Path-Dependent Multicurrency Interest Rate Derivatives

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Abstract
This paper derives the arbitrage-free pricing formulae and hedging ratios of path-dependent multicurrency interest rate derivatives in continuous time. The exchange rates and the zero-coupon bond prices are modelled by multivariate log-normal processes with an arbitrary number of random factors. The exact pricing formulae are obtained by solving the time-evolution equation of the contingent claims described by a multi-dimensional Kolmogorov field equation. Applications to the Asian options and Bermudan options are given as examples. Several European options on currency swaps, forward and futures contracts are also illustrated as special cases. These exact solutions are factor-independent and explicitly depend on the instantaneous covariances of the returns of the exchange rates and the zero-coupon bonds. Since these covariances can be obtained directly from the market, the results can be implemented in a non-parametric way and the usual calibration problems in the finite-factor Gaussian Heath-Jarrow-Morton models can be avoided.

1 Introduction

Gaussian interest rate models in the continuous time setting have been investigated by many authors since Merton [1973] extended the Black-Scholes [1973] European option formula to include stochastic interest rates. The main interest in the Gaussian models is due to the fact that analytic pricing and hedging can be obtained explicitly. In recent years, the multi-factor Heath-Jarrow-Morton (HJM) models [1990, 1992] have become popular among practitioners and theorists. Although the spot yield curve can easily be fitted in these models, it is not easy to calibrate the forward-rate volatilities using a small number of random factors. This difficulty is shown in the empirical studies by Flesaker [1993], Amin and Morton [1994], Brace and Musiella [1994], Pang [1996] and Rebonato [1996]. For a general factor-independent HJM model, Kennedy [1994, 1997] suggested to use Gaussian random fields. As in most of the literature, Kennedy used the Martingale approach to evaluate interest rate options. This approach has a firm mathematical basis, as established by Harrison and Kreps [1979] and by Harrison and Pliska [1981]. But pricings and hedgings often rely on simulation methods for more exotic options.

This paper adopts the PDE (partial differential equation) approach to price and hedge the multicurrency interest rate options, assuming that the yield curves are driven by general multi-factor Gaussian processes (the number of factors can be infinite). The results accommodate all types of yield curves and their volatility surfaces that are observed in the markets. Model users can also choose to parametrise the volatilities with any number of factors and easily recover the results of all finite-factor Gaussian HJM models in the existing literature. This method extends recent work (Chu [1996]) in which the time-evolution equation of contingent claims was derived and an exact pricing formula for any single-currency European type interest rate option was obtained. Prices of cap and caption, swap and swaption, call and compound options were all obtained in the form of multivariate Gaussian integrals. This PDE approach is similar in spirit to the work of Kat and Roozen [1994], who assumed constant interest rates in a multicurrency equity market and derived a general pricing formula for any path-independent equity derivatives.

The main motivation for considering an infinite-factor Gaussian interest rate model is to obtain analytic results which are universal to all finite-factor models, so that the model can be implemented in a non-parametric way. Jamshidian [1993, 1994] derived factor-independent results for several path-independent derivatives but the aspect of non-parametric implementation was not emphasized. For integrated portfolio and risk management, this non-parametric feature is useful because we can price and hedge a wide range of products using one set of input data and avoid the confusion caused by pricing different products with different models and hence different implied volatilities. On the other hand, it is natural from the mathematical point of view to consider stochastic processes such as a Brownian sheet or a random field when modelling assets with a maturity dependence (a continuous variable). Infinite-dimensional stochastic processes have been investigated by mathematicians for a long time starting with the pioneering work of Ito [1978, 1983]. As pointed out by Da Prato and Zabczyk [1992], certain Brownian sheet process can be formulated in terms of an infinite number of Brownian motions. Hence, this paper gives an alternative approach to Kennedy's Gaussian random field model.

There are two main results in this paper. The first result is deriving the time-evolution equation of the multi-currency contingent claims and the second is obtaining the general pricing formulae and hedging ratios of path-dependent interest rate options. Path-independent options are treated as special cases of the general path-dependent formula. The time-evolution equation is a multi-dimensional Kolmogorov field equation which is derived by replicating the contingent claim with a self-financing bond portfolio in a risk neutral world. The general pricing formulae are obtained by solving the Kolmogorov equation in the Markov transition approach. These exact solutions are in the form of multivariate Gaussian integrals which can be integrated analytically or by a straightforward numerical method. The required input data are the spot exchange rates, the zero-coupon bond prices and the instantaneous covariances of their returns. Since these covariances can be determined empirically, there is no need to calibrate the volatility parameters for each factor. The usual calibration problem in the finite-factor Gaussian HJM models can be avoided.

This paper is organised as follows. In section 2, we formulate the dynamics of the multicurrency interest rate market and derive the Kolmogorov field equation which is satisfied by the contingent claims in the arbitrage-free market. Section 3 solves for the pricing formula and the hedging ratios for a general path-independent option. Applications to European options on cross-currency swaps and spreads (diff swaps as in Jamshidian [1994]) are given as examples. Section 4 illustrates European options on the forward and futures of the zero-coupon bonds. In particular, FX options are treated as special cases of these interest rate options. These examples are given in order to enable the reader to become familiar with our approach. These results are then extended to path-dependent options in section 5. The idea is to solve the Kolmogorov equation in the Markov transition approach. Examples given in this section are the FX Asian options and Bermudan swap options. These results can in principle be applied to evaluate American type options, although it is likely to be computationally intensive as the number of exercise dates becomes large. We conclude this paper in section 6 and 7 with discussions on the non-parametric implementation of the model and on the limitations due to the model assumptions. The derivation of the general path-dependent pricing formula and several useful identities can be found in the Appendix.

2 Time Evolution Equation of Multicurrency Contingent Claims

In this section, we introduce the market assumptions and derive the mult-dimensional Kolmogorov field equation which describes the time evolution of the contingent claims. Consider an international interest rate market with $(m+1)$ currencies. The market driving factors are assumed to be the exchange rates and the prices of the zero-coupon (discount) bonds under the following three assumptions.

Assumption 1: The exchange rate for currency D_α with respect to the home currency D_0 at any time s is denoted by $X_\alpha(s)$. The dynamics of the exchange rate is described by a multivariate log-normal process written in terms of m correlated Brownian motions $W_\alpha(s)$,

$$\begin{aligned} dX_\alpha(s) &= X_\alpha(s)\Big(\mu_\alpha(s,X)ds + \xi_\alpha(s)dW_\alpha(s)\Big), \quad \alpha = 1,2,\dots,m \\ \mathrm{E}[W_\alpha(s)] &= 0, \qquad \mathrm{COV}[W_\alpha(s), W_\beta(s)] = \rho_{\alpha,\beta}(s)ds. \end{aligned} \tag{2.1}$$

The diffusion coefficients $\xi_\alpha(s)$ are assumed to be deterministic functions of time. For the home currency, $X_0(s) = 1$ and $\mu_0(s) = \xi_0(s) = 0$ at all time s.

Assumption 2: Let $P_\alpha(s,T)$ denote the present price of a zero-coupon bond (i.e. discount factor) which pays one unit of the currency D_α at its maturity $T \in [0,L]$ and L denote the longest maturity. The dynamics of the zero-coupon bonds is modelled by the following infinite-dimensional stochastic process,

$$\begin{aligned} \frac{dP_\alpha(s,T)}{P_\alpha(s,T)} &= M_\alpha(s,T,P)ds + \sum_{k=1}^{\infty} \sigma_\alpha^{(k)}(s,T)dB_s^{(k)}, \\ P_\alpha(T,T) &= 1(D_\alpha), \qquad M_\alpha(T,T,P) = \sigma_\alpha^{(k)}(T,T) = 0. \end{aligned} \tag{2.2}$$

Since it is convenient to evaluate all the cashflows in terms of the domestic currency, let $\hat{P}_\alpha(s,T) \equiv X_\alpha(s)P_\alpha(s,T)$ denote the domestic values of the zero-coupon bonds. The dynamics of $\hat{P}$ is determined by equation (2.1) and (2.2) according to Ito's Lemma,

$$\frac{d\hat{P}_\alpha(s,T)}{\hat{P}_\alpha(s,T)} = \frac{dX_\alpha(s)}{X_\alpha(s)} + \frac{dP_\alpha(s,T)}{P_\alpha(s,T)} + \mathrm{COV}\left[\frac{dX_\alpha(s)}{X_\alpha(s)}, \frac{dP_\alpha(s,T)}{P_\alpha(s,T)}\right] \tag{2.3}$$

and the covariances of these zero-coupon bond returns are

$$Z_{\alpha,\beta}(s,T_1,T_2) \equiv \frac{1}{ds}\mathrm{COV}\left[\frac{d\hat{P}_\alpha(s,T_1)}{\hat{P}_\alpha(s,T_1)}, \frac{d\hat{P}_\beta(s,T_2)}{\hat{P}_\beta(s,T_2)}\right], \tag{2.4}$$

which in terms of the diffusion coefficients are

$$\begin{aligned} Z_{\alpha,\beta}(s,T_1,T_2) &= \xi_\alpha(s)\xi_\beta(s)\rho_{\alpha\beta}(s) + \sum_{k=0}^{\infty}\sigma_\alpha^{(k)}(s,T_1)\sigma_\beta^{(k)}(s,T_2) \\ &\quad + \sum_{k=0}^{\infty}\left(\sigma_\alpha^{(k)}(s,T_1)\xi_\beta(s)\rho_\beta^{(k)} + \sigma_\beta^{(k)}(s,T_2)\xi_\alpha(s)\rho_\alpha^{(k)}\right) \end{aligned}$$

and $\rho_\alpha^{(k)}$ denotes the correlation between dW_α and $dB^{(k)}$. These diffusion coefficients are assumed to be deterministic functions of time such that the instantaneous covariances of these zero-coupon bond returns and exchange rates are finite.

$$\left|\int_s^{\min(T_1,T_2)} dt\, Z_{\alpha,\beta}(t,T_1,T_2)\right| \quad < \quad \infty.$$

These covariances can also be measured from the yield curve $Y_\alpha(s,T)$, or the forward curve $F_\alpha(s,u)$, according to the following definition,

$$P_\alpha(s,T) = e^{-(T-s)Y_\alpha(s,T)} = e^{-\int_s^T du F_\alpha(s,u)}, \qquad s \leq T.$$

Although we have used an infinite number of Brownian motions, the actual number of the market-driving factors is left open since the extra diffusion coefficients can be set to zero. For example, an M-factor Gaussian HJM model corresponds to a special case when there are M nonzero diffusion terms, which in terms of the forward-rate volatility functions $\sigma_{f,\alpha}^{(k)}(s,u)$ are given by

$$\sigma_\alpha^{(k)}(s,T) = \begin{cases} \int_s^T du\, \sigma_{f,\alpha}^{(k)}(s,u) & \forall \quad k = 1,2,\ldots M \\ 0 & \text{otherwise.} \end{cases}$$

One of the key observations in this paper is that the general pricing formulae and the hedging ratios of the contingent claims depend on the covariances Z as a whole. Since Z can be measured numerically from the market directly, either from the historical discount factors or implied from the market option prices, there is no need to calibrate each individual diffusion coefficient. Further discussion on the implementation of the model will be given in section 6. Since the time-evolution equation (2.9) for the contingent claims does not depend on the drift terms in equation (2.1) and (2.2), we shall not specify them for the purpose of pricing and hedging.

