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Derivatives Research

JPMorgan

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Pricing Exotics under the Smile[1]

Introduction

The volatility implied from the market prices of vanilla options, using the Black Scholes formula, is seen to vary with both maturity and strike price. This surface is known as the volatility smile. It can be considered as a correction for "second order" effects where the market departs in practice from the assumptions underlying the Black Scholes model.

Recent years have seen a surge in the market for exotic path dependent options. Both the liquidity and the volumes of trades of products such as barrier options, compound options and range notes have increased dramatically. These products can have large second order exposures and their traded prices can be significantly offset from the theoretical values calculated under the Black Scholes assumptions.

Considerable research effort has been focused on the search for a consistent framework to value both european and exotic options. The objective being to find a methodology which can be practically implemented in a risk management system. This paper details the Exotic Smile model which has been developed and implemented within J. P. Morgan.

The paper begins with discussion of the market conditions that are behind the volatility smile. The theoretical framework for the model is then presented, leading to a description of the practical implementation. The results from the model are then compared with market exotic prices. Finally, we discuss the implications of the model to the risk management of exotic products.

The Volatility Smile

The Black Scholes model provides a convenient formula for deriving the prices of european style options. The model has a number of useful properties which have led to its universal use. The first such property is that option values are independent of any risk preference. The second is that there is only one input parameter for the formula that is not directly observable in the market – the volatility of the underlying asset.

The key assumption behind the Black Scholes model is that the asset performs a random walk over time, where the returns on the asset are normally distributed with a constant volatility. The implied volatility is the volatility backed out from the market option price using the Black Scholes formula. If the market were consistent with the Black Scholes model then the implied volatility would be the same for all options. In practice the implied volatility is found to vary with both maturity and strike price. This surface is known as the volatility smile. Figure 1 shows the surface of implied volatility for foreign

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exchange options on JPY-USD over a range of strike prices and maturity dates. Similar volatility smile surfaces exist in most liquid options markets.

