Option Pricing Implications of a Stochastic Jump Rate

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Abstract

This paper proposes an alternative option pricing model. The stock price follows a diffusion process with stochastic volatility and random jumps while mean jump rate is modeled as a stochastic process. The model is designed to address the problems with alternative option models, for example, the extreme parameters of stochastic volatility model and the fast convergence to normality of jump model. The proposed general model setting incorporates all current alternative models under Brownian motion framework, including the pure jump model, stochastic volatility model and combination of the two. Option pricing formula for the model is derived. Model parameters are backed out from option prices. In-sample and out-of-sample performance of different sub-models in the framework indicates that the stochastic jump rate is indeed an important improvement to current option pricing models. This paper also first compares these alternative models with Levy process option models, which are characterized by the assumption that stock price is a purely discontinuous Levy process. Empirical results show that generally, Levy process models are limited in their capacity to improve Black-Scholes option pricing behavior.

1 Introduction

1.1 The "Smile"

Since the Black-Scholes option pricing model was introduced in 1973, it has become the most widely used and most powerful tool for trading in option markets. Over the past two decades, however, researchers have found significant deviations of market prices from predictions by the model. Out-of-money options tend to be traded at prices higher than the Black-Scholes predictions. The fact is indicated by the so-called volatility "smile". The Black-Scholes model assumes a constant volatility for options written on one asset, while the implicit volatilities derived from market prices are U-shaped when plotted against the strike price. Evidence of the smile in stock options is documented by, among others, Rubinstein (1985,1994), Dermin and Kani(1994), Dupire (1994). Melino and Turnbull (1990,1991) also found a similar phenomenon with currency options.

Some characteristic features of the smile are also observed . First, for at-the-money options, the implicit volatility increases with time to expiration. This observation is sometimes referred to as the "term structure" of implicit volatilities. Second, the smile has a stronger effect over short-maturity options, and it tends to flatten out monotonically with the increase in option maturity. Third, the shape of the smile is not fixed over time. Bates (1991) found substantial evolution of the smile in S&P 500 futures options over the period 1985-87. It was roughly symmetric during most of 1986 and upward sloping in early 1987 and after the October crash. Many other researchers confirmed that since the market crash in 1987 out-of-money puts have been undervalued by Black-Scholes - - and out-of-money calls overvalued. In another word, the smile changes into a "*smirk*" since 1987.

The inconsistency with the predictions of Black-Scholes raises questions about the theoretical assumptions underlying the model, in which the underlying asset price has a conditional log-normal distribution and returns in different periods are i.i.d normal. The presence of volatility smile indicates that the distribution implied in option prices has longer tails than log-normal distribution and the smirk effects suggests that the underlying distribution is left skewed. The term structure of smile signals a complex evolution of higher moments in the underlying distribution.

1.2 Alternative Models

The recent bad performance of the Black-Scholes model has attracted much attention from financial economists, who have tried to find a thicker-tailed, more left-skewed distribution to better fit the data. These efforts are further justified by evidence from time series studies that stock returns are not i.i.d normal, and there is also the question of whether a continuous model is appropriate as a model for asset prices. Alternative models have been proposed to relax these assumptions.

Eberlein and Keller (1995) and Eberlein, Keller and Prause (1998) proposed a hyperbolic option pricing model, under which stock returns (although still i.i.d.) are modeled as having a hyperbolic distribution. In their model, the price process itself is purely discontinuous, having infinitely many jumps in each finite interval of time. Madan and Seneta (1991) and Madan and Chang (1998) proposed another model featuring discontinuous prices, the variance-gamma model. Both are examples of "Levy" processes and both imply that the distributions of the returns are leptokurtic and skewed.

Merton (1976) also challenged the continuous property of the Brownian motion model, adding random jumps governed by a Poisson process to the continuous path. Merton assumed that jumps are non-systematic and diversifiable, requiring no compensation in average returns. Bates (1991) further discussed non-diversifiable jumps and worked out an elegant solution that takes jump risks into account.