Assumption 3: The multi-currency interest rate market is efficient: namely, there is no arbitrage opportunity and all contingent claims can be replicated by the self-financing portfolios of the zero-coupon bonds in a risk neutral world.

Under these three assumptions, Black-Scholes' methodology can be extended to the interest rates market as follows. Denote the present domestic value of the contingent claim as a function of the zero-coupon bonds, $C[s, \{\hat{P}(s, T_j)\}]$. Since the bond maturity is treated as a continuous variable, the Ito Lemma for the infinitesimal variation of $C[s, \{\hat{P}(s, \cdot)\}]$ involves functional derivatives $\delta/\delta\hat{P}_\alpha(s, u)$ whose properties are explained in the Appendix.

$$dC[s, \{\hat{P}(s, \cdot)\}] = \partial_s C ds + \int_0^L du \sum_{\alpha=0}^m \frac{\delta C}{\delta \hat{P}_\alpha(s, u)} d\hat{P}_\alpha(s, u)$$
$$+\frac{1}{2}\int_0^L du_1 \int_0^L du_2 \sum_{\alpha,\beta=0}^m \frac{\delta^2 C}{\delta\hat{P}_\alpha(s, u_1)\delta\hat{P}_\beta(s, u_2)} \mathrm{COV}[d\hat{P}_\alpha(s, u_1), d\hat{P}_\beta(s, u_2)]. \tag{2.5}$$

In order to replicate the contingent claim, let us invest in a self-financing portfolio which consists one unit of the contingent claim and $h_\alpha(s, u)$ units of the zero-coupon bond $\hat{P}_\alpha(s, u)$ for any maturity $u \in [0, L]$. Since the contract depends on the cashflows in the future, we can set $h_\alpha(s, u) = 0$ for $u < s$. Without external cashflow coming in or out of the self-financing portfolio, the earning or loss comes solely from the price changes. In terms of the domestic currency, this instantaneous gain or loss is

$$dV[s, \{\hat{P}(s, \cdot)\}] = dC[s, \{\hat{P}(s, \cdot)\}] + \int_0^L du \sum_{\alpha=0}^m h_\alpha(s, u) d\hat{P}_\alpha(s, u). \tag{2.6}$$

From equation (2.5), the risky ($d\hat{P}$) terms in this gain/loss cancel if the holding $h_\alpha(s, u)$ is chosen to be

$$\text{(no risk)} \qquad h_\alpha(s, u) = -\frac{\delta C}{\delta\hat{P}_\alpha(s, u)}. \tag{2.7}$$

In other words, the market risk can be hedged away provided that one can trade continuously to adjust the holding of the bond portfolio according to equation (2.7). When the market is efficient (no arbitrage), this instantaneous gain or loss should be the same as offered in the domestic money market

$$dV[s, \{\hat{P}(s, \cdot)\}] = V[s, \{\hat{P}(s, \cdot)\}] \cdot r_0(s)\, ds, \tag{2.8}$$

where the domestic short rate $r_0(s)$ can be determined from the domestic zero-coupon bond according to

$$r_0[s, P_0] = -\lim_{T \to s} \partial_T \ln P_0(s, T).$$

Combining equation (2.5), (2.6) and (2.8) and substituting equation (2.7) for the holding $h_\alpha(s, u)$, we conclude that the contingent claim must satisfy the following multi-dimensional Kolmogorov field equation.

$$\begin{aligned}
&\partial_s C + r_0[s, P_0] \left\{ \int_0^L du \sum_{\alpha=0}^{m} \hat{P}_\alpha(s, u) \frac{\delta C}{\delta \hat{P}_\alpha(s, u)} - C \right\} \\
&\frac{1}{2} \int_0^L du_1 \int_0^L du_2 \sum_{\alpha,\beta=0}^{m} Z_{\alpha,\beta}(s, u_1, u_2) \hat{P}_\alpha(s, u_1) \hat{P}_\beta(s, u_2) \frac{\delta^2 C}{\delta \hat{P}_\alpha(s, u_1) \delta \hat{P}_\beta(s, u_2)} \\
&\qquad = 0
\end{aligned} \tag{2.9}$$

This equation describes how the value of the contingent claim evolves in time when the market players have no risk preference. Its present domestic value can be obtained by solving this equation with a boundary condition given by its terminal payoff together with two sets of input data: the initial domestic values of the zero-coupon bonds (factors) $\{\hat{P}(s, \cdot)\}$ and the covariance functions of the zero-coupon bond returns Z defined in equation (2.4). Formally, a general solution of this equation is given by the Feynman-Kac formula:

$$C[s, \{\hat{P}(s, \cdot)\}] = E_s\Big[C[T, \{\hat{P}(T, \cdot)\}] e^{-\int_s^T r_0(\tau, P_0)\, d\tau} \Big]$$

This formula provides a link between the PDE approach and the Martingale approach. In this paper, we shall evaluate this formula explicitly in terms of the multivariate Gaussian integral.

3 Pricing and Hedging Path-Independent Interest Rate Options

Having derived the time-evolution equation of the contingent claim in equation (2.9), the next step is to find the general solution to it. Let us first consider the case of path-independent options. In general, the terminal payoff of an interest rate contingent claim at the expiry date T depends on the cashflows, say at N discrete intervals $T_1 < \ldots < T_N$. For path-independent options, this payoff is generally given by a value function Φ of the zero-coupon bonds maturing at the future cashflow dates subject to a decision function ϕ:

$$C[T,\{\hat{P}_\alpha(T,T_j)\}] = \Phi[T,\{\hat{P}_\alpha(T,T_j)\}]\,\theta(\phi[T,\{\hat{P}_\alpha(T,T_j)\}]),$$

where $j = 1, 2, \ldots, N$ and θ is the indicator (step) function, defined as

$$\theta\,(\,\phi\,) \;=\; \begin{cases} 1 \;\text{ if } & \phi > 0 \\ 0 \;\text{ otherwise.} \end{cases}$$

If no decision is required at expiry T, the decision function is set to be trivial $\phi = 1$. In the case of a call (put) option, the decision function is the same as the payoff-function,

$$\Big(\Phi\Big)_+ \;\equiv\; \max[\Phi, 0] \;=\; \Phi\,\theta(\Phi).$$

A general characterisation of the contingent claims is that, like money in any currency, values of contingent claims are linearly proportional to the notional amount (the principal) of the contract and they can be added and subtracted. (Throughout this paper, the principal of the contract is assumed to be one D_0.) Mathematically speaking, this means that the prices of the contingent claims are homogeneous functions of degree-one,

$$\sum_{\alpha=0}^{m}\sum_{j=1}^{N} P_\alpha(s,T_j)\frac{\partial}{\partial P_\alpha(s,T_j)}C[s,\{P(s,\cdot)\}] = C[s,\{P(s,\cdot)\}]. \tag{3.1}$$

This homogeneous property consequently simplifies the Kolmogorov equation (2.9), because the two terms proportional to the domestic short rate cancel. Given the path-independent payoff function as the boundary condition, the pricing formula can be solved with the following multivariate Gaussian integral of dimension $M = (m+1)N$,

$$\begin{aligned} C[s,\hat{P}(s,\cdot)] = \int_{-\infty}^{\infty} d^M\vec{x}\;\; g(\vec{x},W)\,\Phi\left[T,\left\{\hat{P}_\alpha(s,T_j)e^{b_{\alpha,j}x_{\alpha,j}-\frac{1}{2}b^2_{\alpha,j}}\right\}\right] \\ \times\theta\left(\phi\left[T,\left\{\hat{P}_\alpha(s,T_j)e^{b_{\alpha,j}x_{\alpha,j}-\frac{1}{2}b^2_{\alpha,j}}\right\}\right]\right) \end{aligned} \tag{3.2}$$

where $g(\vec{x},W)$ is the multivariate normal distribution density

$$g(\vec{x}, W) = \sqrt{\frac{(2\pi)^{-M}}{\det W}} e^{-\frac{1}{2}\sum_{\alpha,\beta=0}^{m}\sum_{i,j=1}^{N} x_{\alpha,i} W^{-1}_{\{\alpha,i\},\{\beta,j\}} x_{\beta,j}} \tag{3.3}$$

and W is the positive definite correlation matrix given by

$$W_{\{\alpha,i\},\{\beta,j\}} = \frac{1}{b_{\alpha,i}b_{\beta,j}} \int_s^T dt\, Z_{\alpha,\beta}(t, T_i, T_j),$$

$$b^2_{\alpha,i} = \int_s^T dt\, Z_{\alpha,\alpha}(t, T_i, T_i), \quad i, j = 1, \ldots, N;\ \alpha, \beta = 0, \ldots, m. \tag{3.4}$$

This pricing formula requires two sets of input: the present zero-coupon bonds and the covariances Z of the bond returns in equation (2.4). This solution is a special case of the path-dependent formula in equation (5.4) when there is only one decision time $Q = 1$ and its proof can found in the appendix. In the case of one currency $\alpha = 0$, equation (3.2) is the previous result obtained by Chu [1996] for the single-currency market.

Since the pricing formula depends explicitly on the present zero-coupon bond prices, the contingent claim can be hedged with a portfolio of zero-coupon bonds. Let $V(s)$ denote the domestic value of the the hedged portfolio, i.e. buying one unit of the contingent claims and selling $h_{\alpha,j}$ units of the zero-coupon bond maturing at T_j in currency D_α:

$$V(s) = C(s) - \sum_{\alpha=0}^{m}\sum_{j=1}^{N} h_{\alpha,j} X_\alpha P_\alpha(s, T_j).$$

This portfolio will be delta neutral when the hedging ratios $h_{\alpha,j}$ are given by the following Gaussian integral according to the no risk condition in equation (2.7)

$$h_{\alpha,j} = \int_{-\infty}^{\infty} d\vec{x}\, g(\vec{x}, W)\, e^{b_{\alpha,j}x_{\alpha,j} - \frac{1}{2}b^2_{\alpha,j}} \frac{\partial}{p_{\alpha,j}} \Big(\Phi[T, \{p\}] \theta\left(\phi[T, \{p\}]\right) \Big), \tag{3.5}$$

where $p_{\alpha,j} \equiv \hat{P}_\alpha(s, T_j) e^{b_{\alpha,j}x_{\alpha,j} - \frac{1}{2}b^2_{\alpha,j}}$. In fact, using the homogeneous property of the pay-off functions, the contingent claim can be exactly replicated by the bond portfolio:

$$C(s) \quad = \quad \sum_{\alpha=0}^{m}\sum_{j=1}^{N} h_{\alpha,j} X_\alpha P_\alpha(s, T_j).$$

In the rest of this section, we apply this result to price several European interest-rate options such as options on currency swaps and spreads. European options on forward and futures contracts will be given separately in section 4. Generalisation to the path-dependent cases such as Asian options and Bermudan options will be given in section 5.