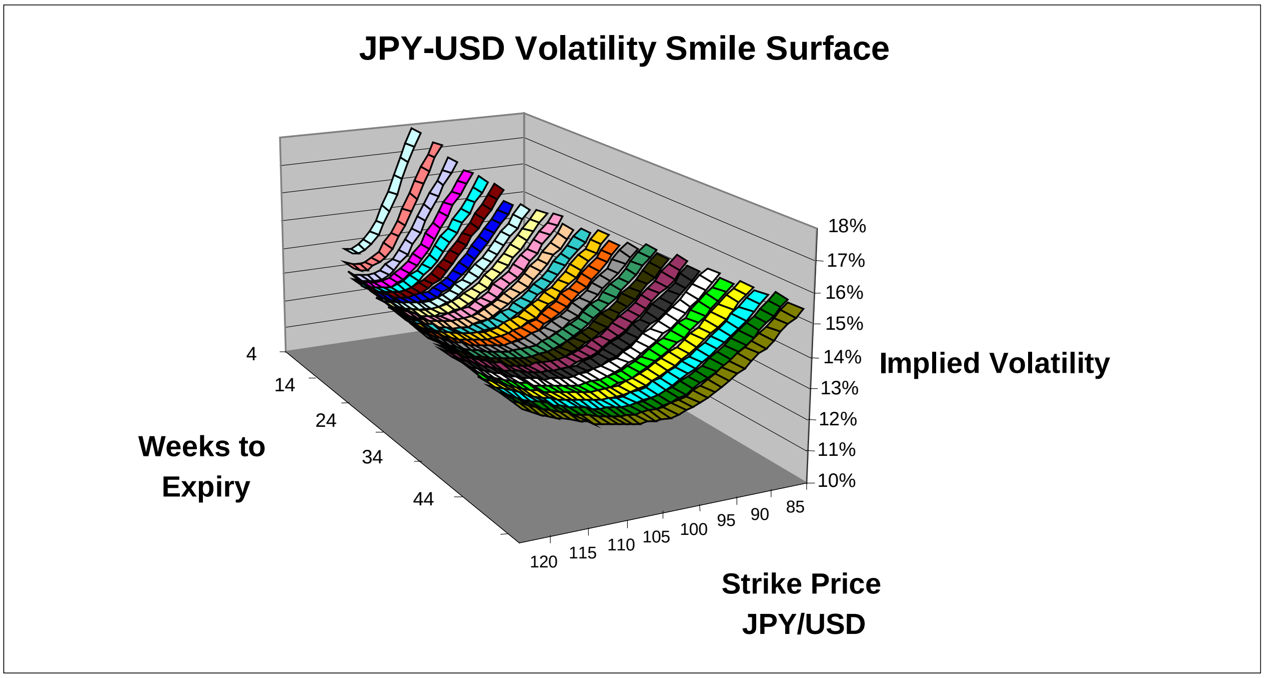

Figure 1 – JPY-USD volatility smile surface

The Black Scholes model gives rise to a lognormal probability distribution for the asset at some future date. However the presence of a volatility smile means that the market's probability distribution deviates from being lognormal. The values of european options at different strike prices depend upon the exact form of the probability distribution for the maturity date. As such the volatility smile provides sufficient information to derive this non-lognormal distribution. Figure 2 shows the market probability distribution derived from the six month volatility smile for JPY-USD. A lognormal distribution based on the at-the-money option volatility is shown for comparison. The graph shows that the market curve has excess probability in the tails of the distribution.

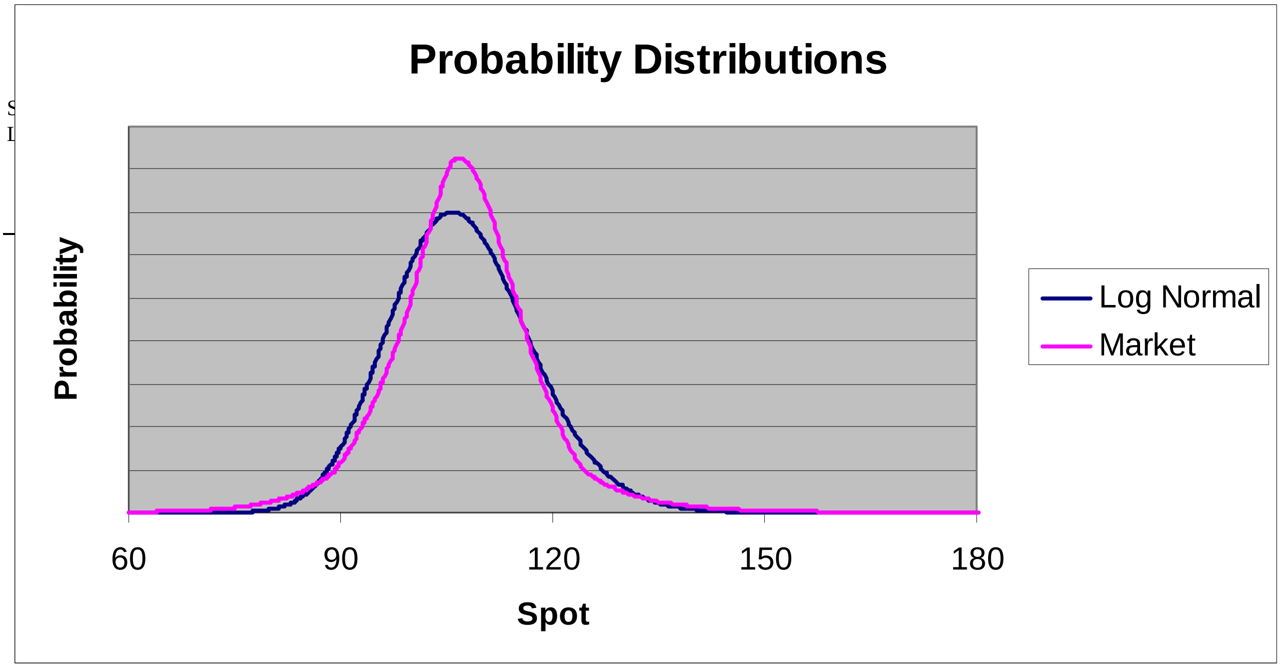

Figure 2 – Probability distributions

The volatility smile can be seen as an adjustment made to the Black Scholes formula to account for “second order” effects where the assumptions behind the model are violated in practice. Analysis of historical data shows that the assumption of constant volatility is inconsistent with the observed market behaviour. Figure 3 shows the value of the JPY-USD exchange rate over the last year. Also shown is the historical series for the implied volatility of a one month at-the-money option. This figure clearly shows that the volatility is not constant with time. In fact the implied volatility is itself highly stochastic. The volatility of implied volatility is in excess of 70% compared with the volatility of the exchange rate which ranges from 10% to an extreme of 40%. It should also be noted that the volatility shows a significant correlation with the underlying exchange rate of -39% over this period.

