ($\xi = 0$).

Maturity	Option Type	$K = 0.08$	$K = 0.09$	$K = 0.10$	$K = 0.11$	$K = 0.12$
$T = 0.10$	Asian Binary cap	0.9306	0.7449	0.4947	0.2445	0.0591
$T = 0.10$	Regular Binary cap	0.8502	0.6973	0.4946	0.2920	0.1393
$T = 0.10$	Asian cap	0.2093	0.1601	0.1014	0.0476	0.0109
$T = 0.10$	Regular cap	0.1913	0.1499	0.1014	0.0569	0.0257
$T = 0.50$	Asian Binary cap	0.7997	0.6556	0.4718	0.2889	0.1470
$T = 0.50$	Regular Binary cap	0.6947	0.5888	0.4727	0.3568	0.2517
$T = 0.50$	Asian cap	0.1799	0.1409	0.0967	0.0563	0.0271
$T = 0.50$	Regular cap	0.1563	0.1266	0.0969	0.0695	0.0465
$T = 1.00$	Asian Binary cap	0.7318	0.6015	0.4444	0.2889	0.1624
$T = 1.00$	Regular Binary cap	0.6405	0.5474	0.4470	0.3471	0.2552
$T = 1.00$	Asian cap	0.1646	0.1293	0.0911	0.0563	0.0300
$T = 1.00$	Regular cap	0.1441	0.1176	0.0916	0.0677	0.0472
$T = 2.00$	Asian Binary cap	0.6632	0.5419	0.3955	0.2519	0.1375
$T = 2.00$	Regular Binary cap	0.5742	0.4913	0.4025	0.3143	0.2329
$T = 2.00$	Asian cap	0.1492	0.1165	0.0810	0.0491	0.0254
$T = 2.00$	Regular cap	0.1291	0.1056	0.0825	0.0612	0.0430

Table 8: **Options Prices for the O-U model** ($\eta = 0.094868$)
This table presents the values of four options: (i) Asian binary cap, (ii) Regular binary cap, (iii) Asian cap and (iv) Regular cap. The parameters that are varied are: (a) exercise price (K), (b) time to maturity (T). The base parameters used are: volatility ($\eta = 0.1$), initial interest rate ($r_0 = 0.1$), mean reversion ($k = 1.5$), mean rate ($\theta = 0.094868$), market price of risk ($\xi = 0$).

Maturity	Option Type	$K = 0.08$	$K = 0.09$	$K = 0.10$	$K = 0.11$	$K = 0.12$
$T = 0.10$	Asian Binary cap	0.8675	0.7030	0.4944	0.2860	0.1217
$T = 0.10$	Regular Binary cap	0.7553	0.6331	0.4944	0.3558	0.2338
$T = 0.10$	Asian cap	0.1952	0.1511	0.1013	0.0557	0.0225
$T = 0.10$	Regular cap	0.1699	0.1361	0.1013	0.0693	0.0432
$T = 0.50$	Asian Binary cap	0.7080	0.5953	0.4700	0.3452	0.2342
$T = 0.50$	Regular Binary cap	0.6244	0.5494	0.4712	0.3933	0.3188
$T = 0.50$	Asian cap	0.1593	0.1280	0.0963	0.0673	0.0433
$T = 0.50$	Regular cap	0.1405	0.1181	0.0966	0.0766	0.0589
$T = 1.00$	Asian Binary cap	0.6458	0.5472	0.4405	0.3349	0.2392
$T = 1.00$	Regular Binary cap	0.5772	0.5118	0.4445	0.3774	0.3128
$T = 1.00$	Asian cap	0.1453	0.1176	0.0903	0.0653	0.0442
$T = 1.00$	Regular cap	0.1298	0.1100	0.0911	0.0735	0.0578
$T = 2.00$	Asian Binary cap	0.5808	0.4883	0.3888	0.2913	0.2042
$T = 2.00$	Regular Binary cap	0.5171	0.4590	0.3994	0.3401	0.2831
$T = 2.00$	Asian cap	0.1306	0.1050	0.0797	0.0568	0.0377
$T = 2.00$	Regular cap	0.1163	0.0987	0.0818	0.0663	0.0523

Table 9: **Range Asian Option Prices**

This table presents the values of the range Asian option. This option is written over a fixed number of days. Every day the option pays off if the average interest rate up to that day lies within a range (a, b). The payoff is a dollar divided by the number of days the option is written for. The values in this table are for a range asian option with maturity $T = 0.2$ years, i.e. 73 days. The parameters that are varied are: (a) mean reversion (k), (b) lower range limit (a) (c) upper range limit (b). The base parameters used are: initial interest rate $(r_0 = 0.1)$, time to maturity $(T = 0.2)$, mean rate $(\theta = 0.1)$, market price of risk $(\xi = 0)$, volatility $(\eta = 0.2)$.

$\theta = 0.05$

Range	$k = 0.5$	$k = 1.5$	$k = 2.5$
$a = 0.09, b = 0.11$	0.3718	0.3539	0.3075
$a = 0.08, b = 0.12$	0.6631	0.6461	0.5923
$a = 0.07, b = 0.13$	0.8388	0.8353	0.8066

$\theta = 0.10$

Range	$k = 0.5$	$k = 1.5$	$k = 2.5$
$a = 0.09, b = 0.11$	0.3818	0.4120	0.4424
$a = 0.08, b = 0.12$	0.6741	0.7132	0.7488
$a = 0.07, b = 0.13$	0.8435	0.8714	0.8922

$\theta = 0.15$

Range	$k = 0.5$	$k = 1.5$	$k = 2.5$
$a = 0.09, b = 0.11$	0.3823	0.3810	0.3406
$a = 0.08, b = 0.12$	0.6717	0.6646	0.6150
$a = 0.07, b = 0.13$	0.8374	0.8265	0.7938

Table 10: **Jump-diffusion Prices for Asian Options**
This table presents prices of bonds, binary Asian options, and standard Asian options in the jump-diffusion model. Results are preented for a range of skewness levels, i.e. as λ varies from 0 to 10.

λ	Bond Price	Binary Asian	Standard Asian
0	0.7409	0.3660	0.0015
1	0.7228	0.6059	0.0063
2	0.7051	0.6737	0.0116
3	0.6878	0.6802	0.0169
4	0.6710	0.6693	0.0219
5	0.6545	0.6541	0.0268
6	0.6385	0.6384	0.0313
7	0.6229	0.6229	0.0357
8	0.6076	0.6076	0.0398
9	0.5927	0.5927	0.0436
10	0.5782	0.5782	0.0473

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Interest Rate Derivatives: A General Approach 56

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