Example 3.1: Currency Swaps

Consider two currencies, D_α and D_0. Following Sandmann and Sondermann [1993] or Brace and Musiela [1994], we express the forward libor rates for the period between $[T_j, T_{j+1}]$ in terms of the zero-coupon bonds as

$$L_\gamma(T_j) \equiv \frac{1}{\delta}\left(\frac{\hat{P}_\gamma(T_j, T_j)}{\hat{P}_\gamma(T_j, T_{j+1})} - 1\right), \quad \delta = T_{j+1} - T_j, \quad \gamma = \alpha, 0. \tag{3.6}$$

If the interest is paid-in-advance, then the cash-flow at the beginning of the period T_j is

$$\text{float-leg} = \delta L_\gamma(T_j)\hat{P}_\gamma(T_j, T_{j+1}), \qquad \text{fixed-leg} = \delta K_\gamma \hat{P}_\gamma(T_j, T_{j+1}).$$

If it is paid-in-arrears, the cashflow at the end of the period T_{j+1} is

$$\text{float-leg} = \delta L_\gamma(T_j)\hat{P}_\gamma(T_{j+1}, T_{j+1}), \quad \text{fixed-leg} = \delta K_\gamma \hat{P}_\gamma(T_{j+1}, T_{j+1}).$$

In this section, we assume that all the payments are paid-in-advance. For contracts paid-in-arrears, the payments will depend on the zero-coupon bonds at the beginning and the end of the period and hence the path-dependent pricing formula in equation (5.4) should be used instead. For most of the swaps discussed below, both paid-in-advance and paid-in-arrears give the same present swap values.

For example, let us consider a forward swap contract for receiving the foreign libor payments in currency D_α and paying domestic libor payments for $(N-1)$ periods from T_1 to T_N with a constant interval δ. The domestic value of the j-th period's cashflows at T_j is the difference of the two libor payments:

$$C^{(j)}_{llcswp}(T_j) = \delta L_\alpha(T_j)\hat{P}_\alpha(T_j, T_{j+1}) - \delta L_0(T_j)\hat{P}_0(T_j, T_{j+1}).$$

According to equation (3.2), the present value of this payment is a four-dimensional integral which can be integrated exactly as

$$C^{(j)}_{llcswp}(s) = \hat{P}_\alpha(s, T_j) - \hat{P}_\alpha(s, T_{j+1}) - \hat{P}_0(s, T_j) + \hat{P}_0(s, T_{j+1}).$$

Therefore, the present value of the $(N-1)$-period swap is simply the sum over the contributions from each period.

$$C_{llcswp}(s) = X_\alpha(s)\Big(P_\alpha(s,T_1) - P_\alpha(s,T_N)\Big) - P_0(s,T_1) + P_0(s,T_N). \quad (3.7)$$

If the forward swap contract holder receives the foreign libor payments in currency D_α and pays domestic fixed-rate interest K_0, the domestic value of the j-th period's cashflow at T_j is the difference of the two interest payments:

$$C^{(j)}_{lfcswp}(T_j) = (T_{j+1} - T_j)\left(L_\alpha(T_j)\hat{P}_\alpha(T_j,T_{j+1}) - K_0\hat{P}_0(T_j,T_{j+1})\right).$$

Assuming constant interval δ for the $(N-1)$ periods from T_1 to T_N, The present value of the forward swap contract is given by

$$C_{lfcswp}(s) = X_\alpha(s)\Big(P_\alpha(s,T_1) - P_\alpha(s,T_N)\Big) - \delta K_0 \sum_{j=2}^{N} P_0(s,T_j). \quad (3.8)$$

Similarly, if the forward swap is between two fixed-rate payments in two currencies, its present value is given by

$$C_{ffcswp}(s) = X_\alpha(s)\delta K_\alpha \sum_{j=2}^{N} P_\alpha(s,T_j) - \delta K_0 \sum_{j=2}^{N} P_0(s,T_j). \quad (3.9)$$

Example 3.2: Currency Swaptions

Having expressed the forward swap contracts as the homogeneous functions of the zero-coupon bonds, we can proceed to evaluate an European call option on the forward swap. Let the option expire at time T which is prior to or at the swap's starting time $T \leq T_1$. The terminal value of the option is the domestic value of the swap value at T if it is positive or zero otherwise. Take for example the $(N-1)$-period forward swap between two libor payments during $[T_1, T_N]$ as described in Example 3.1. The call option pay-off at the expiry is

$$C_{llcswptn}(T) = \Big(\hat{P}_\alpha(T,T_1) - \hat{P}_\alpha(T,T_N) - \hat{P}_0(T,T_1) + \hat{P}_0(T,T_N)\Big)_+.$$

According to equation (3.2), the present value of the currency swaption is given by the following 4-dimensional Gaussian integral

$$C_{llcswptn}(s) = \int_{-\infty}^{\infty} d^4\vec{x}\, g(\vec{x}, W)\Big(X_\alpha(s)P_\alpha(s,T_1)e^{b_{\alpha,1}x_{\alpha,1}-\frac{b^2_{\alpha,1}}{2}}$$
$$-X_\alpha(s)P_\alpha(s,T_N)e^{b_{\alpha,N}x_{\alpha,N}-\frac{b^2_{\alpha,N}}{2}}$$
$$-P_0(s,T_1)e^{b_{0,1}x_{0,1}-\frac{b^2_{0,1}}{2}} + P_0(s,T_N)e^{b_{0,N}x_{0,N}-\frac{b^2_{0,N}}{2}}\Big)_+ . \qquad (3.10)$$

Given the present zero-coupon bond values and the covariances Z of the bond returns, (3.10) can be integrated numerically.

If the forward swap is between the foreign Libor payments and the domestic fixed-rate payments, the present price of the call option is given by the following multivariate Gaussian integral of dimension $(N+1)$ according to equation (3.2),

$$C_{lfcswptn}(s) = \int_{-\infty}^{\infty} d^{N+1}\vec{x}\, g(\vec{x}, W)\left(X_\alpha(s)P_\alpha(s,T_1)e^{b_{\alpha,1}x_{\alpha,1}-\frac{b^2_{\alpha,1}}{2}}\right.$$
$$\left.-X_\alpha(s)P_\alpha(s,T_N)e^{b_{\alpha,N}x_{\alpha,N}-\frac{b^2_{\alpha,N}}{2}} - \delta K_0 \sum_{j=2}^{N} \hat{P}_0(s,T_j)e^{b_{0,j}x_{0,j}-\frac{b^2_{0,j}}{2}}\right)_+ . \qquad (3.11)$$

The "at-the-money" swaption corresponds to the special case when the fixed-rate K_0 is chosen to be the forward swaprate K_*. This swaprate is determined by the zero initial swap value, $C_{lfcswp}(s) = 0$. According to equation (3.8), it is given by

$$K_* = \frac{X_\alpha(s)}{\delta \sum_{j=2}^{N} P_0(s,T_j)}\Big(P_\alpha(s,T_1) - P_\alpha(s,T_N)\Big).$$

When the interest payments are in the same currency, $D_\alpha = D_0$, equation (3.11) recovers the swaption formula obtained by Chu [1996] for the single-currency market.

Finally, if the swap is between two fixed-rate payments in two currencies, the call swaption is given by the following multivariate Gaussian integral of $(2N-2)$ dimensions:

$$C_{ffcswptn}(s) = \int_{-\infty}^{\infty} d^{(2N-2)}\vec{x}\, g(\vec{x}, W)\left(X_\alpha(s)\delta K_\alpha \sum_{j=2}^{N} P_\alpha(s,T_j)e^{b_{\alpha,j}x_{\alpha,j}-\frac{b^2_{\alpha,j}}{2}}\right.$$
$$\left.-\delta K_0 \sum_{j=2}^{N} P_0(s,T_j)e^{b_{0,j}x_{0,j}-\frac{b^2_{0,j}}{2}}\right)_+ . \qquad (3.12)$$

Example 3.3: Libor Spread (Diff Swap) and Spreadtion

A libor spread is the difference between two foreign libor rates L_α and L_β in currency D_α and D_β. Consider a spread contract (Diff swap) which entitles the holder to receive (pay) an interest in the domestic currency D_0 based on the spread if this difference $L_\alpha(T_j) - L_\beta(T_j)$ is positive (negative). If paid-in-advance, the domestic cashflow at the beginning of the j-th period is

$$C^{(j)}_{cspd}(T_j) = \delta \hat{P}_0(T_j, T_{j+1})\Big(\frac{\hat{P}_\alpha(T_j, T_j)}{\hat{P}_\alpha(T_j, T_{j+1})} - \frac{\hat{P}_\beta(T_j, T_j)}{\hat{P}_\beta(T_j, T_{j+1})}\Big)$$

This is homogeneous of degree-one and equation (3.2) involves a 5-dimensional Gaussian integral which can be integrated exactly. After summing over $(N-1)$ periods from T_1 to T_N, the present domestic value of the spread contract is given by

$$\begin{aligned} C_{cspd}(s) &= \delta \sum_{j=1}^{N-1} P_0(s, T_{j+1})\Big\{\frac{P_\alpha(s, T_j)e^{a^{(j)}_\alpha(s)}}{P_\alpha(s, T_{j+1})} - \frac{P_\beta(s, T_j)e^{a^{(j)}_\beta(s)}}{P_\beta(s, T_{j+1})}\Big\} \\ a^{(j)}_\gamma(s) &\equiv \int_s^{T_j} d\tau \Big\{Z_{\gamma,\gamma}(\tau, T_{j+1}, T_{j+1}) - Z_{\gamma,\gamma}(\tau, T_j, T_{j+1}) \\ &\quad -Z_{\gamma,0}(\tau, T_{j+1}, T_{j+1}) + Z_{\gamma,0}(\tau, T_j, T_{j+1})\Big\}, \qquad \gamma = \alpha, \beta. \end{aligned} \tag{3.13}$$

For an European call option on this $(N-1)$-period libor spread contract with an exercise date $T < T_1$ and a strike price K in currency D_0, the pricing formula is given by the following $(3N)$-dimensional Gaussian integral

$$\begin{aligned} C_{cspdtn}(s) = \int_{-\infty}^{\infty} & d^{3N}\vec{x}\, g(\vec{x}, W)\Big\{\delta \sum_{j=1}^{N-1} P_0(s, T_{j+1})e^{b_{0,j+1}x_{0,j+1} - \frac{b^2_{0,j+1}}{2}} \\ & \times\Big(\frac{P_\alpha(s, T_j)}{P_\alpha(s, T_{j+1})}e^{a^{(j)}_\alpha(T) + b_{\alpha,j}x_{\alpha,j} - \frac{b^2_{\alpha,j}}{2} - b_{\alpha,j+1}x_{\alpha,j+1} + \frac{b^2_{\alpha,j+1}}{2}} \\ & \quad - \frac{P_\beta(s, T_j)}{P_\beta(s, T_{j+1})}e^{a^{(j)}_\beta(T) + b_{\beta,j}x_{\beta,j} - \frac{b^2_{\beta,j}}{2} - b_{\beta,j+1}x_{\beta,j+1} + \frac{b^2_{\beta,j+1}}{2}}\Big) \\ & -K\, X_\alpha(s)\, P_0(s, T)e^{b_{0,T}x_{0,T} - \frac{b^2_{0,T}}{2}}\Big\}_+ \end{aligned} \tag{3.14}$$

Notice that the exponents are $a^{(j)}_\gamma(T)$ and not $a^{(j)}_\gamma(s)$. With a change of basis, this integral can be reduced to $(2N-2)$ dimensions and must in general be integrated numerically. If the strike price $K = 0$ or if the option expiry $T = T_1$, then the dimensions reduce further by one. If one of the libor rates is domestic, say $D_\beta = D_0$, then $a_\beta^{(j)}(T) = 0$ and we recover the results obtained by Jamshidian [1994].