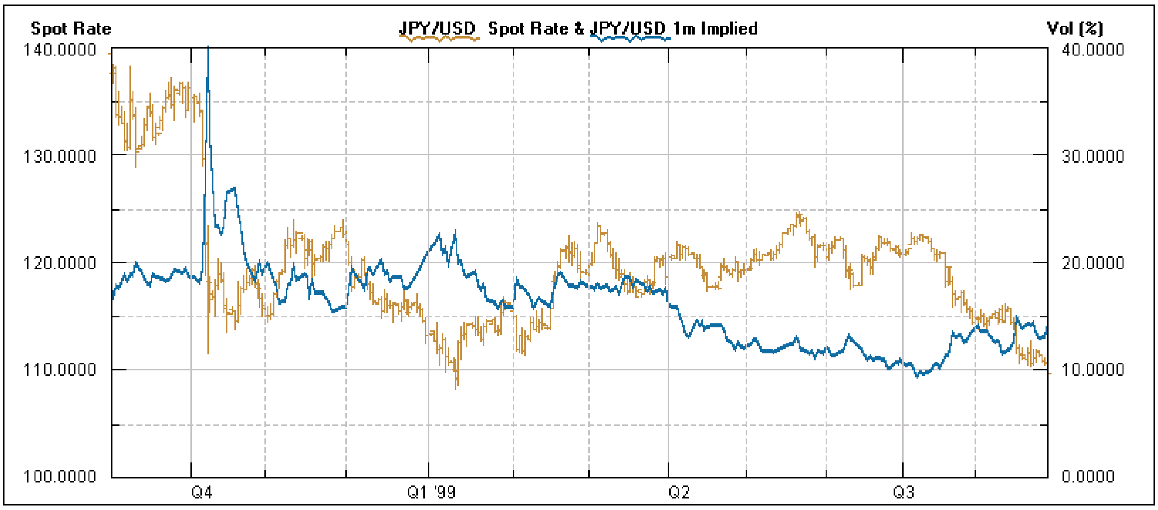

Figure 3 – JPY-USD exchange rate over previous year

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Path Dependent Exotic Options

The market for exotic or path dependent option products has expanded dramatically in recent years in both volume and liquidity. There are now liquid two-way prices observable in the market for a range of exotic option products, and the bid-offer spreads on these products are steadily narrowing. These products can have substantial second order exposures and, as with the vanilla options markets, these exposures have an associated value in a market where the Black Scholes assumptions are violated. The market convention is to calculate the theoretical price for exotic options under the assumptions of the Black Scholes model using the implied volatility for at-the-money vanilla options. However these products are observed to trade at a significant offset to their theoretical value.

Figure 4 shows a comparison between the theoretical Black Scholes price and the mid market prices for a 3 month american binary option on JPY-USD. The topside american binary option pays a fixed USD amount at maturity if at any point during the life of the option the spot exchange rate trades above a fixed level. Conversely a downside binary pays out if a fixed lower level is breached. The figure shows how far the market prices are offset from the Black Scholes theoretical price. For example the topside american binary option with a theoretical value of 50% of the notional actually trades at a discount of 3.75% i.e. at 46.25%.

