The Pricing of Options Depending on a Discrete Maximum

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Abstract

The object of this paper is to provide a price for options depending on the maximum value of the underlying asset when the contract stipulates that the maximum is taken at discrete times during the life of the option. A general method is provided for the computation of a lower bound to the price using results on the Brownian meander complementing the commonly used upper bound which approximates the discrete maximum by the continuous maximum.. Moreover, in the particular case of the European Put Lookback option, a correction factor associated to the lower bound provides a very good numerical approximation to the price of the option which was otherwise obtained by Monte-Carlo methods.

Keywords Brownian Meander, Lookback options, Denisov Decomposition.

1 Introduction

In the last few years the "over the counter" (OTC) market has seen the interest of the public for lookback and barrier options increase. These options have in common the fact that their intrinsic values are functions of the maximum value of the stock over a predetermined length of time (often the life of the option).
The contract of the option often specifies that the maximum value is taken over the daily closing prices of the stock. Therefore, in order to price the option, a discrete maximum needs to be considered. However, (mainly for reasons of practicality and simplicity) the model used by most practitioners follows the Black-Scholes assumptions for the behavior of the stock and approximates the discrete maximum by the continuous maximum of the stock under this model.

We will also assume the Black-Scholes framework as a model for the behavior of the stock which can be formalized as follows:
If we consider a stock whose value is S_t at time t and we define a standard Brownian motion W on a probability space $(\Omega, \mathcal{F}, P)$ where the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is the filtration generated by the Brownian motion augmented to satisfy the usual conditions, the stock price satisfies the following stochastic differential equation (SDE):

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{1}$$

Where μ and σ are constants respectively corresponding to the return and the volatility of the stock S.

Define the risk-neutral probability Q by its Radon-Nikodym derivative with respect to P as follows:

$$\left(\frac{dQ}{dP}\right)_T = \exp\left(-(\frac{\mu - r}{\sigma})\tilde{W}_T + \frac{(\mu - r)^2}{2\sigma^2}\right) \tag{2}$$

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Here r is the instantaneous interest rate and, from Girsanov's theorem, $\tilde{W}_t = W_t + (\frac{\mu - r}{\sigma})t$ for all $t \geq 0$ is a standard Brownian motion under Q.

We can then solve the SDE (1) above and obtain:

$$S_t = S_0 e^{(r - \frac{\sigma^2}{2})t + \sigma \tilde{W}_t} \tag{3}$$

Moreover, Q also being a martingale measure for the process $\{e^{-r \cdot} S_{\cdot}\}_{t \geq 0}$, we obtain the following formula for the price of an option whose payoff is a function of the discrete maximum of the price of the stock on a finite interval $[0, T]$ and the value of the stock at time T:

$$\pi_0 = E_Q \left[e^{-rT} f(\max_{0 \leq k \leq n} S_{kh}, S_T) \right] \tag{4}$$

Where $T = nh$. We assume that the discrete points corresponding to the daily closing prices are evenly spaced with the length of the space interval equal to h.

We now consider, as an example, the pricing of the European Put lookback option which gives to its holder at maturity, the payoff $\max_{0 \leq k \leq n} S_{kh} - S_{nh}$. The usual technique of approximating the discrete maximum $\max_{0 \leq k \leq n} S_{kh}$ by the continuous maximum $\max_{0 \leq t \leq T} S_t$ leads to the following formula for the price of the option(see Hull[10], for instance), using mainly the distributions derived in Shepp[19]:

$$\pi_0 = S_0 e^{-rT} \left[N(b_1) - \frac{\sigma^2}{2r} N(b_1) \right] + S_0 \frac{\sigma^2}{2r} N(-b_2) - S_0 N(b_2) \tag{5}$$

Where $N(x) = \int_{-\infty}^{x} \frac{e^{\frac{-u^2}{2}}}{\sqrt{2\pi}} du$ is the cumulative distribution of a standard normal random variable and:

- $b_1 = \frac{(-r + \frac{\sigma^2}{2})T}{\sigma \sqrt{T}}$,
- $b_2 = b_1 - \sigma \sqrt{T}$.

Clearly, as the discrete maximum was replaced by a continuous one, the formula above only constitutes an upper bound to the price of the option. In general, this is the case as long as the payoff of the option is an increasing function of the maximum.

This paper provides a lower bound to the price of the option by using a different approximation to the discrete maximum. More precisely, instead of considering the continuous maximum, we consider the two discrete times closest to the time of the continuous maximum and use the values of the price of the stock at one or both these times as an approximation of the discrete maximum by default. We then concentrate on the pricing of the European Put Lookback option and obtain numerical results for this lower bound. Moreover, in that particular case, we can derive a correction factor providing very precise approximations when compared to Monte-Carlo simulations.

Section 2 describes the lower bound used to approximate the discrete maximum of a Brownian motion and justifies its appropriateness. Then, Section 3 introduces the concept of Brownian meander. Using the properties of the Brownian meander, we then derive a lower bound for the price of an option whose payoff is an increasing function of the discrete maximum of the stock over a fixed interval. In Section 4, a formula for the lower bound of the price of the European Put Lookback option is obtained using a fairly simple numerical integration. A correction factor is also calculated leading to a more precise approximation. Finally, Section 5 provides tables summarizing numerical results obtained using the formula derived in the previous section.

2 Approximation of the Discrete Maximum

2.1 Another Change of Probability Measure

As we saw in the introduction (Equation 4)

$$\pi_0 = E_Q\left[e^{-rT} f(\max_{0\le k\le n} S_0 e^{(r-\frac{\sigma^2}{2})kh+\sigma\tilde{W}_{kh}}, S_0 e^{(r-\frac{\sigma^2}{2})T+\sigma\tilde{W}_T})\right]$$

We now define the probability Q^*, absolutely continuous with respect to Q, whose Radon-Nikodym derivative with respect to Q has the following expression:

$$\left(\frac{dQ^*}{dQ}\right)_T = \exp[-\frac{r-\frac{\sigma^2}{2}}{\sigma}\tilde{W}_T - \frac{1}{2}(\frac{r-\frac{\sigma^2}{2}}{\sigma})^2 T]$$

If we define the process W^* by:

$$W_t^* = \tilde{W}_t + \frac{r-\frac{\sigma^2}{2}}{\sigma}t \qquad t \ge 0 \tag{6}$$

We know by Girsanov's Theorem (see Revuz and Yor[18]) that W^* is a standard Brownian motion under Q^*.

We can then rewrite π_0 as:

$$\pi_0 = e^{-rT} e^{-\frac{1}{2}(\frac{r-\frac{\sigma^2}{2}}{\sigma})^2 T} E_{Q^*}\left[\exp\left(\frac{r-\frac{\sigma^2}{2}}{\sigma}W_T^*\right) f\left(S_0\exp(\sigma\max_{0\le k\le n} W_{kh}^*), S_0\exp(\sigma W_T^*)\right)\right] \tag{7}$$

From now on, in order to simplify the notation, we will remove the "$*$" as we will work under the probability Q^*.

Computing the price described above basically involves deriving the joint distribution of W_T and $\max_{0\le k\le n} W_{kh}$). This turns out to be very problematic as the difficulty increases with n to the point where it is not feasible numerically (in a reasonable amount of time) for even fairly small values of n. As we saw in the introduction, by replacing the discrete maximum by the continuous maximum, we can use the results derived in Shepp[19] to find a closed formula which constitutes an upper bound to π_0 (when f is an increasing function on its first coordinate). Now, in order to provide a lower bound, which would also be a good approximation, to π_0 we introduce the "quasi-maximum".