4 European Options on the Forward and Futures

This section illustrates various applications of equation (3.2) to price European options on spot, forward and futures contracts of a zero-coupon bond. In particular, FX options are evaluated as interest rate options in this multicurrency model. This feature enhances the view that the FX and fixed-income markets are closely interacted to each other and should be treated as one market. Let us first express the forward and futures contracts in terms of the zero-coupon bonds and then apply equation (3.2) to evaluate the European call and put options.

Consider a forward contract on a zero-coupon bond in currency D_α with maturity T_3. The contract holder is obliged at the delivery date $T_2(\leq T_3)$ to pay the agreed forward price $F_\alpha^{(T_2,T_3)}$ in the domestic currency D_0. At the delivery date T_2, the domestic value of the forward contract is the difference between the actual bond price and the agreed forward price:

$$C_{T_3}(T_2) = \hat{P}_\alpha(T_2, T_3) - F_\alpha^{(T_2,T_3)} \hat{P}_0(T_2, T_2).$$

According to equation (3.2), the present value of the forward contract is simply

$$C_{T_3}(s) = X_\alpha(s) P_\alpha(s, T_3) - F_\alpha^{(T_2,T_3)} P_0(s, T_2).$$

By definition, the forward price is determined such that it costs nothing to enter the contract. This implies that the forward price is

$$F_\alpha^{(T_2,T_3)}(s) = \frac{X_\alpha(s) P_\alpha(s, T_3)}{P_0(s, T_2)}. \tag{4.1}$$

On the other hand, the futures contract is reset continuously so that in the arbitrage-free market its loss or gain should be the same as investing in a self-financing bond portfolio. This implies that the drift term in the Ito expansion in equation (2.5) must vanish and so the futures price satisfies the following diffusion equation,

$$\partial_s f_\alpha^{(T_2,T_3)}(s) + \frac{1}{2}\sum_{\gamma\gamma'=0}^{m}\int_0^L du_1 \int_0^L du_2 \frac{\delta^2 f_\alpha^{(T_2,T_3)}(s)}{\delta \hat{P}_\gamma(s,u_1)\delta \hat{P}_{\gamma'}(s,u_2)} \times \hat{P}_\gamma(s,u_1)\hat{P}_{\gamma'}(s,u_2) Z_{\gamma,\gamma'}(s,u_1,u_2) = 0. \quad (4.2)$$

On the delivery date T_2, the futures price is the same as the forward price of the same day, $f_\alpha^{(T_2,T_3)}(T_2) = F_\alpha^{(T_2,T_3)}(T_2)$. Hence, we can solve for the present futures price from equation (4.2) with the following two-dimensional Gaussian integral

$$\begin{aligned} f_\alpha^{(T_2,T_3)}(s) &= \int_{-\infty}^{\infty} d^2\vec{x}\, g(\vec{x},W)\, \frac{\hat{P}_\alpha(s,T_3)}{\hat{P}_0(s,T_2)} e^{b_{\alpha,T_3}x_{\alpha,T_3} - \frac{1}{2}b^2_{\alpha,T_3}}\, e^{-b_{0,T_2}x_{0,T_2} + \frac{1}{2}b^2_{0,T_2}} \\ &= F_\alpha^{(T_2,T_3)}(s)\, e^{d_\alpha(s,T_2,T_3)}, \\ d_\alpha(s,T_2,T_3) &= \int_s^{T_2} dt \quad (Z_{0,0}(t,T_2,T_2) - Z_{\alpha,0}(t,T_3,T_2)). \end{aligned} \quad (4.3)$$

On the delivery date, $d_\alpha(T_2,T_2,T_3) = 0$ and both forward and futures are given by the domestic spot price of the bond $X_\alpha(T_2)P_\alpha(T_2,T_3)$. When the underlying bond maturity coincides with the delivery date, $T_3 = T_2 = T$, these forward and futures contracts are identical to those of the FX exchange rate, namely the forward currency exchange rate is

$$F_\alpha^{(T,T)}(s) = \frac{X_\alpha(s)P_\alpha(s,T)}{P_0(s,T)}$$

and the futures exchange rate is

$$f_\alpha^{(T,T)}(s) = \frac{X_\alpha(s)P_\alpha(s,T)}{P_0(s,T)} e^{d_\alpha(s,T,T)}.$$

For a single currency market, $F_0^{(T_2,T_3)}(s)$ and $f_0^{(T_2,T_3)}(s)$ correspond to the present forward and futures prices of the domestic zero-coupon bond (Treasury Bills) maturing at T_3 for a delivery date T_2. If the yield curves are both flat, i.e. $Z_{0,0} = Z_{\alpha,0} = 0$, $P_0(s,T) = e^{-r_0(T-s)}$ and $P_\alpha(s,T) = e^{-r_\alpha(T-s)}$ for constant r_0 and r_α, then the futures and forward prices are given by the more primitive formula $X_\alpha(s)e^{r_0(T_2-s) - r_\alpha(T_3-s)}$.

Let us now proceed to price an European call option on the forward or future contract of a foreign zero-coupon bond in currency D_α. Let $T_1 \leq T_2 \leq T_3$ where T_1 is the option exercise date, T_2 the forward's delivery date and T_3 the bond's maturity. Suppose the option strike price in the home currency is K. Then, the option's domestic value on the exercise date T_1 is

$$C^{(T_1,T_2,T_3)}_{f*call}(T_1) = \hat{P}_0(T_1,T_1)\Big(\frac{\hat{P}_\alpha(T_1,T_3)}{\hat{P}_0(T_1,T_2)}e^{d_*} - K\Big)_+$$

$$d_* = \begin{cases} 0 & \text{forward} \\ d_\alpha(T_1,T_2,T_3) & \text{futures} \end{cases}$$

The present domestic value of the call option on the forward/futures contract of the zero-coupon bond maturing at T_3 can then be obtained from integrating a 3-dimensional Gaussian integral according to equation (3.2) which gives the following Black-like formula

$$\begin{aligned} C^{(T_1,T_2,T_3)}_{f*call}(s) = P_0(s,T_1) & \ \frac{X_\alpha(s)P_\alpha(s,T_3)}{P_0(s,T_2)}e^{d_*+B_\alpha(s,T_1,T_2,T_3)}\mathcal{N}(D^{f*}_+) \\ - & \ \ KP_0(s,T_1)\mathcal{N}(D^{f*}_-), \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} D^{f*}_\pm &= \frac{\ln\frac{X_\alpha(s)P_\alpha(s,T_3)}{KP_0(s,T_2)} + d_* + B_\alpha(s,T_1,T_2,T_3) \pm \frac{1}{2}A^2_\alpha(s,T_1,T_2,T_3)}{A_\alpha(s,T_1,T_2,T_3)} \\ B_\alpha(s,T_1,T_2,T_3) &= \int_s^{T_1} dt\Big\{Z_{0,0}(t,T_2,T_2) - Z_{0,0}(t,T_1,T_2) \\ &\qquad - Z_{0,\alpha}(t,T_2,T_3) + Z_{0,\alpha}(t,T_1,T_3)\Big\} \\ A^2_\alpha(s,T_1,T_2,T_3) &= \int_s^{T_1} dt\Big\{Z_{\alpha,\alpha}(t,T_3,T_3) + Z_{0,0}(t,T_2,T_2) \\ &\qquad - 2\,Z_{0,\alpha}(t,T_2,T_3)\Big\} \end{aligned} \tag{4.5}$$

As mentioned before, when the bond maturity date coincides with the delivery date $T_3 = T_2$, the bond forward or futures contracts coincide with the FX forward and futures contracts. When the forward/futures delivery date T_2 coincides with the option exercise date T_1, these forward and futures are the same as the underlying spots. Therefore, we can summarise the European call options on the spot, forward and futures of the zero-coupon bonds (Treasury Bills) and currency exchange rates as follows:

$C_{f*call}^{(T_1,T_2,T_3)}(s)$	call on the forward/futures of $P_\alpha(s,T_3)$ to be delivered at T_2.
$C_{f*call}^{(T_1,T_1,T_3)}(s)$	call on the spot zero-coupon bond $P_\alpha(s,T_3)$.
$C_{f*call}^{(T_1,T_2,T_2)}(s)$	call on the FX forward/futures to be delivered at T_2.
$C_{f*call}^{(T_1,T_1,T_1)}(s)$	call on the FX spot rate at T_1.

The European put options of the spots, forward and futures can be obtained similarly under the put-call parity:

$$C_{f*call}^{(T_1,T_2,T_3)}(s) - C_{f*put}^{(T_1,T_2,T_3)}(s) = P_0(s,T_1)\frac{X_\alpha(s)P_\alpha(s,T_3)}{P_0(s,T_2)}e^{d_*+B_\alpha(s,T_1,T_2,T_3)} - KP_0(s,T_1).$$

These results for the futures and the forward agree with the existing literature, such as the four-factor model by Amin and Jarrow [1991], the two-factor single-currency HJM model by Jarrow and Turnbull [1994] and the general multi-factor model by Jamshidian [1993,1994]. The comparison with the work by Jamshidian [1994] is particularly interesting since his pricing method does not employ PDE or Martingale techniques but relies on explicitly hedging the contract case by case.