Hull and White (1987) and Heston (1993) targeted the constant volatility assumption in the Black-Scholes model. In their models the volatility itself is modeled as a mean-reverting stochastic process driven by a Brownian motion. Heston (1993) generalized Hull and White's model (1987) by allowing the volatility process to be correlated with the stock price process. This seems to be essential in generating excess skewness in the distribution. The stochastic volatility models are related to the ARCH/GARCH models, of which Nelson (1993) showed to be the continuous time limit.

Bates (1996a) applied a stochastic-volatility-with-jump model to the foreign exchange option market, and Bakshi, Cao and Chen (1997) incorporated stochastic interest rates, stochastic volatility and jumps in one general setting to evaluate their performance with S&P 500 options.

1.3 A Quick Evaluation of Alternative Models

Despite the variety of alternative option models, some facts remain unexplained. As Bates (1996b) pointed out, the persistent time evolution of higher moments implicit in option prices could not all be explained away by jumps and stochastic volatility. The problem expresses itself in terms of unstable estimated parameters during different sample periods.

Jump and stochastic volatility models have their own strengths and weaknesses. Jump models easily capture the skewness and excess kurtosis for options with a short maturity. As the maturity increases, the jump component of returns converges to normal. This is consistent with the fact that the smile tends to flatten out for longer maturity options. But according to Das and Sundaram (1998), the convergence rate of the jump component is so fast that the excess skewness and kurtosis become negligible in a short time (in their example, three months), and jump models have a difficult time to fit the smile of options with medium maturity.

Stochastic volatility models, on the other hand, are not capable of generating high levels of skewness and kurtosis at short maturities under reasonable parameterization. For S&P 500 options data, in order to generate sufficient negative skewness, stochastic volatility model requires a negative correlation of about -0.64 between price and volatility process, while the correlation derived from time series is only around -0.12 according to findings by Bakshi, Cao and Chen (1997). However, the stochastic volatility models have rich implications about term structures and are more promising than the jump models when multiple maturity is involved.

The hyperbolic model and the variance-gamma model are shown to outperform the Black-Scholes model using German stock options data (Eberlein, Keller and Prause (1998)) and S&P 500 options data (Madan and Chang (1998)). There is no cross comparison about their competitiveness with Brownian motion type models listed above.

1.4 Theoretical Motivation

My research attempts to reevaluate the jump model by allowing the jump frequency (mean jump rate) to be an independent stochastic process. Including a stochastic feature to the model in addition to stochastic volatility and jumps may help to capture some of the time variation in higher moments implicit in option prices. It also has the potential for improving the traditional jump models, which have already proven effective in correcting the Black-Scholes pricing errors. The main problem with traditional jump models is their quick convergence to normality. The newly added randomness of the jump rate gives the model some features of stochastic volatility models and can be a counterforce to the convergence process.

There is also an economic rationale for this model. If we interpret jumps as market responses to information arrivals, it seems overly simple to assume a constant information arrival rate overtime in today's complex financial market. Moveover, risks brought about by jumps are usually measured as the product of the jump frequency and the mean size of jump. A constant jump intensity implies fixed jump risks over different time periods, which also seems problematic.

The model is specified in a general framework, under which almost all current option models with Brownian motion can be accommodated. The effectiveness of stochastic jump frequency will be studied against traditional jump and stochastic volatility assumptions. I am also interested in comparing the Levy process models with competing alternative option models. This will provide some insights as to the most promising direction in improving the Black-Scholes model. Figure 1 illustrates the relationship between alternative models covered in this paper.

The structure of the paper is as follows. Section II specifies theoretical models to be studied in this research. The first part describes the proposed model and its solution for option price. The second part briefly introduces hyperbolic and variance-gamma models and their solutions. The solutions are proposed in a slightly different manner, which not only facilitates estimation but also makes them more comparable with other models. Section III discusses the data and estimation procedure. Empirical results are presented in section IV. Section V concludes and outlines future research directions.