2.2 The "Quasi Maximum"

We will, from this point further and without loss of generality, assume that $T = 1$ so that we are considering a Brownian motion on $[0, 1]$.

We introduce some new notation:

- $\nu = \inf\{0 \le t \le 1 : W_t = \sup_{0\le s\le 1} W_s\}$
- $k^* = \inf\{0 \le k \le n : W_{kh} = \max_{0\le j\le n} W_{jh}\}$
- $l = [\nu n] = [\frac{\nu}{h}]$

In other words, ν is the time of the continuous maximum, k^* is the time of the discrete maximum and l is the first point in the discretization to the left of ν and is the time of the "quasi-maximum".

The idea of this paper is to use W_{lh} to approximate W_{k^*h}. The first reason for using the quasi-maximum is a practical one. As will be seen in details in section 3, it is possible to rewrite the quasi-maximum W_{lh} using Brownian meanders and then use properties of the latter to compute the expectation of Equation (7). The other reason, and that is what this section is about, comes from the fact that W_{lh} provides a good approximation to W_{k^*h}, especially as n gets larger.

Indeed, some results on the error $\epsilon_n = W_{k^*h} - W_{lh}$ are derived in Nahum[16] that we summarize here:

Proposition 1 *If* $\epsilon_n = W_{\frac{k^*}{n}} - W_{\frac{l}{n}}$

$$\begin{aligned} E[\epsilon_n] &= \int_0^1 \frac{2}{\pi^{\frac{3}{2}}\sqrt{2(1-y)}} \left[Arctan\left(\sqrt{\frac{1-\frac{[yn]}{yn}}{\frac{[yn]}{yn}}}\right) + \sqrt{\frac{[yn]}{yn}(1-\frac{[yn]}{yn})} \right] dy - \frac{0.5826}{\sqrt{n}} + o(\frac{1}{\sqrt{n}}) \\ &= O(\frac{1}{\sqrt{n}}) \end{aligned} \tag{8}$$

So that ϵ_n *converges to* 0 *in* L^1 *at the rate of* $\frac{1}{\sqrt{n}}$.

Proposition 2

$$\sqrt{n}\epsilon_n \xrightarrow[n\to\infty]{d} R(U) - \min_{n=0,\pm1,\pm2,\cdots} R(U+n) \tag{9}$$

Where R is a Bessel process in dimension 3 and U is a Uniform random variable on $[0,1]$ *independent of R.*

Those two results justify the use of the quasi-maximum as an approximation to the discrete maximum since the error made is in the order of $\frac{1}{\sqrt{n}}$. Proposition 1 will also turn out to be useful for the derivation of the correction factor in Section 4 when computing the price of the European Put Lookback.

It would have been possible to define the quasi-maximum as $W_{(l+1)h}$ which corresponds to the value of the Brownian motion at the first point to the right of the time ν of the continuous maximum in the discretization. We then obtain results very similar to the ones described in Proposition 1 and 2. Moreover, it is as practical to compute the price of the option using $W_{(l+1)h}$ as it is with W_{lh}. Therefore, a more precise lower bound to the price of the option can be obtained by computing it using both the quasi-maxima and then considering the maximum of the two prices obtained. An even better lower bound would be obtained by using $\max(W_{lh}, W_{(l+1)h})$ as an approximation. However, (as will be noted in section 3) this would complicate the calculations greatly even though in some particular cases like the European Put Lookback it is still possible to obtain a fairly reasonable two-dimensional integral for the price using this approximation.

We now turn to the practical aspect of approximating the discrete maximum by the quasi-maximum introducing the notion of Brownian meander and Denisov decomposition.

3 The Brownian Meander and General Method

3.1 Definition and Denisov Decomposition

Let $\{W_t\}_{t \geq 0}$ be a standard Brownian motion defined as above.

Definition 1 *If we denote by g the last zero of the Brownian motion before 1: $g = \sup\{t \in [0,1]; W(t) = 0\}$ then the Brownian meander $\{M_t\}_{0 \leq t \leq 1}$ is defined by:*

$$M_t = \frac{|W_{g+(1-g)t}|}{\sqrt{1-g}} \tag{10}$$

Recalling that ν denotes the time the maximum of the Brownian motion on $[0,1]$ is achieved, we get the following theorem due to Denisov[5]:

Theorem 1

$$M_t^+ = \frac{W_\nu - W_{\nu+(1-\nu)t}}{\sqrt{1-\nu}} \qquad 0 \leq t \leq 1 \tag{11}$$

$$M_t^- = \frac{W_\nu - W_{\nu(1-t)}}{\sqrt{\nu}} \qquad 0 \leq t \leq 1 \tag{12}$$

are two independent Brownian meanders and are independent of ν.

This is also referred to as the Denisov decomposition as it decomposes the path of the Brownian motion in two parts, one on each side of the continuous maximum ν. This result is crucial to all the calculations derived throughout this paper especially because of the independence property it brings into light.

We now introduce the following notation:

$$n_s(x) = \frac{e^{-\frac{x^2}{2s}}}{\sqrt{2\pi s}}$$

$$N_s(a,b) = \int_a^b n_s(x)dx$$

Note that $N_1(-\infty, b) = N(b)$.

A Brownian meander is an inhomogeneous Markov process (see Durrett, Iglehart and Miller[7]) with marginal density:

$$d(t,y) = P(M_t \in dy) = \frac{2y}{t^{\frac{3}{2}}} e^{-\frac{y^2}{2t}} N_{1-t}(0,y)dy \qquad 0 \leq t \leq 1, y \geq 0 \tag{13}$$

and transition density:

$$tr(s,x,t,y) = P(M_t \in dy | M_s \in dx) = (n_{t-s}(y-x) - n_{t-s}(y+x)) \frac{N_{1-t}(0,y)}{N_{1-s}(0,x)} \qquad 0 \leq s < t \leq 1, x, y \geq 0 \tag{14}$$

We have now established the necessary background to allow for the calculations of the lower bound.

3.2 General Method

The key to computing a lower bound to the price of path-dependent options based on a discrete maximum using the quasi-maximum described above is to "translate" the expressions containing values of a Brownian motion into expressions only written in terms of the Brownian meanders M^+ and M^- in order to be able to use the independence property of the Denisov decomposition. Towards this objective, we can easily notice the following:

$$W_1 = \sqrt{\nu} M_1^- - \sqrt{1-\nu} M_1^+ \tag{15}$$

$$W_{lh} = \sqrt{\nu}\left[M_1^- - M_{1-\frac{lh}{\nu}}^-\right] \tag{16}$$

$$W_{(l+1)h} = \sqrt{\nu} M_1^- - \sqrt{1-\nu} M_{\frac{(l+1)h-\nu}{1-\nu}}^+ \tag{17}$$

It makes a lot of sense to go from the Brownian motion to the Brownian meander as we are studying the behavior of the path of the Brownian motion near its maximum and the Denisov decomposition allows us to look at both sides of the path (to the left and to the right of ν) independently of each other and independently of ν.