5 Pricing and Hedging Path-Dependent Interest Rate Options

Let us now proceed to solve the Kolmogorov field equation (2.9) when the boundary condition is path-dependent. The main idea is to adopt the so-called Markov transition approach to solve Kolmogorov's equation as outlined by Williams [1991]. The first step is to reformulate the path-independent solution in equation (3.2) as a transition process which propagates the contingent claim from $\tau_1 = T_1$ back to the present $\tau_0 = s$ as follows:

$$\begin{aligned} C[\tau_0, p^{(0)}] &= \int_{-\infty}^{\infty} d\vec{x}^{(1)} g(\vec{x}^{(1)}, W^{(1)})\, C\left[\tau_1, \left\{p_{\alpha,j}^{(0)} e^{b_{\alpha,j}^{(1)} x_{\alpha,j}^{(1)} - \frac{1}{2}(b_{\alpha,j}^{(1)})^2}\right\}\right] \\ &\equiv \int \mathcal{D}p^{(1)}\, G(\tau_0, p^{(0)}; \tau_1, p^{(1)})\, C\left[\tau_1, p^{(1)}\right], \end{aligned}$$

where the transition density function $G(\tau_0, p^{(0)}; \tau_1, p^{(1)})$ is given by the multi-variate Gaussian density function with a correlation matrix given in equation

(3.4), namely

$$W^{(1)}_{\{\alpha,i\}.\{\beta,j\}} = \frac{1}{b^{(1)}_{\alpha,i} b^{(1)}_{\beta,j}} \int_{\tau_0}^{\tau_1} dt Z_{\alpha,\beta}(t,T_i,T_j), \quad \left(b^{(1)}_{\alpha,j}\right)^2 = \int_{\tau_0}^{\tau_1} dt Z_{\alpha,\alpha}(t,T_j,T_j).$$

To simplify the notation, $p^{(q)}$ stands for a set of zero-coupon bonds at time τ_q,

$$p^{(q)} \equiv \left\{ p^{(q)}_{\alpha,j} = p^{(q-1)}_{\alpha,j} e^{b^{(q)}_{\alpha,j} x^{(q)}_{\alpha,j} - \frac{1}{2}(b^{(q)}_{\alpha,j})^2}, \forall T_j \geq \tau_q; \alpha = 0, \ldots, m \right\},$$
$$p^{(q)}_{\alpha,j} \equiv X_\alpha(\tau_q) P_\alpha(\tau_q, T_j), \qquad \forall \quad q = 0, 1.$$

If the option expiry is at some later time, say $T = \tau_Q$ for $\tau_Q > \ldots > \tau_2 > \tau_1$, then this transition approach is consistent provided that

$$C[\tau_0, p^{(0)}] = \int \mathcal{D}p^{(1)} \ldots \mathcal{D}p^{(Q)}\, G(\tau_0, p^{(0)}; \tau_1, p^{(1)}) \ldots$$
$$G(\tau_{Q-1}, p^{(Q-1)}; \tau_Q, p^{(Q)})\, C\left[\tau_Q, p^{(Q)}\right],$$

where the transition density function from τ_q back to τ_{q-1} is defined as

$$\int \mathcal{D}p^{(q)} G(\tau_{q-1}, p^{(q-1)}; \tau_q, p^{(q)})\, C\left[\tau_q, p^{(q)}\right]$$
$$\equiv \int_{-\infty}^{\infty} d\vec{x}^{(q)}\, g(\vec{x}^{(q)}, W^{(q)})\, C\left[\tau_q, \left\{ p^{(q-1)}_{\alpha,j} e^{\vec{b}^{(q)}_{\alpha,j} \vec{x}^{(q)}_{\alpha,j} - \frac{1}{2}\left(\vec{b}^{(q)}_{\alpha,j}\right)^2} \right\}\right] \tag{5.1}$$

and the variances and the correlation matrices are given by

$$W^{(q)}_{\{\gamma,j\},\{\gamma',j'\}} \equiv \left(\frac{1}{b^{(q)}_{\gamma,j} b^{(q)}_{\gamma',j'}}\right) \int_{\tau_{q-1}}^{\tau_q} dt\, Z_{\gamma,\gamma'}(t, T_j, T_{j'}),$$
$$\left(b^{(q)}_{\alpha,j}\right)^2 \equiv \int_{\tau_{q-1}}^{\tau_q} dt\, Z_{\alpha,\alpha}(t, T_j, T_j), \qquad \forall T_i, T_j \geq \tau_q. \tag{5.2}$$

When the covariances Z are deterministic, this consistency is ensured by the following Markov property (see Appendix):

$$C[\tau_{q-1}, p^{(q-1)}] = \int \mathcal{D}p^{(q)} \int \mathcal{D}p^{(q+1)} G\Big(\tau_{q-1}, p^{(q-1)}; \tau_q, p^{(q)}\Big) \times$$

$$G\Big(\tau_q, p^{(q)}; \tau_{q+1}, p^{(q+1)}\Big) C[\tau_{q+1}, p^{(q+1)}]$$
$$= \int \mathcal{D}p^{(q+1)} G\Big(\tau_{q-1}, p^{(q-1)}; \tau_{q+1}, p^{(q+1)}\Big) C[\tau_{q+1}, p^{(q+1)}],$$
$$\forall\, q = 1, \dots, Q-1. \tag{5.3}$$

Together with the "dimensional-reduction" property of the multivariate Gaussian density given in equation (A.2), this Markov transition property enables us to evaluate any path-dependent option as follows. Let the payment dates in the contract be at $T_1 < \ldots < T_N$ and the cashflows be subject to Q different decision or exercise dates, $\tau_1 < \ldots < \tau_Q$, where the last decision date τ_Q is the option expiry date. On each decision date τ_q, the payment function and the decision function depend on the future cashflows and so they are functions of the zero-coupon bonds with maturity $T_{j_q} \geq \tau_q$ only. Therefore, the payoff at the option expiry τ_Q can be written as

$$C[\tau_Q] = \Phi\left[\tau_Q, \left\{\hat{P}_{\alpha_1}(\tau_1, T_{j_1}), \ldots, \hat{P}_{\alpha_Q}(\tau_Q, T_{j_Q})\right\}\right] \times$$
$$\theta\left(\phi_1\left[\tau_1, \left\{\hat{P}_{\alpha_1}(\tau_1, T_{j_1})\right\}\right]\right) \ldots \theta\left(\phi_Q\left[\tau_Q, \left\{\hat{P}_{\alpha_Q}(\tau_Q, T_{j_Q})\right\}\right]\right)$$

If no decision is required at τ_q, then $\phi_q = 1$. The present value of this path-dependent option can be obtained by successively propagating this terminal value from τ_Q back to τ_0. As shown in the Appendix, the following multivariate Gaussian integral solves the Kolmogorov equation for the above boundary condition:

$$C[s, \{\hat{P}(s, \cdot)\}] = \int \mathcal{D}p^{(1)} \ldots \mathcal{D}p^{(Q)}\, G\Big(\tau_0, p^{(0)}; \tau_1, p^{(1)}\Big) \ldots$$
$$\times G\Big(\tau_{Q-1}, p^{(Q-1)}; \tau_Q, p^{(Q)}\Big)\, \Phi\left[\tau_Q, \left\{p^{(1)}, \ldots, p^{(Q)}\right\}\right]$$
$$\times \theta\left(\phi_1\left[\tau_1, p^{(1)}\right]\right) \ldots \theta\left(\phi_Q\left[\tau_Q, p^{(Q)}\right]\right)$$
$$= \int_{-\infty}^{\infty} d\vec{x}^{(1)} \ldots \int_{-\infty}^{\infty} d\vec{x}^{(Q)}\, g(\vec{x}^{(1)}, W^{(1)}) \ldots g(\vec{x}^{(Q)}, W^{(Q)})$$
$$\Phi\left[\tau_Q, \left\{p^{(1)}, \ldots, p^{(Q)}\right\}\right] \theta\left(\phi_1\left[\tau_1, p^{(1)}\right]\right) \ldots \theta\left(\phi_Q\left[\tau_Q, p^{(Q)}\right]\right), \tag{5.4}$$

where the variances $b^{(q)}$ and the correlation matrix $W^{(q)}$ for each period $[\tau_q, \tau_{q+1}]$ are given in equation (5.2). $p^{(q)}$ is now a set of all the zero-coupon bonds that are "propagated" from τ_q back to the present $s = \tau_0$,

$$p^{(q)} \equiv \left\{ p^{(q)}_{\alpha,j} = X_\alpha(s) P_\alpha(s,T_j) e^{\sum_{n=1}^{q} \left(b^{(n)}_{\alpha,j} x^{(n)}_{\alpha,j} - \frac{1}{2}(b^{(n)}_{\alpha,j})^2 \right)}, \quad \forall\, T_j \geq \tau_q \right\}. \quad (5.5)$$

The hedging ratios (deltas) can be obtained by the following multivariate Gaussian integral according to the no risk condition in equation (2.7),

$$h_{\alpha,j} = \int_{-\infty}^{\infty} d\vec{x}^{(1)} \ldots \int_{-\infty}^{\infty} d\vec{x}^{(Q)}\, g(\vec{x}^{(1)}, W^{(1)}) \ldots g(\vec{x}^{(Q)}, W^{(Q)}) \times$$
$$\sum_{q=1}^{Q} \sum_{\alpha=0}^{m} \sum_{j=q}^{N} e^{\sum_{n=1}^{q} \left(b^{(n)}_{\alpha,j} x^{(n)}_{\alpha,j} - \frac{1}{2}(b^{(n)}_{\alpha,j})^2 \right)} \frac{\partial}{p^{(q)}_{\alpha,j}}$$
$$\left(\Phi \left[\tau_Q, \left\{ p^{(1)}, \ldots, p^{(Q)} \right\} \right] \theta \left(\phi_1[\tau_1, p^{(1)}] \right) \ldots \theta \left(\phi_Q[\tau_Q, p^{(Q)}] \right) \right). \quad (5.6)$$

In the rest of this section, we apply this result to evaluate FX Asian options and Bermudan options, both with a finite number of decision times. Although this formula is very general, for options with continuous exercise time such as American options, the number of decision times Q becomes very large and the numerical integration will be computationally intensive. Bunch and Johnson [1992] suggested an approximation method to evaluate this type of multivariate Gaussian integrals. Alternatively, a path-integral formulation such as the one proposed by Dash [1988, 1989] will be worth investigating.