2 Model Specifications

2.1 A Stochastic Volatility and Stochastic Jump Frequency Model

The model with stochastic jump frequency and stochastic volatility is specified as follows:

$$
\begin{aligned}
\frac{ds}{s} &= (\mu - d - \lambda\kappa) \cdot dt + \sqrt{v} \cdot dZ_t + u \cdot dq \\
dv &= (\alpha - \beta v) \cdot dt + \sigma_v \sqrt{v} \cdot dW_t \\
d\lambda &= (\theta - \eta\lambda) \cdot dt + \sigma_\lambda \sqrt{\lambda} \cdot dM_t
\end{aligned}
\tag{1a}
$$

μ : mean return of stock

$d : dividend$

$prob(dq = 1) = \lambda \cdot dt$

$corr(dZ_t, dW_t) = \rho$

$ln(1 + u)\tilde{\ } N(ln(1 + \kappa) - \frac{1}{2}\delta^2, \delta^2)$

Here s is the current stock price; v is the square of stock volatility; Z, W and M are Brownian motions. M is independent of Z and W and all three are independent of the jump u. The stock price is characterized as a diffusion process driven by the Brownian motion Z but with random jumps following a *Poisson* distribution with parameter λ. The jump frequency, λ, follows a mean-reverting process driven by an independent Brownian motion . A jump has a mean size of κ and variance δ^2 given that occurs. The stock price process is correlated with the volatility process, as captured by a non-zero correlation coefficient ρ.

Under this model the market is not complete. The volatility risk and the ever-changing jump risk can't be hedged away using traded asset. We no longer have risk-neutral equivalence as in the Black-Scholes world. Fortunately, the risk-neutral transformation of the model can be obtained by explicitly taking risks into account. Applying the Cox, Ingersoll and Ross (1985b) general equilibrium results and assuming log utility of market participants, the model can be rewritten in a slightly different way:

$$\begin{aligned} \frac{ds}{s} &= (r - d - \lambda\kappa^*) \cdot dt + \sqrt{v} \cdot dZ_t + u \cdot dq \\ dv &= (\alpha - \beta^* v) \cdot dt + \sigma_v \sqrt{v} \cdot dW_t \\ d\lambda &= (\theta - \eta^* \lambda) \cdot dt + \sigma_\lambda \sqrt{\lambda} \cdot dM_t \end{aligned} \tag{2}$$

r : interest rate

$ln(1 + u)\tilde{\ } N(ln(1 + \kappa^*) - \frac{1}{2}\delta^2, \delta^2)$

These starred variables are risk-neutral parameters, and they differ from their counterparts by an implicit risk premium. We work mainly with this model.

This model specification can accommodate all the previous models under the Brownian motion framework. (1) Black-Scholes model (B-S). Upon setting $\alpha = \beta^* = \theta = \eta^* = \sigma_v = \sigma_\lambda = \kappa^* = \delta = \rho = 0$; (2) Merton's (1976) pure jump model (Merton) by $\alpha = \beta^* = \theta = \eta^* = \sigma_v = \sigma_\lambda = \rho = 0$; (3) Heston's (1993) stochastic volatility model (Heston), by allowing $\theta = \eta^* = \sigma_\lambda = \kappa^* = \delta = 0$; (4) Bates jump-diffusion with stochastic volatility (Bates) model, with $\theta = \eta^* = \sigma_\lambda = 0$; (5) The stochastic jump model (S-J), if we let $\alpha = \beta^* = \sigma_v = \rho = 0$; (6) The general stochastic jump & volatility model (SV_SJ) with none of the above restrictions are imposed on the parameters.