According to Equation 7 and replacing W_{k^*h} by the quasi-maximum W_{lh} we need to compute the following expectation E:

$$E = E\left[\exp\left(cW_1\right) f\left(S_0 \exp(\sigma W_{lh}), S_0 \exp(\sigma W_1)\right)\right] \tag{18}$$

With $c = \frac{r - \frac{\sigma^2}{2}}{\sigma}$.

This expectation can now be written using the equalities (15) and (16) which leads to:

$$\begin{aligned} E \;=\; & E\Big[\exp\Big(c\left[\sqrt{\nu}M_1^- - \sqrt{1-\nu}M_1^+\right]\Big) \\ & f\left(S_0 \exp\left(\sigma\sqrt{\nu}\left[M_1^- - M_{1-\frac{lh}{\nu}}^-\right]\right), S_0 \exp\left(\sigma\left[\sqrt{\nu}M_1^- - \sqrt{1-\nu}M_1^+\right]\right)\right)\Big]\end{aligned}$$

Now, by conditioning on ν, which follows the Arcsine law (see Karatzas and Shreve[13]) and using the independence of the two Brownian meanders with ν (see Theorem 1), we get:

$$\begin{aligned} E \;=\; & \int_0^1 \frac{dy}{\pi\sqrt{y(1-y)}} E\Big[\exp\Big(c\left[\sqrt{y}M_1^- - \sqrt{1-y}M_1^+\right]\Big) \\ & f\left(S_0 \exp\left(\sigma\sqrt{y}\left[M_1^- - M_{1-\frac{\lceil yn\rceil}{yn}}^-\right]\right), S_0 \exp\left(\sigma\left[\sqrt{y}M_1^- - \sqrt{1-y}M_1^+\right]\right)\right)\Big]\end{aligned} \tag{19}$$

Because of the independence of the two Brownian meanders M^+ and M^- and the equalities 13 and 14 giving, respectively, the marginal and transition densities of these processes, the expression above is (at worst) a 4-dimensional integral. It is therefore possible to numerically integrate the expression (and in a very reasonable amount of time). As will be described in details in the following question, the expression can be greatly simplified for "nice" enough functions f. This is the case when considering lookback options and even common barrier options.

Before getting into the particular case of the European Put Lookback option, note that we obtain a very similar expression when using $W_{(l+1)h}$ as an approximation. Indeed the expectation then has the following form (using Equation (17)):

$$
\begin{aligned}
E &= \int_0^1 \frac{dy}{\pi\sqrt{y(1-y)}} E\Big[\exp\left(c\left[\sqrt{y}M_1^- - \sqrt{1-y}M_1^+\right]\right) \\
& f\left(S_0\exp\left(\sqrt{y}M_1^- - \sqrt{1-y}M^+_{\frac{[yn]-ny+1}{n(1-y)}}\right), S_0\exp\left(\sigma\left[\sqrt{y}M_1^- - \sqrt{1-y}M_1^+\right]\right)\right)\Big] \quad (20)
\end{aligned}
$$

Taking the maximum value of the expectations 19 and 20 then provides the lower bound sought after.
Now, if we use $\max(W_{lh}, W_{(l+1)h})$ as an approximation to the discrete maximum, we can easily see from the calculations above that we then get the sum of two 5-dimensional integrals (at worst). Indeed, it just involves introducing two indicators specifying the set over which W_{lh} is greater than $W_{(l+1)h}$ and its compliment. This "detail" adds greatly to the difficulty of the problem as it slows down the integration. However, in the case of the Put Lookback, on which we now focus our attention, some tedious calculations (see 4.2) lead to the reduction of the expression to a 2-dimensional integral.

4 Pricing The European Put Lookback Option

The European Put Lookback option is an option that gives to its holder the possibility to sell the underlying stock at maturity T for the price of the maximum value the stock has reached during the life of the option. It contains a very attractive feature in the fact it prevents the investor from having any regrets for not having sold at the maximum possible. In this paper we are concerned with options whose contract stipulates that the maximum is considered among the closing prices everyday during the life of the option. Therefore, the payoff is: $\sup_{0\le k\le n} S_{kh} - S_n$.

Using the same method described in the beginning of this paper we easily obtain the following expression for the price π_0 of this option:

$$
\begin{aligned}
\pi_0 &= e^{-rT}e^{-\frac{1}{2}(\frac{r-\frac{\sigma^2}{2}}{\sigma})^2 T} E\left[S_0\exp\left(\frac{r-\frac{\sigma^2}{2}}{\sigma}W_1\right)\exp\left(\sigma\max_{0\le k\le n} W_{kh}\right)\right] - S_0 \\
&= S_0 e^{-rT}e^{-\frac{1}{2}(\frac{r-\frac{\sigma^2}{2}}{\sigma})^2 T} E\left[\exp\left(cW_1\right)\exp\left(\sigma W_{k^*h}\right)\right] - S_0 \quad (21)
\end{aligned}
$$

The problem is therefore reduced to computing the expectation:

$$E\left[\exp\left(cW_1\right)\exp\left(\sigma W_{k^*h}\right)\right] \quad (22)$$

4.1 A first Approximation

The first approximation corresponds to the lower bound described in the general method (see 3.2). In this particular case, the function f we need to consider is defined by $f(x, y) = x$.

In the following, we will denote by $NN(x, y, \rho)$ the cumulative distribution function of a bivariate normal with correlation coefficient ρ.

Proposition 3 *with the notation defined above:*

$$E\left[e^{cW_1}e^{\sigma W_{lh}}\right] = \int_0^1 \frac{1}{\pi\sqrt{y(1-y)}}\psi[(c+\sigma)\sqrt{y},\sigma\sqrt{y},1-\frac{lh}{y}]\phi(c\sqrt{1-y})dy \tag{23}$$

$$E\left[e^{cW_1}e^{\sigma W_{(l+1)h}}\right] = \int_0^1 \frac{1}{\pi\sqrt{y(1-y)}}\psi[-c\sqrt{1-y},\sigma\sqrt{1-y},\frac{(l+1)h-y}{1-y}]\phi(-(c+\sigma)\sqrt{y})dy \tag{24}$$

With:

$$\begin{aligned}\psi[\beta,\gamma,s] &= E\left[E[e^{\beta M_1}|M_s]e^{-\gamma M_s}\right]\\ &= \sqrt{2\pi}e^{\frac{\beta^2(1-s)}{2}}[(\beta-\gamma)e^{\frac{(\beta-\gamma)^2 s}{2}}NN((\beta-\gamma)\sqrt{s},\beta-\gamma s,\sqrt{s})\\ &+ (\beta+\gamma)e^{\frac{(\beta+\gamma)^2 s}{2}}NN(-(\beta+\gamma)\sqrt{s},\beta+\gamma s,-\sqrt{s})]+2e^{\frac{\gamma^2 s(1-s)}{2}}N(-\gamma\sqrt{s(1-s)}) \end{aligned} \tag{25}$$

(26)

and,

$$\begin{aligned}\phi(\lambda) &= E\left[e^{-\lambda M_1}\right]\\ &= 1-\lambda\sqrt{2\pi}e^{\frac{\lambda^2}{2}}N(-\lambda)\end{aligned} \tag{27}$$

Proof:

We will only prove the first equation (23) of the proposition as the second one follows exactly the same pattern.