Example 5.1: FX Asian Options (Geometric and Arithmetic)

Consider the FX Asian option which is a call option on the geometric average of the exchange rate X_α at Q different times, $\tau_1 < \ldots < \tau_Q$. Let the strike price at the option expiry τ_Q be some constant $K(D_0)$. Expressing the exchange rate in terms of the zero-coupon bonds as $X_\alpha(\tau_q) = \hat{P}_\alpha(\tau_q, \tau_q)/\hat{P}_0(\tau_q, \tau_q)$, the terminal domestic value of the contract is

$$C_{Agcall}(\tau_Q) = \hat{P}_0(\tau_Q, \tau_Q) \left(\left(\frac{\hat{P}_\alpha(\tau_1, \tau_1) \ldots \hat{P}_\alpha(\tau_Q, \tau_Q)}{\hat{P}_0(\tau_1, \tau_1) \ldots \hat{P}_0(\tau_Q, \tau_Q)} \right)^{\frac{1}{Q}} - K \right)_+ .$$

According to equation (5.4), the pricing formula of the geometric Asian option is given by the following of $Q(Q+1)$-dimensional Gaussian integral,

$$C_{Agcall}(s) = \int_{-\infty}^{\infty} d^{2Q} \vec{x}^{(1)} g(\vec{x}^{(1)}, W^{(1)}) \ldots \int_{-\infty}^{\infty} d^{2} \vec{x}^{(Q)} g(\vec{x}^{(Q)}, W^{(Q)}) \times$$

$$
\Big\{ X_\alpha(s) \Big(\frac{P_\alpha(s,\tau_1)\dots P_\alpha(s,\tau_Q)}{P_0(s,\tau_1)\dots P_0(s,\tau_Q)} \Big)^{\frac{1}{Q}} \times e^{\frac{1}{Q}\sum_{j=1}^{Q}\sum_{q=1}^{j}\left(b_{\alpha,j}^{(q)} x_{\alpha,j}^{(q)} - \frac{1}{2}(b_{\alpha,j}^{(q)})^2 - b_{0,j}^{(q)} x_{0,j}^{(q)} + \frac{1}{2}(b_{0,j}^{(q)})^2 \right)} - K \Big\}_+ P_0(s,\tau_Q)\, e^{\sum_{q=1}^{Q}\left(b_{0,Q}^{(q)} x_{0,Q}^{(q)} - \frac{1}{2}(b_{0,Q}^{(q)})^2 \right)}.
$$

This integral can be evaluated explicitly by first replacing $x_{\alpha,Q}^{(Q)}$ and $x_{0,Q}^{(Q)}$ with two new variables $y_{\alpha,Q}$ and $y_{0,Q}$,

$$
y_{0,Q} \equiv \sum_{q=1}^{Q} b_{0,Q}^{(q)} x_{0,Q}^{(q)}, \quad y_{\alpha,Q} \equiv \frac{1}{Q} \sum_{j=1}^{Q} \sum_{q=1}^{j} \left(b_{\alpha,j}^{(q)} x_{\alpha,j}^{(q)} - b_{0,j}^{(q)} x_{0,j}^{(q)} \right)
$$

and then using equation (A.2) to yield the following Black's like formula,

$$
\begin{aligned}
C_{Agcall}(s) &= P_0(s,\tau_Q) X_\alpha(s) \Big(\frac{P_\alpha(s,\tau_1)\cdot\cdot P_\alpha(s,\tau_Q)}{P_0(s,\tau_1)\cdot\cdot P_0(s,\tau_Q)} \Big)^{\frac{1}{Q}} \exp(-\psi_Q)\, \mathcal{N}(D_+^Q) \\
&\quad - K P_0(s,\tau_Q) \mathcal{N}(D_-^Q) \\
D_\pm^Q &= \frac{1}{\zeta_Q} \left(\ln \frac{X_\alpha(s)}{K} + \frac{1}{Q} \ln \left(\frac{P_\alpha(s,\tau_1)\dots P_\alpha(s,\tau_Q)}{P_0(s,\tau_1)\dots P_0(s,\tau_Q)} \right) - \psi_Q \pm \frac{1}{2}\zeta_Q^2 \right) \\
\zeta_Q^2 &= \frac{1}{Q^2} \sum_{j,k=1}^{Q} \int_s^{\min(\tau_j,\tau_k)} dt \Big\{ Z_{\alpha,\alpha}(t,\tau_j,\tau_k) + Z_{0,0}(t,\tau_j,\tau_k) - 2Z_{\alpha,0}(t,\tau_j,\tau_k) \Big\} \\
\psi_Q &= -\frac{1}{2}\zeta_Q^2 + \frac{1}{2Q} \sum_{j=1}^{Q} \int_s^{\tau_j} dt \Big\{ Z_{\alpha,\alpha}(t,\tau_j,\tau_j) - Z_{0,0}(t,\tau_j,\tau_j) \\
&\qquad + 2Z_{0,0}(t,\tau_j,\tau_Q) - 2Z_{\alpha,0}(t,\tau_j,\tau_Q) \Big\}.
\end{aligned} \tag{5.7}
$$

From the multivariate Gaussian integral for C_{Agcall}, it is easy to see that the put option on the FX geometric average can be evaluated from the following put-call parity:

$$
C_{Agcall} - C_{Agput} = P_0(s,\tau_Q) \Big\{ X_\alpha(s) \left(\frac{P_\alpha(s,\tau_1)\cdot\cdot P_\alpha(s,\tau_Q)}{P_0(s,\tau_1)\cdot\cdot P_0(s,\tau_Q)} \right)^{\frac{1}{Q}} e^{-\psi_Q} - K \Big\}.
$$

If the foreign and domestic yields are both constant (flat) and the exchange rate volatilities are constant, $Z_{\alpha,\alpha}(t,\tau_j,\tau_k) = \xi_\alpha^2$, then we recover the results by Jarrow and Rudd [1983] and Vorst [1992]. For constant intervals δ,

$$\psi_Q = \frac{\xi_\alpha^2 \delta}{12Q}(Q^2-1), \qquad \zeta_Q^2 = \frac{\xi_\alpha^2\delta}{6Q}(Q+1)(2Q+1).$$

A more popular type of FX Asian option is the call(put) option on the arithmetic average of the exchange rate at Q different times, $\tau_1 < \ldots < \tau_Q$. Let the strike price at the option expiry τ_Q be $K(D_0)$. The terminal domestic value of the call option is

$$C_{Aacall}(\tau_Q) = \hat{P}_0(\tau_Q,\tau_Q)\Big(\frac{1}{Q}\sum_{q=1}^{Q}\left(\frac{\hat{P}_\alpha(\tau_q,\tau_q)}{\hat{P}_0(\tau_q,\tau_q)}\right) - K\Big)_+.$$

Hence, according to the pricing formula in equation (5.4), the present domestic value of the arithmetic Asian option is given by the following Gaussian integral of dimension $Q(Q+1)$.

$$\begin{aligned} C_{Aacall}(s) = &\int_{-\infty}^{\infty} d^{2Q}\vec{x}^{(1)} g(\vec{x}^{(1)}, W^{(1)}) \ldots \int_{-\infty}^{\infty} d^2\vec{x}^{(Q)} g(\vec{x}^{(Q)}, W^{(Q)}) \times \\ &\Big\{\frac{X_\alpha(s)}{Q}\sum_{q=1}^{Q}\frac{P_\alpha(s,\tau_q)}{P_0(s,\tau_q)} e^{\sum_{n=1}^{q}\left(b_{\alpha,q}^{(n)}x_{\alpha,q}^{(n)} - \frac{1}{2}(b_{\alpha,q}^{(n)})^2 - b_{0,q}^{(n)}x_{0,q}^{(n)} + \frac{1}{2}(b_{0,q}^{(n)})^2\right)} \\ &-K\Big\}_+ P_0(s,\tau_Q) e^{\sum_{n=1}^{Q}\left(b_{0,Q}^{(n)}x_{0,Q}^{(n)} - \frac{1}{2}(b_{0,Q}^{(n)})^2\right)}. \end{aligned} \tag{5.8}$$

Under a change of basis, this integral can be reduced to Q dimensions and so the integration must in general be carried out numerically. The put-call parity for the arithmetic Asian option can be deduced from equation (5.8) as

$$C_{Aacall}(s) - C_{Aaput}(s) = P_0(s,\tau_Q)\Big\{\frac{X_\alpha(s)}{Q}\sum_{q=1}^{Q}\frac{P_\alpha(s,\tau_q)}{P_0(s,\tau_q)}e^{a_q} - K\Big\}$$

$$a_q = \int_0^{\tau_q} dt\Big\{Z_{\alpha,0}(t,\tau_q,\tau_Q) - Z_{\alpha,0}(t,\tau_q,\tau_q) - Z_{0,0}(t,\tau_q,\tau_Q) + Z_{0,0}(t,\tau_q,\tau_q)\Big\}$$

In the case of constant interest rates, many authors, including Turnbull and Wakeman [1991], Levy [1992], Vorst [1992] and Zhang [1995], have suggested to approximate the arithmetic average with the geometric average. For stochastic interest rates, Nielsen and Sandmann [1996] used the Monte-Carlo simulations based on a one-factor HJM model and approximated the single-currency Asian equity options according to Turnbull and Wakeman [1991] and Vorst [1992]. The result in equation (5.8) is more general and the numerical integration may be less computationally intensive as long as the dimension Q is not too large.

Example 5.2: Bermudan Options

The Bermudan contract, also known as the schedule contract or the Geske-Johnson [1984] procedure, is half-way between European and American options. It allows the buyer to exercise on any one of the discrete payment dates $\tau_1 < \ldots < \tau_Q$ prior to the maturity date of the underlying product. Let $B(s)$ denote the present price of the Bermudan contract on some contingent claim C_B, e.g. caps, floors, swaps and swaptions etc. It is most profitable to exercise the option at τ_q when the underlying contingent claim $C_B(\tau_q)$ is worth more than the value of the Bermudan option $B(\tau_q)$, i.e. the anticipated return for later exercises. During the lifetime of the contract $\tau_n \in [\tau_0, \tau_Q]$, the value of the Bermudan option is the sum of the option values which are optimally exercised at each $\tau_q(> \tau_n)$:

$$B[\tau_n, p^{(n)}] = \sum_{q=n+1}^{Q} \tilde{B}_q(\tau_n, p^{(n)}). \tag{5.9}$$

According to equation (5.4), $\tilde{B}_q(\tau_n, \{\hat{P}(s,\cdot)\})$, which denotes the value at τ_n for optimally exercising at τ_q, is given by the following Gaussian integral:

$$\begin{aligned}\tilde{B}_q(\tau_n, p^{(n)}) = \int \mathcal{D}p^{(n+1)} \ldots \int \mathcal{D}p^{(q)} G\left(\tau_n, p^{(n)}; \tau_{n+1}, p^{(n+1)}\right) \times \\ \ldots G\left(\tau_{q-1}, p^{(q-1)}; \tau_q, p^{(q)}\right) C_B\left[\tau_q, p^{(q)}\right] \times \\ \theta\left(C_B[\tau_q, p^{(q)}] - B[\tau_q, p^{(q)}]\right) \theta\left(B[\tau_{q-1}, p^{(q-1)}] - C_B[\tau_{q-1}, p^{(q-1)}]\right) \times \\ \ldots \theta\left(B[\tau_{n+1}, p^{(n+1)}] - C_B[\tau_{n+1}, p^{(n+1)}]\right)\end{aligned} \tag{5.10}$$

and the present value of the Bermudan option is given by equation (5.9) for $n = 0$, $\tau_0 = s$ and $p^{(0)} = \{\hat{P}(s,\cdot)\}$. Equation (5.10) can be integrated numerically, after expressing the underlying C_B as a homogeneous function of the zero-coupon bonds. For example, let C_B be the currency swap over $(N-1)$ periods in $[\tau_1, \tau_N]$ between two libor payments given in equation (3.7) and the Bermudan option's holder can choose whether or not to enter the swap on the first $Q(< N)$ payment dates. Then C_B in equation (5.10) corresponds to

$$C_{llcswap}\left[\tau_q, p^{(q)}\right] = p^{(q)}_{\alpha,q} - p^{(q)}_{\alpha,N} - p^{(q)}_{0,q} + p^{(q)}_{0,N}, \qquad \forall \quad q = 1, \ldots, Q.$$

6 Nonparametric Implementation

One of the key observations in this paper is that fixed-income options and FX options are homogeneous degree-one functions of the zero-coupon bonds and their prices can be determined completely by the deterministic covariance functions of the zero-coupon bond returns, $Z_{\alpha,\beta}(s, T_1, T_2)$, defined in equation (2.4).

$$Z_{\alpha,\beta}(s, T_i, T_j) = \mathrm{COV}\left[\left(\frac{dX_\alpha(s)}{X_\alpha(s)} + \frac{dP_\alpha(s, T_i)}{P_\alpha(s, T_i)}\right), \left(\frac{dX_\beta(s)}{X_\beta(s)} + \frac{dP_\beta(s, T_j)}{P_\beta(s, T_j)}\right)\right] \tag{6.1}$$

These covariance functions (surfaces) can be measured empirically from the market data and there is no need to calibrate the diffusion coefficients (volatility parameters) as usually suggested in the existing literature.