2.2 Solution for Option Price

The value of an European call option on the stock with strike price X and expiration date T can be expressed as:

$$\begin{aligned} C(S_t, X, T) &= e^{-r(T-t)} E_t^*(S_T - X)^+ \\ &= e^{-r(T-t)}[(\int_X^\infty S_T P_t^*(S_T) dS_T - X \int_X^\infty P_t^*(S_T) dS_T)\ ] \\ &= S_t \cdot \pi_1 - B(t,T) \cdot X \cdot \pi_2 \end{aligned} \tag{3}$$

where E_t^* is the risk-neutral expectation conditional on what is known at time t; $P_t^*(\cdot)$ is the risk-neutral density function for S_T; $B(t,T) = e^{-r(T-t)}$ is the risk neutral discount factor from time t to T; and $\pi_2 = \Pr(S_T > X)$ is a risk-neutral probability. The quantity $\pi_1 \equiv B(t,T) \cdot S_t^{-1} \int_X^\infty S_T P_t^*(S_T) dS_T$ can also be viewed as a probability if we express the cumulative distribution function of S_T , $F_{S_T}(s)$ as:

$$F_{S_T}(s) = \int_0^s w P_t^*(w) dw \tag{4}$$

and define a new cumulative distribution function as:

$$\widetilde{F}_{S_T}(s) = \frac{\int_0^s w P_t^*(w) dw}{\int_0^\infty w P_t^*(w) dw} = B(t,T) \cdot S_t^{-1} \int_0^s w P_t^*(w) dw \tag{5}$$

for $s \geq 0$ and $\widetilde{F}_{S_T}(s) = 0$ elsewhere. The second equality in (5) comes from the fact that $P_t^*(\cdot)$ is the risk-neutral density function of S_T , and the expected return of any asset in this risk-neutral world should be risk-free interest rate, so that

$$E^*(S_T) = B^{-1}(t,T) \cdot S_t \tag{6}$$

Applying the new cumulative distribution function, $\widetilde{F}_{S_T}(s)$, we easily see that

$$\pi_1 = \int_X^\infty \frac{S_T}{E^*(S_T)} P_t^*(S_T) dS_T = 1 - \widetilde{F}_{S_T}(X) \tag{7}$$

The two probabilities π_1 and π_2 can be derived by inverting the corresponding characteristic functions. The characteristic function of $s = \log(S_T/S_t)$ that is associated with π_2, $\phi_2(\xi; v, \lambda, \Theta)$, is:

$$
\begin{aligned}
\phi_2(\xi; v, \lambda, \Theta) = \exp\{&(r-d)(T-t)i\xi - \frac{\alpha(T-t)}{\sigma_v^2}(\rho\sigma_v i\xi - \beta^* - \gamma_2) \\
&- \frac{2\alpha}{\sigma_v^2} ln(1 + \frac{(\rho\sigma_v i\xi - \beta^* - \gamma_2)(1 - e^{\gamma_2(T-t)})}{2\gamma_2}) - \frac{\theta(T-t)}{\sigma_\lambda^2}(-\eta^* - \gamma_2^{'}) \\
&- \frac{2\theta}{\sigma_\lambda^2} ln(1 + \frac{(-\eta^* - \gamma_2')(1 - e^{\gamma_2'(T-t)})}{2\gamma'}) \ + \frac{i\xi + \xi^2}{\rho\sigma_v i\xi - \beta^* + \gamma_2 \frac{1+e^{\gamma_2(T-t)}}{1-e^{\gamma_2(T-t)}}} \cdot v \\
&- \frac{2[(1+k^*)^{i\xi} e^{\frac{\delta^2}{2}(i\xi - \xi^2)} - 1 - \kappa^* i\xi]}{-\eta^* + \gamma_2' \frac{1+e^{\gamma_2'(T-t)}}{1-e^{\gamma_2'(T-t)}}} \cdot \lambda\},
\end{aligned} \tag{8}
$$

where

$$
\begin{aligned}
i &= \sqrt{-1} \\
\gamma_2 &= \sqrt{(\rho\sigma_v - \beta^*)^2 + \sigma_v^2(i\xi - \xi^2)} \\
\gamma_2^{'} &= \sqrt{\eta^{*2} - 2\sigma_\lambda^2[(1+\kappa^*)^{i\xi} e^{\frac{\delta^2(-\xi^2 - i\xi)}{2}} - 1 - \kappa^* i\xi]}
\end{aligned} \tag{9}
$$