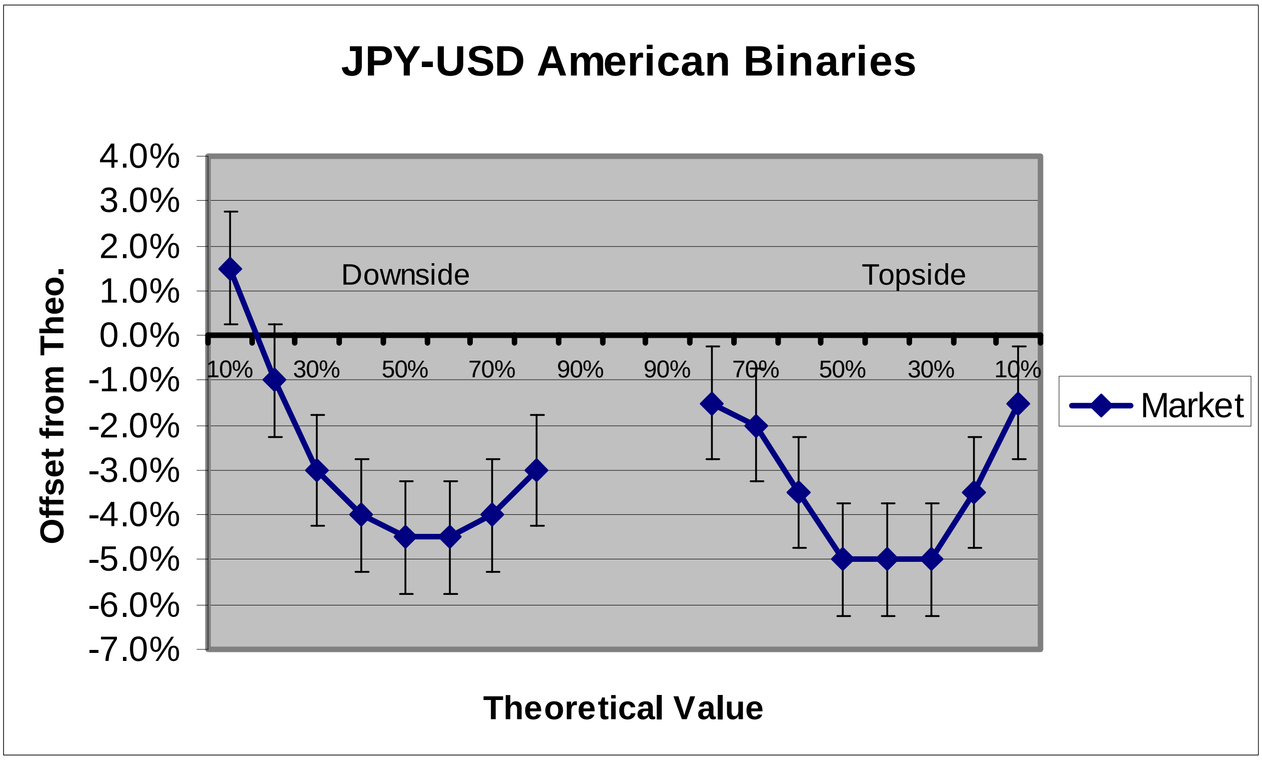

Figure 4 – JPY-USD American binaries

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As mentioned before the volatility smile is the observable impact of second order effects on the prices of vanilla options. European option prices depend only upon the expected distribution for the asset value at the maturity for the option, and as such provides information about how this distribution differs from the Black Scholes lognormal distribution. The smile does not directly provide information about the process that leads to this non-lognormal distribution. In contrast path dependent options products are sensitive not only to the distribution of the asset value at any one time but to how this distribution evolves from one time to the next. A number of different processes could be postulated which would match the observed volatility smile and yet give different values for the same path dependent option.

Aims of the Model

The objective for this work is to develop a model for pricing path dependent exotic options in a manner consistent with the volatility smile. Since exotic option prices are sensitive to the exact process behind the smile we need to find a process which is both representative of the dynamics of the market and which reproduces the volatility smile observed for vanilla options. This process can then be used to price path dependent options.

There are a number of existing published models which seek to describe the process behind the volatility smile. These fall into three broad categories – stochastic volatility, deterministic state-dependent volatility and discrete jump models. The discrete jump models are applicable to markets which are dominated by the risk of a sudden and significant move in the asset value. This model is discussed in detail in the Optional events and jumps article in this series (JP Morgan, 1999).