Theoretically, Z is the expected future covariance of the zero-coupon bond returns over the life of the option. Given the nature of market uncertainty, we can at best measure Z either by evaluating equation (6.1) from historical data for the exchange rates and the zero-coupon bonds or by calculating the implied values from the market prices of fairly liquid options. For historical covariances, the only freedom here is how many observations to take in evaluating the covariances. This amounts to finding out the period when the revealed (past) information is most influential to the market expectation. Practitioners prefer mark-to-market and this requires extracting the implied covariance (volatility) surfaces from a wide range of market option prices, such as caps, floors, and options on the futures and forwards of government bonds and exchange rates.

Whether historical or implied, this data is usually given in discrete maturities in monthly, quarterly or semi-annual intervals. Hence, they define a set of discrete points, e.g. $Z_{\alpha,\beta}(s, T_i, T_j)$ is a point (i, j) on the (α, β) volatility surface. In order to find other points on the surfaces, we interpolate these points with a smooth function $F_{\alpha,\beta}(s, T_i, T_j)$, such that it agrees with $Z_{\alpha,\beta}$ at point (i, j). If the shapes of the covariance surfaces do not change much in time, we can assume that the covariance functions are time-stationary,

$$F_{\alpha,\beta}(s + h, T_i + h, T_j + h) = F_{\alpha,\beta}(T_i - s, T_j - s), \quad \forall \quad h > 0.$$

To improve the approximation, we can also multiply a time-dependent factor $H(s)$ to scale up and down the covariance surfaces day-by-day or month-by-month:

$$Z_{\alpha,\beta}(s,T_i,T_j) \sim H_{\alpha,\beta}(s) \times F_{\alpha,\beta}(T_i - s, T_j - s).$$

Having these interpolating covariance functions, the variances $b^{(q)}$'s and the correlation matrices $W^{(q)}$'s can be obtained numerically from integrating Z from τ_{q-1} to τ_q. A systematic empirical study on the volatility surfaces will help to determine how stable the scaling factor is under the time-stationary assumption. Please notice that the model validity does not rely on the time-stationary assumption although it does make the implementation much easier.

7 Conclusions

This paper derives the general arbitrage-free pricing formulae and hedging ratios for interest rate derivatives in the multicurrency market. These formulae are obtained by solving the multi-dimensional Kolmogorov field equation when the covariances of the zero-coupon bond returns are deterministic functions of time. The prices of both path-dependent and path-independent contingent claims can be evaluated in terms of multivariate Gaussian integrals with two groups of input data: (1) the spot exchange rates and the present zero-coupon bonds of all currencies. (2) the covariances of these zero-coupon bond returns in their domestic values. The results are universal to all Gaussian interest rate models regardless of the number of market-driving factors and they can be implemented in non-parametric way. This non-parametric feature is particularly useful for an integrated portfolio management involving a wide range of FX and fixed-income products.

The main criticism of this model is the assumption that the covariance (volatility) surfaces of the zero-coupon bond returns are deterministic in time. As in most Gaussian interest rate models, we bear the risk of generating negative interest rates. Several pricing models of positive interest rates have been suggested in the literature; for example, the affine multi-factor model by Duffie and Kan [1996] which generalises the short rate model of Cox, Ingersoll and Ross [1985], the lognormal libor rate model of Sandmann and Sondermann [1993], the potential model by Rogers [1997] and a positive supermartingale model proposed by Flesaker and Hughston [1996]. The trade-offs for using a factor-independent Gaussian HJM model as shown in this paper are: (1) the spot yield curves and their volatility surfaces can be fitted easily and (2) exact and analytic pricings and hedgings of more complicated contingent claims can be easily evaluated. Empirical testing on a wide range of interest rate derivatives is now possible to be carried out without imposing too many parametric assumptions on the volatility surfaces.

A Appendix

A.1 Useful Identities

Let $g(\vec{x}, W)$ be a M-dimensional Gaussian density with a positive definite $M \times M$ correlation matrix W,

$$g(\vec{x}, W) \equiv \frac{1}{\sqrt{(2\pi)^M \det W}} e^{-\frac{1}{2}\vec{x}^T W^{(-1)} \vec{x}}, \qquad \int_{-\infty}^{\infty} d^M \vec{x}\, g(\vec{x}, W) = 1$$

Let the vector space be decomposed into $\mathbb{R}^M = \mathbb{R}^k \times \mathbb{R}^{M-k}$ such that $\vec{x} = (y_1, \ldots, y_k, z_{k+1}, \ldots, z_M)$ and the correlation matrix is decomposed as

$$W_{M\times M} = \begin{pmatrix} Y_{k\times k} & V_{k\times(M-k)} \\ V^T_{(M-k)\times k} & Z_{(M-k)\times(M-k)} \end{pmatrix}.$$

The multivariate Gaussian density satisfies the following properties:

$$\int_{-\infty}^{\infty} d^M \vec{x}\, g(\vec{x}, W) \exp(\vec{b}\cdot\vec{x}) = \exp(\frac{1}{2}\vec{b}^T W \vec{b}) \tag{A.1}$$

$$\int_{-\infty}^{\infty} d^k \vec{y}\, g(\vec{x}, W)\Phi(\vec{z}) = g(\vec{z}, Z)\Phi(\vec{z}) \tag{A.2}$$

$$\int_{-\infty}^{\infty} \frac{dz}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\, \Phi(z) z = \int_{-\infty}^{\infty} \frac{dz}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\, \partial_z \Phi(z) \tag{A.3}$$

The first identity gives the moment generating function and the second is the "dimensional-reduction" property. Both can be found in the book by Miller [1964]. The last identity is obtained from integration by part.

A.2 Proof of the Markov Transition Property

Let us rewrite the left hand side of equation (5.3) in terms of the following Gaussian integral,

$$\text{LHS} = \int_{-\infty}^{\infty} d\vec{x}^{(q)} \int_{-\infty}^{\infty} d\vec{x}^{(q+1)} g(\vec{x}^{(q)}, W^{(q)}) g(\vec{x}^{(q+1)}, W^{(q+1)}) \times$$

$$C\left[\tau_{q+1},\left\{\hat{P}_\alpha(\tau_{q-1},T_j)e^{\sum_{n=q}^{q+1} b_{\alpha,j}^{(n)}x_{\alpha,j}^{(n)}-\frac{1}{2}\left(b_{\alpha,j}^{(n)}\right)^2}\right\}\right]. \tag{A.4}$$

To evaluate this integral, let us introduce the following new variables

$$b_{\alpha,j}^2 \equiv \sum_{n=q}^{q+1}\left(b_{\alpha,j}^{(n)}\right)^2 = \int_{\tau_{q-1}}^{\tau_{q+1}} dt Z_{\alpha,\alpha}(t,T_j,T_j) \tag{A.5}$$

$$\mathcal{Z}_{\{\alpha,i\},\{\beta,j\}} \equiv \sum_{n=q}^{q+1}\mathcal{Z}_{\{\alpha,i\},\{\beta,j\}}^{(n)} = \int_{\tau_{q-1}}^{\tau_{q+1}} dt\, Z_{\alpha,\beta}(t,T_i,T_j) \tag{A.6}$$

$$b_{\alpha,j}\, y_{\alpha,j} \equiv \sum_{n=q}^{q+1} b_{\alpha,j}^{(n)}\, x_{\alpha,j}^{(n)} \tag{A.7}$$

then, the exponents in the integrand of equation (A.4) is

$$\sum_{n=q}^{q+1}\left(b_{\alpha,j}^{(n)}x_{\alpha,j}^{(n)}-\frac{1}{2}\left(b_{\alpha,j}^{(n)}\right)^2\right) = b_{\alpha,j}y_{\alpha,j}-\frac{1}{2}b_{\alpha,j}^2. \tag{A.8}$$

Replacing $\vec{x}^{(q+1)}$ by the new variables $\vec{y}$ and integrating over $d\vec{x}^{(q)}$, we have

$$\begin{aligned} d\vec{x}^{(q+1)}\int_{-\infty}^{\infty} d\vec{x}^{(q)}\, g(\vec{x}^{(q)},W^{(q)})g(\vec{x}^{(q+1)},W^{(q+1)})\, C[\tau_{q+1},\{p_{\alpha,j}^{(q+1)}\}] \\ = d\vec{y}\, g(\vec{y},Y)\, C\left[\tau_{q+1},\{p_{\alpha,j}^{(q-1)}e^{b_{\alpha,j}y_{\alpha,j}-\frac{1}{2}b_{\alpha,j}^2}\}\right], \end{aligned} \tag{A.9}$$

where the new correlation matrix Y is given by

$$\vec{b}^T\cdot Y\cdot\vec{b} = \left(\mathcal{Z}^{(q)}\right)^{-1}\frac{1}{\left(\mathcal{Z}^{(q)}\right)^{-1}+\left(\mathcal{Z}^{(q+1)}\right)^{-1}}\left(\mathcal{Z}^{(q+1)}\right)^{-1} = \frac{1}{\mathcal{Z}} \tag{A.10}$$

From equation (A.6), $g(\vec{y},Y)$ is the transition density from τ_{q+1} to τ_{q-1}. Thus equation (5.3) is established.