From Equation (19):

$$E\left[e^{cW_1}e^{\sigma W_{lh}}\right] = \int_0^1 \frac{dy}{\pi\sqrt{y(1-y)}}E\Big[\exp\left(c\left[\sqrt{y}M_1^- - \sqrt{1-y}M_1^+\right]\right)\exp\left(\sigma\sqrt{y}\left[M_1^- - M^-_{1-\frac{[yn]}{yn}}\right]\right)\Big]$$

Using the Denisov Decomposition Theorem (Theorem 1) we know that the processes M^+ and M^- are independent, which leads to:

$$E\left[e^{cW_1}e^{\sigma W_{lh}}\right] = \int_0^1 \frac{dy}{\pi\sqrt{y(1-y)}}E\left[e^{-c\sqrt{1-y}M_1^+}\right]E\left[e^{(c+\sigma)\sqrt{y}M_1^- - \sigma\sqrt{y}M^-_{1-\frac{lh}{y}}}\right]$$

Finally by conditioning on $M^-_{1-\frac{lh}{y}}$ in the second expectation above we get:

$$E\left[e^{cW_1}e^{\sigma W_{lh}}\right] = \int_0^1 \frac{dy}{\pi\sqrt{y(1-y)}}E\left[e^{-c\sqrt{1-y}M_1^+}\right]E\left[E[e^{(c+\sigma)\sqrt{y}M_1^-}|M^-_{1-\frac{lh}{y}}]e^{-\sigma\sqrt{y}M^-_{1-\frac{lh}{y}}}\right]$$

which provides Equation (23).

The expressions for $\phi(\lambda)$ and $\psi(\beta,\gamma,s)$ are derived in details in **Appendix A**.

□

By taking the maximum of the two lower bounds calculated in the proposition above we obtain the lower bound sought after. As will be observed in Section 5, this approximation turn out to be closer to the Monte-Carlo simulation than the usual approximation using the continuous maximum, for most of the examples covered in the table. Moreover, this approximation has the advantage of being extremely fast to compute as it only requires a numerical integration in one dimension with a fairly smooth integrand which contains cumulative normal and bivariate normal functions (for which it is possible to use very accurate approximations).

4.2 A more precise lower bound

Clearly, a better approximation to W_{k^*h} would be $\max(W_{lh}, W_{(l+1)h})$. Although it makes the calculations more difficult, it is still possible to derive a relatively simple expression which leads to a fast numerical integration.

Before establishing the formula for this approximation, we first need to define a few functions:

$$\begin{aligned}\xi(\lambda,s,v) &= E[e^{-\lambda M_1}|M_s\in dv]\\ &= \frac{e^{\frac{\lambda^2(1-s)}{2}}}{2N_{1-s}(0,v)}\left[e^{-\lambda v}N[(-\lambda+\frac{v}{1-s})\sqrt{1-s}]-e^{\lambda v}N[-(\lambda+\frac{v}{1-s})\sqrt{1-s}]\right] \quad (28)\end{aligned}$$

$$\begin{aligned}\psi_1(\beta,\gamma,s,b) &= E\left[E[e^{\beta M_1}|M_s]e^{-\gamma M_s}I_{M_s\le b}\right]\\ &= \sqrt{2\pi}e^{\frac{\beta^2(1-s)}{2}}\Bigg[(\beta-\gamma)e^{\frac{(\beta-\gamma)^2 s}{2}}[NN\Big((\beta-\gamma)\sqrt{s},\beta-\gamma s,\sqrt{s}\Big)-NN\Big(\frac{(\beta-\gamma)s-b}{\sqrt{s}},\beta-\gamma s,\sqrt{s}\Big)]\\ &+ (\beta+\gamma)e^{\frac{(\beta+\gamma)^2 s}{2}}[NN\Big(-(\beta+\gamma)\sqrt{s},\beta+\gamma s,-\sqrt{s}\Big)-NN\Big(-\frac{(\beta+\gamma)s+b}{\sqrt{s}},\beta+\gamma s,-\sqrt{s}\Big)]\Bigg]\\ &+ 2e^{\frac{\gamma^2 s(1-s)}{2}}[N(\frac{b+\gamma s(1-s)}{\sqrt{s(1-s)}})-N(-\gamma\sqrt{s(1-s)})]\\ &+ \frac{e^{\frac{\beta^2(1-s)}{2}}e^{-\frac{b^2}{2s}}}{\sqrt{s}}\left[e^{-(\beta+\gamma)b}N\left(\frac{\beta(1-s)-b}{\sqrt{1-s}}\right)-e^{(\beta-\gamma)b}N\left(\frac{\beta(1-s)+b}{\sqrt{1-s}}\right)\right] \quad (29)\end{aligned}$$

and,

$$\begin{aligned}\psi_2(\beta,\gamma,s,b) &= E\left[E[e^{\beta M_1}|M_s]e^{-\gamma M_s}I_{M_s>b}\right]\\ &= \psi(\beta,\gamma,s)-\psi_1(\beta,\gamma,s,b) \quad (30)\end{aligned}$$

Proposition 4 *With the same notation as above:*

$$E\left[e^{cW_1}e^{\sigma\max(W_{lh},W_{(l+1)h})}\right]=E_1+E_2 \quad (31)$$

where,

$$E_1 = \int_{y=0}^{1}\int_{v=0}^{\infty}\frac{d\left(\frac{(l+1)h-y}{1-y},v\right)\xi\left(c\sqrt{1-y},\frac{(l+1)h-y}{1-y},v\right)\psi_1\left((c+\sigma)\sqrt{y},\sigma\sqrt{y},1-\frac{lh}{y},\sqrt{\frac{1-y}{y}}v\right)}{\pi\sqrt{y(1-y)}}dvdy \quad (32)$$

$$E_2 = \int_{y=0}^{1}\int_{v=0}^{\infty}\frac{e^{-\sigma\sqrt{1-y}v}d\left(\frac{(l+1)h-y}{1-y},v\right)\xi\left(c\sqrt{1-y},\frac{(l+1)h-y}{1-y},v\right)\psi_2\left((c+\sigma)\sqrt{y},0,1-\frac{lh}{y},\sqrt{\frac{1-y}{y}}v\right)}{\pi\sqrt{y(1-y)}}dvdy \quad (33)$$

Proof:

The method used here is very similar to the one described in the proof of Proposition 3. The idea being to replace the expressions depending on the values of the Brownian motion by expressions only depending on values of Brownian meanders and ν.

$$\begin{aligned} E\left[e^{cW_1}e^{\sigma\max(W_{lh},W_{(l+1)h})}\right] &= E\left[e^{cW_1}e^{\sigma W_{lh}}I_{W_{lh}\geq W_{(l+1)h}}\right] + E\left[e^{cW_1}e^{\sigma W_{(l+1)h}}I_{W_{lh}<W_{(l+1)h}}\right] \\ &= E_1 + E_2 \end{aligned}$$

We will concentrate on E_1 as the formula for E_2 can be obtained in exactly the same manner. By using Equations 15, 16 and 17 we get:

$$\begin{aligned} E_1 &= E\left[e^{c[\sqrt{\nu}M_1^- - \sqrt{1-\nu}M_1^+]}e^{\sigma\sqrt{\nu}[M_1^- - M^-_{1-\frac{lh}{\nu}}]}I_{\sqrt{\nu}[M_1^- - M^-_{1-\frac{lh}{\nu}}] > \sqrt{\nu}M_1^- - \sqrt{1-\nu}M^+_{\frac{(l+1)h-\nu}{1-\nu}}}\right] \\ &= E\left[e^{(c+\sigma)\sqrt{\nu}M_1^-}e^{-c\sqrt{1-\nu}M_1^+}e^{-\sigma\sqrt{\nu}M^-_{1-\frac{lh}{\nu}}}I_{\sqrt{\frac{\nu}{1-\nu}}M^-_{1-\frac{lh}{\nu}} < M^+_{\frac{(l+1)h-\nu}{1-\nu}}}\right] \end{aligned}$$