Stochastic volatility models for the smile have been discussed by Heston (1993) and Stein and Stein (1991), among others. The approach taken by Heston involved defining a stochastic process for the instantaneous volatility with a number of free parameters. These free parameters can then be chosen so that the model gives the best possible fit to the vanilla market prices. In practice the smile is not well matched over all option maturities by a single volatility process. For markets where the bid-offer spreads for european options are at the basis point level any best-fit approach with a limited number of free parameters is likely to be outside the spread for most options.

Derman and Kani (1994) and Dupire (1994) have developed a state dependent volatility model where the volatility smile can used to uniquely determine a deterministic surface for the local instantaneous volatility as a function of the asset value and time. These models use the volatility smile as an input and so by virtue of their construction are guaranteed to match the market smile. However the deterministic process may not be a good model for the actual market dynamics and as such may not give realistic exotic option values.

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The methodology we have employed here uses a combination of these two types of model. The gross properties of the smile are matched by a stochastic volatility process. Whilst a second deterministic component is introduced which is calibrated to ensure that the market smile is exactly matched. The calculation of this deterministic component comes from a constraint derived from the state dependent volatility models.

State Dependent Volatility Model.

The risk-neutral value for european options depends only upon the expected probability distribution for the underlying asset at the maturity of the option. Under the assumptions of the Black Scholes model the asset will have a lognormal distribution at any future date. The volatility smile provides information about how the expected probability distribution differs from lognormal. It does not hold enough information to uniquely determine the process by which this non-lognormal distribution arises. However if the assumption is made that the local instantaneous volatility is a purely deterministic function of the asset value and time, this function can be uniquely determined from the volatility smile.

The models of Dupire (1994) and Derman and Kani (1994) assume that the asset value undergoes a random walk with the returns being normally distributed but where the instantaneous local volatility is a deterministic function of the asset value and time – $\sigma_D(S,t)$.

$$\frac{dS}{S} = \mu dt + \sigma_D(S,t)dZ \qquad \text{(Equation 1)}$$

where S is the asset value, μ is the drift and dZ is a Wiener process with a mean of zero and a variance of dt.

They go on to show that it is possible to uniquely determine the function $\sigma_D(S,t)$ from the volatility smile by construction of an implied binomial tree. In the continuous time limit the formula for $\sigma_D(S,t)$ becomes

$$\sigma_D^2(K,T) = 2\frac{\{\frac{\partial C_{KT}}{\partial T} + \mu K \frac{\partial C_{KT}}{\partial K} + rfC_{KT}\}}{K^2 \frac{\partial^2 C_{KT}}{\partial K^2}} \qquad \text{(Equation 2)}$$

where C_{KT} is the current market value of an option with strike price K and maturity T.

This deterministic model may not realistically represent the market dynamics, however it does provide a useful constraint that must apply to any model if it is to match the market prices for all european options. The deterministic local volatility function, $\sigma_D(S,t)$, represents a market consensus estimate for the instantaneous volatility at some future

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time *t* with the asset at a value *S*. In fact in can be shown that the local variance $\sigma_D{}^2(S,t)$ is the expectation of the future instantaneous variance $\sigma^2(t)$ at time *t* conditional on the asset having value *S* at time *t*.

$$\sigma_D^2(K,T) = \mathrm{E}[\sigma^2(T) \mid S(T) = K] \qquad \text{(Equation 3)}$$

It follows then that whatever process we chose to model volatility must satisfy this constraint on the conditional expectation of the instantaneous volatility for the prices of all european options are to be matched.

We can consider a class of models where this constraint on the expectation of future variance is met but where there is some stochastic distribution around this expected value. All models in the class would match the european option prices by virtue of the constraint. The Dupire and Derman models are a special case of this class where there is no stochastic distribution of σ^2.

Full Model Process

For the methodology presented here we have modelled the asset value as a stochastic process driven by a local instantaneous volatility. This volatility is modelled by two components, *X* and *Y*, where *Y* is a stochastic mean reverting process which is correlated with the asset value and *X(S,t)* is a deterministic component which is calibrated to ensure that the prices of vanilla options are matched with the market. The full process is defined below.

$$\frac{dS}{S} = (r_d - r_f)dt + \sqrt{X(S,t)Y}\,dZ_1 \qquad \text{(Equation 4)}$$

$$dY = \alpha(Y_* - Y)dt + \xi\sqrt{Y}\,dZ_2 \qquad \text{(Equation 5)}$$

$$X^2(K,T) = \frac{\sigma_D^2(K,T)}{\mathrm{E}[Y^2(T) \mid S(T) = K]} \qquad \text{(Equation 6)}$$

where α is the rate at which the stochastic component *Y* reverts towards a long-term equilibrium level of Y_*, ξ is the instantaneous volatility of *Y* and dZ_1 and dZ_2 are Wiener processes which have correlation ρ. Equation 6 comes from applying the constraint in equation 3 and rearranging.