The details of deriving the function are given in Appendix 1. A similar method is applied to get the characteristic function associated with π_1, $\phi_1(\xi; v, \lambda, \Theta)$. Alternatively we can take a short cut by making use of the relationship between ϕ_1 and ϕ_2.

$$
\begin{aligned}
\phi_1(\xi; v, s) &= \int_0^\infty e^{i\xi \ln(s)} d\widetilde{F}_{S_T}(s) \\
&= B(t,T) S_t^{-1} \int_0^\infty e^{i\xi \ln(s)} s \cdot dF_{S_T}(s) \\
&= B(t,T) \int_0^\infty e^{(i\xi+1)\ln(s)} dF_{S_T}(s) \\
&= B(t,T) \phi_2(\xi - i)
\end{aligned} \tag{10}
$$

The probabilities can be calculated by finding the inverse Fourier transform of the characteristic functions.

$$
prob(S_T > X|\phi_i) = \pi_i = \frac{1}{2} + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\phi_i \cdot e^{-\ln(\frac{X}{s_t})\iota\xi}}{\iota\xi} d\xi \tag{11}
$$

Following Kendall, Ord, Stuart(1987), after some algebraic manipulation, the integration can be transformed into:

$$prob(S_T > X|\phi_i) = \pi_i = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \frac{imag(\phi_i)\cdot e^{-\ln(\frac{X}{s_t})i\xi}}{\xi}d\xi \tag{12}$$
$$= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \frac{\mathrm{real}(\phi_i)\cdot e^{-\ln(\frac{X}{s_t})i\xi}}{\iota\xi}d\xi$$

2.3 Models Based on Levy Processes

2.3.1 The Hyperbolic Model

The hyperbolic distribution was introduced into the financial context by Eberlein and Keller (1995). It is specified as follows:

$$dS_t/S_{t^-} = \mu dt + dY_t + e^{\sigma\triangle Y_t} - 1 - \sigma \triangle Y_t \tag{13}$$

where Y_t is a random variable with a hyperbolic density function:

$$f_{(\alpha,\beta,\delta,\mu)}(y) = \frac{\sqrt{\alpha^2-\beta^2}}{2\alpha\delta K_1(\delta\sqrt{\alpha^2-\beta^2})}e^{-\alpha\sqrt{\delta^2+(y-\mu)^2}+\beta(y-\mu)} \tag{14}$$

Here K_1 denotes the modified Bessel function of the third kind with index unity. The moment generating function takes the following form

$$M(\xi) = e^{\mu\xi}\frac{\sqrt{\alpha^2-\beta^2}}{K_1(\delta\sqrt{\alpha^2-\beta^2})}\frac{K_1(\delta\sqrt{\alpha^2-(\beta+\xi)^2}}{\sqrt{\alpha^2-(\beta+\xi)^2}}. \tag{15}$$

The hyperbolic process has almost surely infinitely many discontinuities in any finite interval of time. When the stock returns follow the hyperbolic distribution as specified above, the payoffs of options can't be replicated with portfolios of riskless bonds and the underlying asset alone. Thus the equivalent martingale measure is not unique. Eberlein and Keller (1995) proposed an "Esscher transform" approach to find an equivalent measure, but their "statistical approach" is more comparable to other methods that I shall apply. Risk-neutrality imposes the following relationship among parameters:

$$\mu = r - \ln \frac{K_1(\delta\sqrt{\alpha^2 - (\beta+1)^2})}{K_1(\delta\sqrt{\alpha^2 - \beta^2})} + \frac{1}{2} \ln \frac{\alpha^2 - (\beta+1)^2}{\alpha^2 - \beta^2} \tag{16}$$

Substituting for μ in (15) and converting $M(\xi)$ to the characteristic function, as $\phi(\xi) = M(i\xi)$, the price of an European option can be derived by the same method discussed in the last section.