First, we condition on ν, which we know follows the Arcsine law. We then further condition on the couple $(M^+_{\frac{(l+1)h-\nu}{1-\nu}}, M_1^+)$ and we use the Theorem by Denisov (Theorem 1) to get the following expression:

$$\begin{aligned} E_1 &= \int_0^1 \frac{dy}{\pi\sqrt{y(1-y)}}\int_0^{+\infty} P(M^+_{\frac{(l+1)h-\nu}{1-\nu}} \in dv)E[e^{-c\sqrt{1-\nu}M_1^+}|M^+_{\frac{(l+1)h-\nu}{1-\nu}} \in dv] \\ &\quad E[e^{(c+\sigma)\sqrt{\nu}M_1^-}e^{-\sigma\sqrt{\nu}M^-_{1-\frac{lh}{\nu}}}I_{\sqrt{\frac{\nu}{1-\nu}}M^-_{1-\frac{lh}{\nu}}<v}] \\ &= \int_0^1 \frac{dy}{\pi\sqrt{y(1-y)}}\int_0^{+\infty} d\left(\frac{(l+1)h-\nu}{1-\nu}, v\right) E[e^{-c\sqrt{1-\nu}M_1^+}|M^+_{\frac{(l+1)h-\nu}{1-\nu}} \in dv] \\ &\quad E\left[E[e^{(c+\sigma)\sqrt{\nu}M_1^-}|M^-_{1-\frac{lh}{\nu}} \in dv]e^{-\sigma\sqrt{\nu}M^-_{1-\frac{lh}{\nu}}}I_{\sqrt{\frac{\nu}{1-\nu}}M^-_{1-\frac{lh}{\nu}}<v}\right] dv \end{aligned}$$

which provides the equation sought after.
The calculations leading to the formulas 28, 29 and 30 are detailed in **Appendix B**.

□

4.3 Correction Factor

The results on the first moment and the asymptotic distribution of the error $\epsilon_n = W_{k^*h} - W_{lh}$ described in Proposition 1 and 2 provide a multiplicative correction factor to the expectation 23 which, as can be seen in section 5 leads to an extremely good approximation of the price otherwise computed by Monte-Carlo simulations.

But first, to justify the use of the correction factor we prove the following lemma (which is adapted from a result in Asmussen, Glynn and Pitman[1] (AGP)):

Lemma 1 *For any $\beta < \infty$ the family $\{\exp(\beta n^{\frac{1}{3}}\epsilon_n)\}$ is uniformly integrable. In particular the families $\{n^{\frac{p}{3}}\epsilon_n^p\}$ are uniformly integrable for $p < \infty$.*

Proof:

$\delta_n = W_\nu - W_{lh} = W_\nu - W_{\frac{[\nu n]}{n}}$ is clearly an upper bound for ϵ_n, therefore it suffices to prove that:

$$\limsup_{n\to+\infty} E\left[e^{2\beta n^{\frac{1}{3}}\delta_n}\right] < \infty \tag{34}$$

Conditionally on $\nu = y, W_\nu = m, W_1 = m - x$, we know from Proposition 2 of AGP that $m - W_{y-\frac{[yn]}{n}}$ is a Bessel bridge $BB(3, y, m)$ so that:

$$E\left[e^{2\beta n^{\frac{1}{3}}\delta_n}|\nu = y, W_\nu = m, W_1 = m - x\right] = E\left[e^{2\beta n^{\frac{1}{3}}R_{y-\frac{[yn]}{n}}}|R_y = m\right]$$

Where R is a Bessel process in dimension 3. If we define three independent standard Brownian motion processes: B^1, B^2 and B^3 then,

$$R_t = \sqrt{(B_t^1)^2 + (B_t^2)^2 + (B_t^3)^2} \qquad t \geq 0 \tag{35}$$

Set $\alpha = 2\beta n^{\frac{1}{3}}$.

Now, for $t \leq y$:

$$\begin{aligned} E\left[e^{\alpha R_t}|R_y = m\right] &= E\left[e^{\alpha\sqrt{(B_t^1)^2+(B_t^2)^2+(B_t^3)^2}}|B_y^1 = m, B_y^2 = 0, B_y^3 = 0\right] \\ &\leq E\left[e^{\alpha|B_t^1|}|B_y^1 = m\right]\left(E\left[e^{\alpha|B_t^1|}|B_y^1 = 0\right]\right)^2 \end{aligned}$$

Moreover, conditionally on $B_y^1 = m$, B_t^1 is a normal random variable: $B_t^1 \sim N(\frac{mt}{y}, \frac{(y-t)t}{y})$. So,

$$\begin{aligned} E\left[e^{\alpha|B_t^1|}|B_y^1 = m\right] &\leq E\left[e^{\alpha B_t^1}|B_y^1 = m\right] + E\left[e^{-\alpha B_t^1}|B_y^1 = m\right] \\ &\leq 2e^{\frac{\alpha m t}{y} + \frac{\alpha^2(y-t)t}{2y}} \end{aligned}$$

and,

$$E\left[e^{\alpha|B_t^1|}|B_y^1 = 0\right] \leq 2e^{\frac{\alpha^2(y-t)t}{2y}}$$

Using those upper bounds we get:

$$E\left[e^{2\beta n^{\frac{1}{3}}\delta_n}\right] \leq 8E\left[e^{2\beta n^{\frac{1}{3}}\frac{t}{\nu}W_\nu}e^{2\beta^2 n^{\frac{2}{3}}\frac{(\nu-t)t}{\nu}}\right]\left(E\left[e^{2\beta^2 n^{\frac{2}{3}}\frac{(\nu-t)t}{\nu}}\right]\right)^2$$

With $t = \nu - \frac{[\nu n]}{n}$. In particular,

- $t = \nu - \frac{[\nu n]}{n} \leq \frac{1}{n}$

- $\frac{\nu-t}{\nu} \leq 1 \Longrightarrow \frac{(\nu-t)t}{\nu} \leq \frac{1}{n}$

Which further leads to:

$$E\left[e^{2\beta n^{\frac{1}{3}}\delta_n}\right] \leq 8\exp\left(\frac{6\beta^2}{n^{1/3}}\right) E\left[\exp\left(2\beta n^{\frac{1}{3}}\frac{t}{\nu}W_\nu\right)\right]$$

But,

$$E\left[\exp\left(2\beta n^{\frac{1}{3}}\frac{t}{\nu}W_\nu\right)\right] = E\left[\exp\left(2\beta n^{\frac{1}{3}}\frac{t}{\nu}W_\nu\right) I_{\nu\leq\frac{1}{n^{2/3}}}\right] + E\left[\exp\left(2\beta n^{\frac{1}{3}}\frac{t}{\nu}W_\nu\right) I_{\nu>\frac{1}{n^{2/3}}}\right]$$

- On $\{\nu > \frac{1}{n^{2/3}}\}$, $\frac{t}{\nu} < \frac{1}{n^{1/3}}$

So,

$$E\left[\exp\left(2\beta n^{\frac{1}{3}}\frac{t}{\nu}W_\nu\right) I_{\nu\leq\frac{1}{n^{2/3}}}\right] \leq E\left[\exp\left(2\beta W_\nu\right)\right] < \infty$$

- On $\{\nu \leq \frac{1}{n^{2/3}}\}$, $W_\nu = \sup_{0\leq s\leq\frac{1}{n^{2/3}}} W_s = \sup_{0\leq s\leq 1} W_{\frac{s}{n^{2/3}}}$

So, $n^{1/3}W_\nu =^d |W_1|$ by Levy's Theorem (see Durett [6] for instance).