This process, without the deterministic part *X(S,t)*, is the same as that presented by Heston (1993). The Heston process has the advantage that an analytic solution exists for

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options prices under this model. As mentioned before, the parameters α, Y_* and ξ in the Heston process can be chosen so that options prices calculated using the model provide the best fit to the current market prices. A similar methodology is used to derive these parameters for this more general model.

There are two limiting cases that should be considered for this process. Firstly when ξ is zero the volatility becomes completely deterministic and the process reduces to be the same as Derman and Dupire models. The other case is when the market option prices can be exactly matched by the pure stochastic volatility process. In this situation the deterministic component is redundant and the calibration results in a $X(S,t)$ function which is unity for all values of S and t. In this case we are left with the Heston model.

This methodology should be contrasted with the stochastic implied tree presented by Derman and Kani (1997), which models the stochastic evolution of the whole smile surface in a manner similar to the treatment of interest rates in Heath, Jarrow and Morton (1992). Here we construct a process for the stochastic behaviour of the instantaneous local volatility using an approach similar to Hull and White (1990).

Implementation

The model is implemented as a two dimensional trinomial tree. One dimension represents the asset value, the second represents Y. For each node in the tree the transition probabilities to nine nodes in the following time slice are calculated. These probabilities are chosen so that the drift and variance in both dimensions as well as the covariance are consistent with the process. This still leaves a number of free parameters that are judiciously chosen to ensure that the transition probabilities are all positive and less than one. At the edges of the tree there are regions of both extreme high and low local volatility. In order to ensure that the transition probabilities remain well behaved the spacing of the nodes in the first dimension of the tree need to be adjusted to a size suited to the local volatility. The transition probabilities must remain positive to eliminate regions of arbitrage within the tree.

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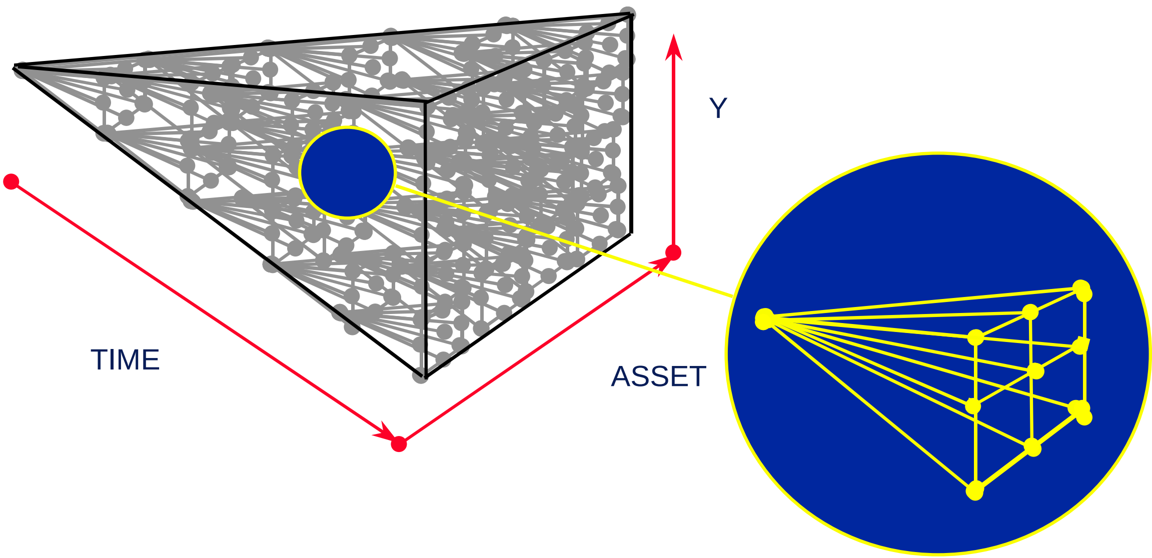