2.3.2 Variance-Gamma Model

Madan and Seneta (1990) and Madan and Milne (1991) proposed a model under which stock prices follow another purely discontinuous process — variance-gamma process. This process has two parameters (θ and ν) that account for skewness and kurtosis in addition to the volatility parameter (σ) in the Black-Scholes model.

The risk-neutral process of stock price is specified as

$$\begin{aligned} S(T) &= S(t)\exp(r(T-t) + X(\sigma, v, \theta) + \varpi(T-t)) \\ \varpi &= \frac{1}{v}\ln(1 - \theta v - \sigma^2 v/2) \end{aligned} \tag{17}$$

and $X(\sigma, v, \theta)$ is a variance-gamma distribution — a mixture of normals with gamma distributed variance – with density function

$$f(x) = \int_0^\infty \frac{1}{\sigma\sqrt{2\pi g}} \exp[-\frac{(x-\theta g)^2}{2\sigma^2 g}] \frac{g^{\frac{(T-t)}{v}-1}\exp(-\frac{g}{v})}{v^{\frac{t}{v}}\Gamma(\frac{T-t}{v})} dg \tag{18}$$

The characteristic function for the variance gamma process is

$$\phi(\xi) = (1 - \iota\xi\theta v + (\sigma^2 v/2)\xi^2)^{-(T-t)/v} \tag{19}$$

As before, knowing the characteristic function, we can use Fourier inversion to get the probabilities in the option pricing formula (3), as described in equation (12).

2.3.3 Fast Fourier Transform (FFT)

Given a set of model parameters Θ, the crucial step of calculating an option price is to compute the two probabilities in equation (12), which requires evaluating the infinite integrals involved in the Fourier inversion. The integrals can be evaluated numerically. The IMSL Fortran library provides a subroutine DQDAGS that can handle quite complicated integrands. Although DQDAGS is efficient in approximating one integral, the Fast Fourier Transform (FFT) is a better technique when we need to price many options with different strike prices all at once. Since the probabilities derived from the FFT may not be at the exact point of price/strike ratio for each option individually, we employ a two-point extrapolation rule to get the approximate probability for each price/strike ratio. Table 1 compares accuracies of FFT and DQDAGS and shows that the difference is acceptably small. Calculation time can be reduced by more than half using the FFT.

Table 1

Difference Between Option Prices from DQDAGS and FFT

Number of Options	Range of log(s/x)	Max. Diff.	Mean Diff.	Min. Diff.
5852 (t=0.25)	[-0.78, 0.42]	0.01	0.004	0.0
5852 (t=0.35)	[-0.78, 0.42]	0.007	0.002	0.0
5852 (t=0.50)	[-0.78, 0.42]	0.02	0.004	0.0

Note: The check is done with current strike price set at 100.00 and stock prices calculated according to the strike to price ratio. The t value in the parenthesis is time to expiration of options. The exercise is about the Bates_SJ model with hypothetical model parameters.

3 Data and Estimation

3.1 Data Description

This research applies the models to individual daily CBOE transaction data during the month of July 1996. Each transaction record includes the time at which the transaction takes place, the strike price, time to maturity, the bid and ask option price, and the matching stock price. We expect the non-synchronous trading problem to be minimal. The analysis is applied to a representative individual stock option and S&P 500 index option. IBM is selected for the individual stock option analysis because of its active trading. Most individual stock options traded on CBOE are of the American type, which should be valued more than their European counterparts. However, if we restrict our subjects to call options only and if there is no dividend payment during the lifetime of the option, then the value of an American option is not different from that of a corresponding European option.