A.3 Proof of the path-dependent pricing formula in equation (5.4)

Let us first rewrite the path-dependent pricing formula in equation (5.4) in terms of the multivariate Gaussian integral:

$$C(s) = \int_{-\infty}^{\infty} d\vec{x}^{(1)} \dots \int_{-\infty}^{\infty} d\vec{x}^{(Q)}\, g(\vec{x}^{(1)}, W^{(1)}) \dots g(\vec{x}^{(Q)}, W^{(Q)}) \times$$
$$\Phi\left[\tau_Q, \left\{p^{(1)}, \dots p^{(Q)}\right\}\right] \theta\left(\phi_1(p^{(1)})\right) \dots \theta\left(\phi_Q(p^{(Q)})\right) \qquad \text{(A.11)}$$

where $p^{(q)}$ denotes the collection of all zero-coupon bonds at time τ_q.

$$p^{(q)} \equiv \left\{ p^{(q)}_{\alpha,i} = \hat{P}_\alpha(s, T_i)\, e^{\sum_{n=1}^{q}\left(b^{(n)}_{\alpha,i} x^{(n)}_{\alpha,i} - \frac{1}{2}\left(b^{(n)}_{\alpha,i}\right)^2\right)}, \right.$$
$$\left. \forall\, \alpha = 0, \dots, m; \quad i = q, \dots, N \right\}$$

If no decision needs to be made at τ_q, then $\phi_q = 1$. To check the boundary condition at the option expiry τ_Q, we can use the definitions of the b's and W's in equation (3.4). At the option expiry τ_Q, we have $\vec{b}^{(q)}(\tau_Q) = 0$ and $W^{(q)}$'s become the identity matricies. Hence $p^{(q)}_{\alpha,i} = \hat{P}_\alpha(s, T_i)$ and the integral reproduces the required terminal pay-off

$$C(\tau_Q) = \Phi(\tau_Q, \{\hat{P}_\alpha(\tau_q, \cdot)\})\theta\left(\phi_1[\tau_1, \hat{P}_\alpha(\tau_1, \cdot)]\right) \dots \theta\left(\phi_Q[\tau_Q, \hat{P}_\alpha(\tau_Q, \cdot)]\right).$$

To check that the Kolmogorov equation is satisfied by equation (5.4), we first evaluate the functional derivatives in terms of the ordinary differentiation multiplied by a Dirac's delta function,

$$\frac{\delta \Phi[p_\alpha(s,T)]}{\delta \hat{P}_\beta(s,u)} = \delta(u-T)\delta_{\alpha,\beta}\left(\frac{\partial p_\alpha(s,T)}{\partial \hat{P}_\alpha(s,T)}\right)\frac{\partial \Phi[p_\alpha(s,T)]}{\partial p_\alpha(s,T)}, \quad \forall\, u, T \in [s, L]$$

where the Kronecker delta function $\delta_{\alpha,\beta}$ and the Dirac delta function $\delta(u-T)$ are defined as follows

$$\delta_{\alpha,\beta} = \begin{cases} 1 & \text{if} \quad \alpha = \beta \\ 0 & \text{otherwise} \end{cases}$$
$$\int_0^L du\; H(u)\delta(u-T) = \begin{cases} H(T) & \text{for} \quad T \in [0, L] \\ 0 & \text{otherwise} \end{cases}$$

Then using the chain rule of ordinary differentiation and the homogeneous property of the contingent claim,

$$\Phi = \sum_{q=1}^{Q}\sum_{\alpha=0}^{m}\sum_{j=q}^{N}\hat{P}_\alpha(\tau_q,T_j)\frac{\partial}{\partial\hat{P}_\alpha(\tau_q,T_j)}\Phi\left[\tau_Q,\left\{p^{(1)},\ldots,p^{(Q)}\right\}\right]$$

$$\phi_q = \sum_{\alpha=0}^{m}\sum_{j=q}^{N}\hat{P}_\alpha(\tau_q,T_j)\frac{\partial}{\partial\hat{P}_\alpha(\tau_q,T_j)}\phi_q\left[\tau_q,\left\{p^{(q)}\right\}\right]$$

it is easy to see that the two terms which depend on the domestic short rate in Kolmogorov's equation cancel each other

$$\begin{aligned}
& r_0\int_0^L du\,\sum_{\alpha=0}^{m}\hat{P}_\alpha(s,u)\frac{\delta C}{\delta\hat{P}_\alpha(s,u)} \\
&= r_0\int_{-\infty}^{\infty}d\vec{x}^{(1)}\ldots\int_{-\infty}^{\infty}d\vec{x}^{(Q)}\,g(\vec{x}^{(1)},W^{(1)})\ldots g(\vec{x}^{(Q)},W^{(Q)})\times \\
&\qquad \sum_{q=1}^{Q}\sum_{\alpha=0}^{m}\sum_{i=q}^{N}\Big\{\theta(\phi_1)\ldots\theta(\phi_Q)\,p^{(q)}_{\alpha,i}\frac{\partial}{\partial p^{(q)}_{\alpha,i}}\Phi \\
&\qquad\qquad +\Phi\frac{\theta(\phi_1)\ldots\theta(\phi_Q)}{\theta\left(\phi_q\right)}\,\delta\left(\phi_q\right)\,p^{(q)}_{\alpha,i}\frac{\partial}{\partial p^{(q)}_{\alpha,i}}\phi_q\Big\} \\
&= r_0\int_{-\infty}^{\infty}d\vec{x}^{(1)}\ldots\int_{-\infty}^{\infty}d\vec{x}^{(Q)}\,g(\vec{x}^{(1)},W^{(1)})\ldots g(\vec{x}^{(Q)},W^{(Q)})\,\Phi\,\theta(\phi_1)\ldots\theta(\phi_Q).
\end{aligned} \tag{A.12}$$

Note that the second term in the sum drops out because the first derivative of the step function gives a Dirac delta function, $\partial_\phi\theta(\phi) = \delta(\phi)$ and the integration over $\phi\delta(\phi)$ gives zero.

Similarly, the diffusion term in Kolmogorov's equation can be evaluated with the chain rule:

$$\begin{aligned}
& \frac{1}{2}\int_0^L du_1\ldots\int_0^L du_2\sum_{\alpha,\beta}Z_{\alpha,\beta}(s,u_1,u_2)\hat{P}_\alpha(s,u_1)\hat{P}_\beta(s,u_2)\frac{\delta^2 C}{\delta\hat{P}_\alpha(s,u_1)\delta\hat{P}_\beta(s,u_2)} \\
&= \int_{-\infty}^{\infty}d\vec{x}^{(1)}\ldots\int_{-\infty}^{\infty}d\vec{x}^{(Q)}\,g(\vec{x}^{(1)},W^{(1)})\ldots g(\vec{x}^{(Q)},W^{(Q)})\times \\
& \frac{1}{2}\sum_{\alpha,\beta=0}^{m}\sum_{q,q'=1}^{Q}\sum_{i=q}^{N}\sum_{j=q'}^{N}\,Z_{\alpha,\beta}(s,T_i,T_j)\,p^{(q)}_{\alpha,i}p^{(q')}_{\beta,j}\frac{\partial}{\partial p^{(q)}_{\alpha,i}}\frac{\partial}{\partial p^{(q')}_{\beta,j}}\left(\Phi\,\theta(\phi_1)\ldots\theta(\phi_Q)\right)
\end{aligned} \tag{A.13}$$

Finally, to calculate the time derivative $\partial_s C$, it is easier to first diagonalise each correlation matrix $W^{(q)}$ into its eigenvalues $\lambda^{(q)}_{\alpha,j}$ by changing the basis from $\vec{x}^{(q)}$ to $\vec{z}^{(q)}$ via $\vec{x}^{(q)} = S^{(q)} D^{(q)} \vec{z}^{(q)}$, such that

$$W^{(q)} = S^{(q)} D^{(q)} \left(D^{(q)}\right)^T \left(S^{(q)}\right)^T, \quad D^{(q)} \equiv \begin{pmatrix} \frac{1}{\sqrt{\lambda^{(q)}_{0,q}}} & 0 & \dots & 0 \\ 0 & \frac{1}{\sqrt{\lambda^{(q)}_{0,q+1}}} & \dots & 0 \\ . & . & . & . \\ 0 & 0 & \dots & \frac{1}{\sqrt{\lambda^{(q)}_{m,N}}} \end{pmatrix}$$

In the new basis, the density function can be rewritten in terms of uncorrelated Gaussian densities, $\hat{g}(z) \equiv \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$ and the time dependence is transformed into the exponents in the integrand. We can then take the time-derivatives and re-arrange the integral using integration by part as in equation (A.3). The result in terms of the original variables is

$$\begin{aligned} \partial_s C = & \int_{-\infty}^{\infty} d\vec{z}^{(1)} \dots \int_{-\infty}^{\infty} d\vec{z}^{(Q)} \left(\prod_{\alpha=0}^{m} \prod_{q=1}^{Q} \prod_{j=q}^{N} \hat{g}(z^{(q)}_{\alpha,j}) \right) \times \\ & \sum_{\alpha=0}^{m} \sum_{q=1}^{Q} \sum_{i=q}^{N} \partial_s \Big\{ b^{(q)}_{\alpha,i} \left(S^{(q)} D^{(q)} \vec{z}^{(q)} \right)_{\alpha,i} - \frac{\left(b^{(q)}_{\alpha,i}\right)^2}{2} \Big\} p^{(q)}_{\alpha,i} \frac{\partial}{\partial p^{(q)}_{\alpha,i}} (\Phi \theta(\phi_1) \dots \theta(\phi_Q)) \\ = & \int_{-\infty}^{\infty} d\vec{x}^{(1)} g(\vec{x}^{(1)}, W^{(1)}) \dots \int_{-\infty}^{\infty} d\vec{x}^{(Q)} g(\vec{x}^{(Q)}, W^{(Q)}) \sum_{\alpha,\beta=0}^{m} \sum_{q,q'=1}^{Q} \sum_{i=q}^{N} \sum_{j=q'}^{N} \\ & \frac{1}{2} \partial_s \Big\{ \sum_{n=1}^{\min(q,q')} b^{(n)}_{\alpha,i} W^{(n)}_{\{\alpha,i\},\{\beta,j\}} b^{(n)}_{\beta,j} \Big\} p^{(q)}_{\alpha,i} p^{(q')}_{\beta,j} \frac{\partial}{\partial p^{(q)}_{\alpha,i}} \frac{\partial}{\partial p^{(q')}_{\beta,j}} (\Phi \theta(\phi_1) \dots \theta(\phi_Q)). \end{aligned} \tag{A.14}$$

From equation (3.4),

$$\sum_{n=1}^{\min(q,q')} b^{(n)}_{\alpha,i} W^{(n)}_{\{\alpha,i\},\{\beta,j\}} b^{(n)}_{\beta,j} = \int_{s}^{\min(\tau_q,\tau_{q'})} dt\, Z_{\alpha,\beta}(t, T_i, T_j).$$

Thus (A.14) cancels with (A.13) and the pricing formula in equation (5.4) is the desired solution of Kolmogorov's equation (2.9).

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