Figure 5 – Schematic of a two-dimensional trinomial tree

The function $X(S,t)$ is calculated by an initial calibration pass forwards though the tree. Starting with a probability of one at the first node the transition probabilities are used to propagate the probability to the nodes in the next time slice. Equation 6 can then be used to determine the values of X for this slice and consequently the transition probabilities to the next slice. This procedure can then be repeated for each subsequent slice until $X(S,t)$ has been determined for the complete tree.

The function $\sigma_D(S,t)$ is evaluated using equation 2. In practice not all option prices are observed in the market and so the function C_{KT} is not completely determined. Only certain liquid benchmark maturities and strike prices are used as inputs and the function C_{KT} is determined by interpolation and extrapolation. The form of the interpolation is chosen to make $\sigma_D(S,t)$ smooth and well behaved.

One of the requirements of the model is that it should be able to exactly reproduce the prices for european options from the input market smile. The accuracy of the model depends upon the number of time slices used in the tree. Increasing the number of time slices increases the accuracy but also increases the computational workload. In the full implementation the tree prices for european options converge to within 1/10 of a basis point of the market prices.

Results

The two-dimensional tree model has been fully implemented and is currently being used for both live pricing and risk management by our foreign exchange options trading desk. The model shows a very close agreement with the market prices over the whole spectrum

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of liquid path dependent option products. These include barrier knock out options, american options and american binary options.