The IBM options are actively traded. To avoid noisy information, all options with less than one week to expire are discarded. There was a dividend payment in the amount of \$0.35 on Sept. 10, 1996. To get around the problem of possible early exercise of American calls, we only use IBM call options maturing in July and August. There are around 1,000 transactions for each maturity on an average day.

S&P 500 options are European options, an advantage that provides us with a wide range of strike prices and maturity date to work with. Four maturity periods are selected for estimation on each trading day—July, August, December and March before July options mature on the 18th. After July options mature, October options are added to the analysis. The maturity of the options range from 8 days to 185 days. Options with maturity less than a week and price less than \$1 are disregarded in order to avoid possible noises.

Trading in S&P 500 options are extremely intensive. To alleviate the computational burden, we only include call options in this stage. On each trading day, we pick the first valid transaction record for each strike price and maturity date as a representative data point. There are typically 60 to 100 options data available everyday.

Treasury bills yields with corresponding expiration times are used as a proxy for the interest rate. Although the interest rate is assumed constant over the lifetime of each option, the yield information was manually collected for each trading day in July, 1996. We take the average of the bid and ask yield as our interest rate. Treasury bills are due on the Thursdays of each month, so we pick the rate corresponding to the third Thursday of each month.

Time to maturity of both options are calculated in terms of trading days. In the case of S&P 500 options, the annual dividend rate is extracted from the daily S&P 500 future price and S&P 500 closing price by the following relationship:

$$d = r - \ln(F/S)/t \tag{20}$$

where d is the annual dividend rate, r is the interest rate, F is the future price, S is the current stock price and t is time to maturity. We use the average of the calculated dividend rate.

To see whether the IBM and S&P 500 data are appropriate for a study designed to capture the smile effect, we first construct the implicit volatility curve from option data on a typical day. Figure 2 and Figure 3 clearly show that a smile does exist in those data, signaling a failure of the Black-Scholes pricing formula.

3.2 Estimation Procedure

The theoretical models for prices pertain to a risk-neutral world. Neither the risk-neutral stochastic process nor the timing and size of jumps is directly observable. It is thus impossible to estimate the model directly from observed stock prices. However, the parameters can be backed out from option prices via nonlinear least square (NLS) method by solving

$$\min_{\Theta} \sum_j [C_j(S_t, X, T-t) - \widehat{C}_j(S_t, X, T-t; \Theta)]^2, \tag{21}$$

where $C_j(S_t, X, T-t)$ is the actual price of an option, struck at X and expiring at the $T-t$ units of time, and $\widehat{C}_j(S_t, X, T-t; \Theta)$ is the price of the same option predicted by the model, given parameters Θ.

The parameters are the ones that minimize the mean squared errors of market prices from predicted prices. Defining the mean squared errors as the absolute difference between theoretical price and market price would put more weight on high-priced options — usually in-the-money options and options with a longer maturity — than on low-priced options. Nevertheless, a lot of researchers use this simple method. Our choice is consistent with that of Bakshi, Cao and Chen (1997) and Bates (1996a).

4 Empirical Results

4.1 In-Sample Fit

The estimation procedure described above is repeated for each model with each trading day's data. The averages of all parameters are reported in Table 2 and Table 3, for IBM data and S&P 500 data respectively. The resulting sum of mean squared errors (SSE) can be viewed as a measure of in-sample fit. Since there are different numbers of observations on different trading day, we report the ratio of sum of mean squared errors for each model to that of the Black-Scholes model. Roughly speaking, the lower the ratio, the better the model corrects the mispricing of Black-Scholes.