Therefore,

$$E\left[\exp\left(2\beta n^{\frac{1}{3}}\frac{t}{\nu}W_\nu\right) I_{\nu>\frac{1}{n^{2/3}}}\right] \leq E\left[\exp\left(2\beta|W_1|\right)\right] < \infty.$$

We now clearly have:

$$\limsup_{n\to+\infty} E\left[e^{2\beta n^{\frac{1}{3}}\delta_n}\right] < \infty$$

□

This lemma now leads to the following theorem:

Theorem 2 *With the above notation,*

$$E\left[e^{cW_1}e^{\sigma W_{k^*h}}\right] \approx E\left[e^{cW_1}e^{\sigma W_{lh}}\right]\left(1+\sigma E[\epsilon_n]\right) \tag{36}$$

The approximation can then be computed using Equations 1 and 23.

Proof:

As is described in Nahum[16], the construction of the proof of Proposition 2 actually shows that

$$(W, \sqrt{n}\epsilon_n) \xrightarrow[n\to+\infty]{d} (W, R(U) - \min_{n=0,\pm1,\pm2,\cdots} R(U+n))$$

Which implies that W and ϵ_n are asymptotically independent. So for large n,

$$E\left[e^{cW_1}e^{\sigma W_{k^*h}}\right] = E\left[e^{cW_1}e^{\sigma W_{lh}}e^{\sigma\epsilon_n}\right] \approx E\left[e^{cW_1}e^{\sigma W_{lh}}\right]E\left[e^{\sigma\epsilon_n}\right]$$

And from Lemma 1 we know that:

$$E\left[e^{\sigma\epsilon_n}\right] = 1 + \sigma E[\epsilon_n] + \sum_{i=2}^{\infty}\frac{\sigma^i E[\epsilon_n^i]}{i!} = 1 + \sigma E[\epsilon_n] + o(n^{-2/3})$$

□

5 Numerical Results

The tables below give the different prices of a European Put Lookback Option on an asset that is worth \$100 at time 0. The prices are computed for different values of the interest rate, volatility and maturity of the option using four different methods. The methods correspond to the ones described in the paper, the first one computes the lower bound using the quasi-maximum as an approximation for the discrete maximum, the second one adds the correction factor to the first method, the third one is obtained via Monte-Carlo simulations and the last one uses the continuous maximum as an approximation for the discrete maximum.

Note that the price using the "Quasi-maximum+correction factor" takes less than a second to be computed on a Ultra Sparc 2 whereas the Monte-Carlo method takes more than 2 hours.

Table 1: annual interest rate = 10%, maturity = 1 year: 250 days

σ	Quasi-maximum	Quasi-maximum+correction factor	Monte-Carlo	Continuous maximum
0.1	3.72	4.03	4.03	4.41
0.2	10.73	11.41	11.42	12.23
0.3	18.48	19.58	19.60	20.90

Table 2: annual interest rate = 7%, maturity = 1 year: 250 days

σ	Quasi-maximum	Quasi-maximum+correction factor	Monte-Carlo	Continuous maximum
0.1	4.57	4.88	4.88	5.27
0.2	11.93	12.63	12.63	13.45
0.3	19.88	21.02	21.02	22.33

Table 3: annual interest rate = 10%, maturity = 6 months: 125 days

σ	Quasi-maximum	Quasi-maximum+correction factor	Monte-Carlo	Continuous maximum
0.1	3.05	3.34	3.34	3.71
0.2	7.98	8.60	8.62	9.40
0.3	13.24	14.21	14.24	15.47

Table 4: annual interest rate = 7%, maturity = 6 months: 125 days

σ	Quasi-maximum	Quasi-maximum+correction factor	Monte-Carlo	Continuous maximum
0.1	3.55	3.84	3.85	4.22
0.2	8.62	9.24	9.26	10.04
0.3	13.94	14.92	14.95	16.19

Table 5: annual interest rate = 10%, maturity = 3 months: 62 days

σ	Quasi-maximum	Quasi-maximum+correction factor	Monte-Carlo	Continuous maximum
0.1	2.34	2.56	2.61	2.97
0.2	5.66	6.21	6.28	7.03
0.3	9.11	10.00	10.09	11.27

Table 6: annual interest rate = 7%, maturity = 3 months: 62 days

σ	Quasi-maximum	Quasi-maximum+correction factor	Monte-Carlo	Continuous maximum
0.1	2.60	2.85	2.89	3.25
0.2	5.97	6.55	6.60	7.36
0.3	9.45	10.37	10.44	11.62

6 Conclusion

After having approximated the discrete maximum of a Brownian motion on a finite interval by what is defined as the Quasi-maximum, this paper has used the Denisov decomposition which provides one of the main properties of the Brownian meander to provide a lower bound (easily computed numerically) for the price of an option depending on the discrete maximum. Moreover, in the particular case of the European Put Lookback option, the lower bound reduces to a simple integration and we have computed a correction factor associated to this lower bound which provides (as it is shown by the numerical examples above) a very good approximation to the price of that option otherwise computed using Monte-Carlo methods.

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APPENDIX A

1. $\phi(\lambda) = E\left[e^{-\lambda M_1}\right]$

Just using Equation 13:

$$E\left[e^{-\lambda M_1}\right] = \int_0^{+\infty} e^{-\lambda u} u e^{-\frac{u^2}{2}} du$$

Then by a simple integration by parts we get:

$$\begin{aligned} E\left[e^{-\lambda M_1}\right] &= 1 - \lambda e^{\frac{\lambda^2}{2}} \int_0^{+\infty} e^{-\frac{(\lambda+u)^2}{2}} du \\ &= 1 - \lambda\sqrt{2\pi} e^{\frac{\lambda^2}{2}} N(-\lambda) \end{aligned}$$

2. $\psi(\beta,\gamma,s) = E\left[E[e^{\beta M_1}|M_s]e^{-\gamma M_s}\right]$

First, recall Equations 13 and 14:

$$\begin{aligned} d(t,y) &= P(M_t \in dy) = \frac{2y}{t^{\frac{3}{2}}} e^{-\frac{y^2}{2t}} N_{1-t}(0,y)dy \qquad 0 \le t \le 1, y \ge 0 \\ tr(s,x,t,y) &= P(M_t \in dy | M_s \in dx) = (n_{t-s}(y-x) - n_{t-s}(y+x)) \frac{N_{1-t}(0,y)}{N_{1-s}(0,x)} \qquad 0 \le s < t \le 1, x, y \ge 0 \end{aligned}$$

We now look at $E[e^{\beta M_1}|M_s]$:

$$E[e^{\beta M_1}|M_s \in du] = \int_0^{+\infty} e^{\beta v} tr(s,u,1,v)dv$$

so that,

$$2N_{1-s}(0,u)E[e^{\beta M_1}|M_s \in du] = \frac{1}{\sqrt{2\pi(1-s)}} \int_0^{+\infty} e^{\beta v}\left[e^{-\frac{(u-v)^2}{2(1-s)}} - e^{-\frac{(u+v)^2}{2(1-s)}}\right] dvdu$$