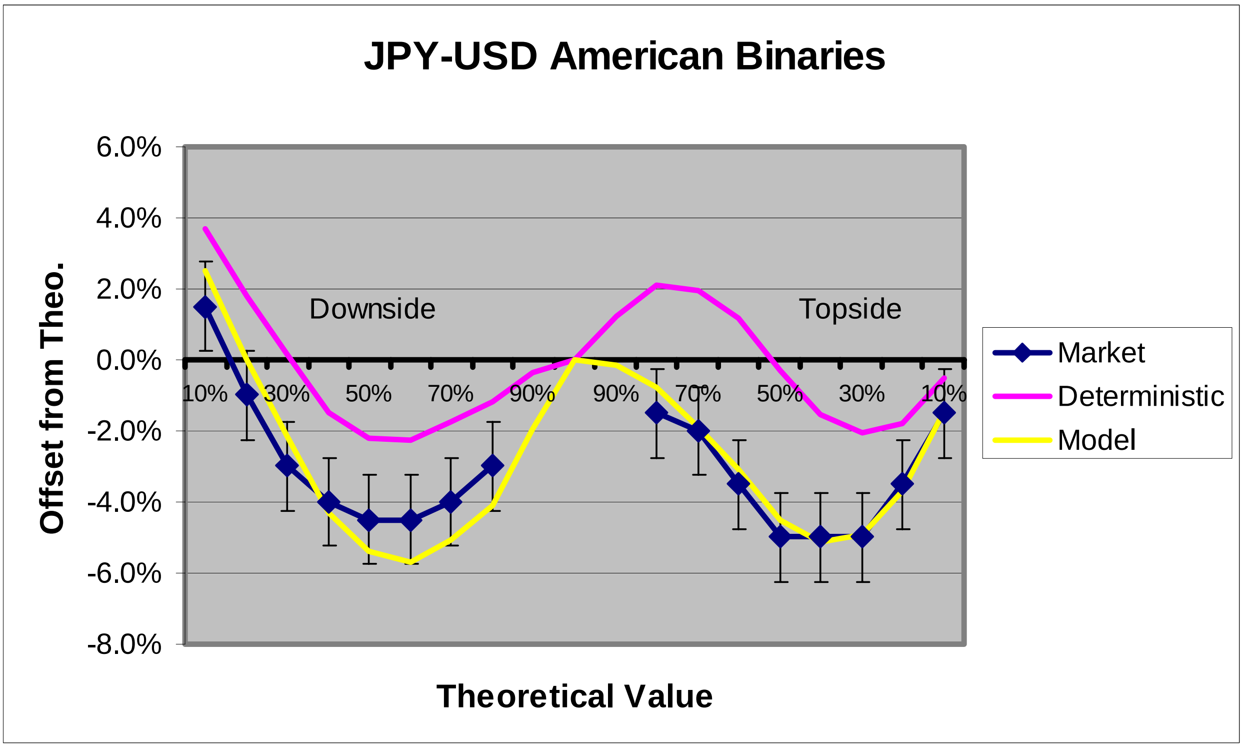

Figure 6 – JPY-USD American binaries with model prices

The results for the american binary options are shown in Figure 6. This figure shows the same graph as Figure 4 but with the model prices superimposed on the market data. The values calculated using a deterministic implied tree model are also shown for reference. The model shows very good agreement with the market prices. The results have the correct functional form and the correct magnitude. In contrast the implied tree results are a long way from both the model and the market.

The american binary options have a very significant exposure to one particular second order term. This term is the change in the option's vega as the volatility changes. In a stochastic volatility environment this exposure has a associated value which gives rise to the significant offset from the Black Scholes price. The implied tree model, being deterministic, does not assign much value to this second order exposure which explains why its results are so far from the model.

Risk Management.

The purpose of developing this model is not just to be able to price exotic price path dependent options in a manner consistent with the market, but also to be able to manage the risk on these trades. The second order effects which impact the exotic option prices also have a significant impact on the basic option sensitivities such as delta and theta. In

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order to hedge these deals in a smile environment it is necessary to be able to calculate the correct sensitivities.

An insight into the effect that the smile has on the option sensitivities can be gained by looking at the option breakeven situations. Consider a portfolio consisting of a long option position that is delta hedged. In a non-smile environment the Black Scholes equation shows how the portfolio value changes with time and with moves in the asset value. Ignoring the effects of carry, funding and slide, the Black Scholes contingent claims equation can be written schematically as

$$\Theta + \frac{1}{2}\Gamma_s \Delta S^2 = 0 \qquad \text{(Equation 7)}$$

where Θ is the time decay of the portfolio, Γ_s is the gamma with respect to the asset value and ΔS is a change in the asset value consistent with the volatility of the asset. You would expect the portfolio to lose value over time due to the time decay of the option, however you would expect to be able to make back this value by trading the gamma position as the asset value moves. In order to breakeven the market must experience the asset value moves expected from the option volatility.

Hull and White (1990) show how the Black Scholes equation should be modified in order to take into account stochastic volatility. The schematic equation now becomes

$$\Theta + \frac{1}{2}\Gamma_S \Delta S^2 + \frac{1}{2}\Gamma_\sigma \Delta\sigma^2 + \Gamma_{\sigma S}\Delta\sigma\Delta S = 0 \qquad \text{(Equation 8)}$$

where Γ_σ is the gamma of the portfolio with respect to volatility and $\Gamma_{\sigma S}$ is the cross gamma with respect to both the asset value and volatility and $\Delta\sigma$ is a change in volatility consistent with the volatility of volatility. These second order terms have a substantial impact on the breakeven conditions and consequently affect the values of the usual first order sensitivities. In order to breakeven in this environment the market must also experience the moves in volatility expected from the volatility of volatility as well as the expected correlated moves in volatility and the asset value.

The pricing model can be used to calculate the option sensitivities in a smile environment by tweaking the input market parameter. The basic first order sensitivities such as delta and gamma can be significantly different from their value in a Black Scholes environment.

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Conclusions

The model presented here provides a way to price exotic options in a manner consistent with the process driving the vanilla smile. This methodology has been shown to match closely the market prices for exotic options. The development of this model not only provides the opportunity to analytically price path dependent options in a manner consistent with the european market but also to be able to manage the risk on these products more efficiently.

The fact that the model agrees to such a degree with the market prices provides confirmation that the assumptions behind the model are close to the actual market mechanism. Certainly, it adds weight to the argument that the volatility smile is driven chiefly by the market assigning value to the risks associated with a stochastic volatility. It should also be noted that the model is using information purely derived from the vanilla options market and there is no fitting to the exotic market data. The results therefore demonstrate that both the exotic and vanilla markets are consistent in their approach to valuing smile exposures.

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