Thus,

$$\begin{aligned} \psi(\beta,\gamma,s) &= \int_0^{+\infty} e^{-\gamma u} d(s,u) E[e^{\beta M_1}|M_s \in du] \\ &= \int_{u=0}^{+\infty} e^{-\gamma u} \frac{u}{s^{\frac{3}{2}}} e^{-\frac{u^2}{2s}} 2N_{1-s}(0,u) E[e^{\beta M_1}|M_s \in du] \\ &= \int_{u=0}^{+\infty} \frac{u}{s^{\frac{3}{2}}} e^{-\gamma u} e^{-\frac{u^2}{2s}} \frac{1}{\sqrt{2\pi(1-s)}} \int_{v=0}^{+\infty} e^{\beta v}\left[e^{-\frac{(u-v)^2}{2(1-s)}} - e^{-\frac{(u+v)^2}{2(1-s)}}\right] dvdu \end{aligned}$$

We now rewrite the above expression as the difference of two integrals, I_1 and I_2:

$$\begin{aligned} \psi(\beta,\gamma,s) &= I_1 - I_2 \\ &= \int_{u=0}^{+\infty} \frac{u}{s^{\frac{3}{2}}} e^{-\gamma u} e^{-\frac{u^2}{2s}} \frac{1}{\sqrt{2\pi(1-s)}} \int_{v=0}^{+\infty} e^{\beta v} e^{-\frac{(u-v)^2}{2(1-s)}} dvdu \\ &\quad - \int_{u=0}^{+\infty} \frac{u}{s^{\frac{3}{2}}} e^{-\gamma u} e^{-\frac{u^2}{2s}} \frac{1}{\sqrt{2\pi(1-s)}} \int_{v=0}^{+\infty} e^{\beta v} e^{-\frac{(u+v)^2}{2(1-s)}} dvdu \end{aligned}$$

In I_1 we make the change of variable $z = u - v - \beta(1-s)$ and in I_2 we set $z = u + v - \beta(1-s)$, we then get:

$$\begin{aligned} I_1 &= \frac{1}{\sqrt{2\pi(1-s)}} \int_{u=0}^{+\infty} \frac{u}{s^{\frac{3}{2}}} e^{-\frac{u^2}{2s}} e^{(\beta-\gamma)u} e^{\frac{\beta^2(1-s)}{2}} \int_{z > -(\beta(1-s)+x)} e^{-\frac{z^2}{2(1-s)}} dzdu \\ I_2 &= \frac{1}{\sqrt{2\pi(1-s)}} \int_{u=0}^{+\infty} \frac{u}{s^{\frac{3}{2}}} e^{-\frac{u^2}{2s}} e^{-(\beta+\gamma)u} e^{\frac{\beta^2(1-s)}{2}} \int_{z > -(\beta(1-s)-x)} e^{-\frac{z^2}{2(1-s)}} dzdu \end{aligned}$$

For both those integrals we now use an integration by parts to get the following expressions:

$$
\begin{aligned}
I_1 &= \frac{1}{\sqrt{2\pi s(1-s)}} e^{\frac{\beta^2(1-s)}{2}} \int_{u=0}^{+\infty} (\beta-\gamma) e^{(\beta-\gamma)u} e^{-\frac{u^2}{2s}} \int_{z>-(\beta(1-s)+x)} e^{-\frac{z^2}{2(1-s)}} dz du \\
&+ \frac{1}{\sqrt{2\pi s(1-s)}} e^{\frac{\beta^2(1-s)}{2}} \int_{u=0}^{+\infty} e^{(\beta-\gamma)u} e^{-\frac{u^2}{2s}} e^{-\frac{(\beta(1-s)+u)^2}{2(1-s)}} du \\
&+ \frac{1}{\sqrt{2\pi s(1-s)}} e^{\frac{\beta^2(1-s)}{2}} \int_{z>-\beta(1-s)} e^{-\frac{z^2}{2(1-s)}} dz \\
&= \sqrt{2\pi}(\beta-\gamma) e^{\frac{\beta^2(1-s)}{2}} e^{\frac{(\beta-\gamma)^2 s}{2}} \int_0^{+\infty} n_s(u-(\beta-\gamma)s) N\left(\frac{u+\beta(1-s)}{\sqrt{1-s}}\right) du \\
&+ e^{\frac{\gamma^2 s(1-s)}{2}} N\left(-\gamma\sqrt{s(1-s)}\right) \\
&+ \frac{e^{\frac{\beta^2(1-s)}{2}}}{\sqrt{s}} N\left(\beta\sqrt{1-s}\right)
\end{aligned}
$$

and, using the exact same method,

$$
\begin{aligned}
I_2 &= -\sqrt{2\pi}(\beta+\gamma) e^{\frac{\beta^2(1-s)}{2}} e^{-\frac{(\beta+\gamma)^2 s}{2}} \int_0^{+\infty} n_s(u+(\beta+\gamma)s) N\left(\frac{-u+\beta(1-s)}{\sqrt{1-s}}\right) du \\
&- e^{\frac{\gamma^2 s(1-s)}{2}} N\left(-\gamma\sqrt{s(1-s)}\right) \\
&+ \frac{e^{\frac{\beta^2(1-s)}{2}}}{\sqrt{s}} N\left(\beta\sqrt{1-s}\right)
\end{aligned}
$$

It is fairly easy to derive the following useful result by some simple calculations:

$$
\int_0^{+\infty} n_{c^2}(x-b) N\left(\frac{x-a}{c_0}\right) dx = NN\left(\frac{b}{c}, \frac{b-a}{c_0\sqrt{1+\left(\frac{c}{c_0}\right)^2}}, \frac{\frac{c}{c_0}}{\sqrt{1+\left(\frac{c}{c_0}\right)^2}}\right) \tag{37}
$$

We then obtain:

$$
\begin{aligned}
\psi(\beta,\gamma,s) &= I_1 - I_2 \\
&= \sqrt{2\pi} e^{\frac{\beta^2(1-s)}{2}} \Big[(\beta-\gamma) e^{\frac{(\beta-\gamma)^2 s}{2}} NN\left((\beta-\gamma)\sqrt{s}, \beta-\gamma s, \sqrt{s}\right) \\
&+ (\beta+\gamma) e^{\frac{(\beta+\gamma)^2 s}{2}} NN\left(-(\beta+\gamma)\sqrt{s}, \beta+\gamma s, -\sqrt{s}\right) \Big] + 2 e^{\frac{\gamma^2 s(1-s)}{2}} N\left(-\gamma\sqrt{s(1-s)}\right)
\end{aligned}
$$

APPENDIX B

1. $\xi(\lambda, s, v) = E[e^{-\lambda M_1} | M_s \in dv]$

Here we just use Equation 14 to get:

$$\begin{aligned}
\xi(\lambda, s, v) &= \int_{u=0}^{+\infty} tr(s, v, 1, u) e^{-\lambda u} du \\
&= \frac{1}{2\sqrt{2\pi(1-s)}N_{1-s}(0,v)} \int_{u=0}^{+\infty} e^{-\lambda u} \left(e^{-\frac{(v-u)^2}{2(1-s)}} - e^{-\frac{(v+u)^2}{2(1-s)}} \right) du
\end{aligned}$$

By rearranging and completing the squares in the exponentials we get to:

$$\begin{aligned}
\xi(\lambda, s, v) &= \frac{e^{\frac{\lambda^2(1-s)}{2}}}{2\sqrt{2\pi(1-s)}2N_{1-s}(0,v)} \left[e^{-\lambda v} \int_0^{+\infty} e^{-\frac{[u+(1-s)(\lambda - \frac{v}{1-s})]^2}{2(1-s)}} du - e^{\lambda v} \int_0^{+\infty} e^{-\frac{[u+(1-s)(\lambda + \frac{v}{1-s})]^2}{2(1-s)}} du \right] \\
&= \frac{e^{\frac{\lambda^2(1-s)}{2}}}{2N_{1-s}(0,v)} \left[e^{-\lambda v} N\left(\sqrt{1-s}(\frac{v}{1-s} - \lambda) \right) - e^{\lambda v} N\left(-\sqrt{1-s}(\frac{v}{1-s} + \lambda) \right) \right]
\end{aligned}$$

2. $\psi_1(\beta, \gamma, s, b) = E\left[E[e^{\beta M_1} | M_s] e^{-\gamma M_s} I_{M_s \leq b} \right]$

The technique used to compute ψ_1 is identical to the one used to compute ψ which is described in details in Appendix A. The change of variables used as well as the integration by parts are the same, the only difference being that the integration on u is made from 0 to b instead of going to $+\infty$. We, therefore, get to the following expression:

$$\psi_1(\beta, \gamma, s, b) = I_1^1 - I_2^1$$

with:

$$\begin{aligned}
I_1^1 &= \sqrt{2\pi}(\beta - \gamma) e^{\frac{\beta^2(1-s)}{2}} e^{\frac{(\beta-\gamma)^2 s}{2}} \int_0^b n_s(u - (\beta - \gamma)s) N\left(\frac{u + \beta(1-s)}{\sqrt{1-s}} \right) du \\
&+ \frac{1}{\sqrt{2\pi s(1-s)}} e^{\frac{\beta^2(1-s)}{2}} \int_0^b e^{(\beta-\gamma)u} e^{-\frac{u^2}{2s}} e^{-\frac{(\beta(1-s)+u)^2}{2(1-s)}} du \\
&- \frac{e^{\frac{\beta^2(1-s)}{2}} e^{-\frac{b^2}{2s}} e^{-(\beta+\gamma)b}}{\sqrt{2\pi s(1-s)}} \int_{z > -(\beta(1-s)-b)} e^{-\frac{z^2}{2(1-s)}} dz \\
&+ \frac{e^{\frac{\beta^2(1-s)}{2}}}{\sqrt{2\pi s(1-s)}} \int_{z > -\beta(1-s)} e^{-\frac{z^2}{2(1-s)}} dz
\end{aligned}$$

and,

$$\begin{aligned}
I_2^1 &= -\sqrt{2\pi}(\beta + \gamma) e^{\frac{\beta^2(1-s)}{2}} e^{-\frac{(\beta+\gamma)^2 s}{2}} \int_0^b n_s(u + (\beta + \gamma)s) N\left(\frac{-u + \beta(1-s)}{\sqrt{1-s}} \right) du \\
&- \frac{1}{\sqrt{2\pi s(1-s)}} e^{\frac{\beta^2(1-s)}{2}} \int_0^b e^{-(\beta+\gamma)u} e^{-\frac{u^2}{2s}} e^{-\frac{(\beta(1-s)+u)^2}{2(1-s)}} du
\end{aligned}$$

$$
\begin{aligned}
&-\frac{e^{\frac{\beta^2(1-s)}{2}}e^{-\frac{b^2}{2s}}e^{-(\beta+\gamma)b}}{\sqrt{2\pi s(1-s)}}\int_{z>-(\beta(1-s)+b)}e^{-\frac{z^2}{2(1-s)}}dz\\
&+\frac{e^{\frac{\beta^2(1-s)}{2}}}{\sqrt{2\pi s(1-s)}}\int_{z>-\beta(1-s)}e^{-\frac{z^2}{2(1-s)}}dz
\end{aligned}
$$

which then leads to:

$$
\begin{aligned}
\psi_1(\beta,\gamma,s,b) &= \sqrt{2\pi}e^{\frac{\beta^2(1-s)}{2}}\Bigg[(\beta-\gamma)e^{\frac{(\beta-\gamma)^2 s}{2}}[NN\Big((\beta-\gamma)\sqrt{s},\beta-\gamma s,\sqrt{s}\Big)-NN\Big(\frac{(\beta-\gamma)s-b}{\sqrt{s}},\beta-\gamma s,\sqrt{s}\Big)]\\
&\quad+(\beta+\gamma)e^{\frac{(\beta+\gamma)^2 s}{2}}[NN\Big(-(\beta+\gamma)\sqrt{s},\beta+\gamma s,-\sqrt{s}\Big)-NN\Big(-\frac{(\beta+\gamma)s+b}{\sqrt{s}},\beta+\gamma s,-\sqrt{s}\Big)]\Bigg]\\
&\quad+2e^{\frac{\gamma^2 s(1-s)}{2}}\left[N\left(\frac{b+\gamma s(1-s)}{\sqrt{s(1-s)}}\right)-N(-\gamma\sqrt{s(1-s)})\right)\Bigg]\\
&\quad+\frac{e^{\frac{\beta^2(1-s)}{2}}e^{-\frac{b^2}{2s}}}{\sqrt{s}}\left[e^{-(\beta+\gamma)b}N\left(\frac{\beta(1-s)-b}{\sqrt{1-s}}\right)-e^{(\beta-\gamma)b}N\left(\frac{\beta(1-s)+b}{\sqrt{1-s}}\right)\right]
\end{aligned}
$$

We can then easily derive the formula for $\psi_2(\beta,\gamma,s,b)$:

$$
\begin{aligned}
\psi_2(\beta,\gamma,s,b) &= \psi(\beta,\gamma,s)-\psi_1(\beta,\gamma,s,b)\\
&= \sqrt{2\pi}e^{\frac{\beta^2(1-s)}{2}}\Bigg[(\beta-\gamma)e^{\frac{(\beta-\gamma)^2 s}{2}}NN\Big(\frac{(\beta-\gamma)s-b}{\sqrt{s}},\beta-\gamma s,\sqrt{s}\Big)\\
&\quad+(\beta+\gamma)e^{\frac{(\beta+\gamma)^2 s}{2}}NN\Big(-\frac{(\beta+\gamma)s+b}{\sqrt{s}},\beta+\gamma s,-\sqrt{s}\Big)\Bigg]\\
&\quad+2e^{\frac{\gamma^2 s(1-s)}{2}}\left[N\left(-\frac{b+\gamma s(1-s)}{\sqrt{s(1-s)}}\right)\right]\\
&\quad-\frac{e^{\frac{\beta^2(1-s)}{2}}e^{-\frac{b^2}{2s}}}{\sqrt{s}}\left[e^{-(\beta+\gamma)b}N\left(\frac{\beta(1-s)-b}{\sqrt{1-s}}\right)-e^{(\beta-\gamma)b}N\left(\frac{\beta(1-s)+b}{\sqrt{1-s}}\right)\right]
\end{aligned}
$$