The empirical results from this research are quite interesting. For IBM data, we only focus on five sub-models of the general framework: the Black-Scholes model (B-S), Merton's pure jump model (Merton), Heston's stochastic volatility model (Heston), Bates' stochastic-volatility -with-jump model (Bates) and the constant-volatility-with-stochastic-jump model (S-J). Hyperbolic model is also estimated with IBM data.

The results from IBM data are generally consistent with the findings of previous research. The Merton pure jump model by itself reduces Black-Scholes squared pricing error by more than 50 percent. Stochastic volatility is still a more powerful model, cutting Black-Scholes errors by more than 60 percent and thus explaining a dramatic part of the smile. Combing stochastic volatility and jump models does a better job than either alone. As predicted, the stochastic volatility model implies a strongly negative correlation, ρ between stock prices and volatility. Having jumps in addition to stochastic volatility allows some negative skewness to be absorbed by the negative jump mean κ^* . For this reason, we get a smaller value of ρ for the Bates model.

One interesting observation is that the S-J model with constant volatility, which has one less parameter than the Bates model, turns out to give a slightly better in-sample fit. The tentative conclusion is that, in the case of an individual stock option, although the traditional jump model can not outperform the stochastic volatility model, it can do a lot better when the jump density is allowed to vary.

The hyperbolic model, which has never before been applied to US data, can reduce the Black-Scholes squared pricing errors by only a marginal 30 percent.

We conduct the same estimation procedure with the S&P 500 data. Due to the availability of more strike price range and more maturities, all the models, including the general SV_SJ model have been estimated. In the case of S&P 500 options, sub-model S-J still outperforms both the Merton model and the Heston model, but the Bates model, with one more parameter than the S-J model, turns out to be the best among all sub models. Parameter values are generally consistent with previous study applied to the same data in different time periods (Bakshi, Cao and Chen 1997), though the mean-reverting parameter β for stochastic volatility process is relatively large in this particular data set.

The SV_SJ model does improve the in-sample fit even further compared with the Bates model. This justifies our motivation of analyzing a stochastic jump rate process. As we should expect, compared with the simple stochastic volatility model, the Heston model, the Bates model fits the data better with a smaller negative correlation between stock price process and volatility process. The SV_SJ model attributes more negative skewness to the stochastic negative jump component and results in a even smaller correlation between stock price process and volatility process ρ. The SV_SJ model also improves the S-J model in the sense that it reduces the absolute value of mean jump rate required by channeling some of the smile effect through the stochastic volatility process.

From the in-sample fit results we can conclude that the stochastic jump rate improves the traditional jump diffusion model with constant jump rate. It makes jump-type models a competitive alternative against models based on stochastic volatility assumption. The improvement with S&P 500 data is significant, though it is not as great as with the IBM data. It may have something to do with the different natures of IBM options data and S&P 500 options data.

First, the jump type model still holds an advantage in fitting shorter maturity options. To some extent, the superior performance of S-J model in the IBM case may contribute to the relatively shorter maturity data we used in the estimation process. Second and more important, from our economic rationale where our jump type models are rooted, the source of jumps and stochastic movements in jump rate is information. In the individual stock option case, the information pertaining to the particular firm in question affects directly investors' expectations about the firm's future. S&P 500 option, on the other hand, consists of representative stocks from all industries, information affecting different firms or different industries tend to cancel out. Therefore it is hard for jump type models to capture the behavior of the stock in general with complicated co-movement between individual stock component.

Table 2

Estimated model parameters and in sample fit-IBM

	B-S	Merton	Heston	Bates	S-J	Hyp
α			0.57	0.34		545.34
β^*			6.15	8.23		-6.37
σ_v			0.005	0.003		
ρ			-0.74	-0.68		
λ		0.659		1.8	0.89	
κ^*		-0.0716		-0.027	-0.065	
δ^2		0.023		0.034	0.043	0.2
θ					2.04	
η^*					12.4	
σ_λ					0.45	
$\sqrt{\nu}$	0.305	0.28	0.28	0.274	0.276	
SSE (%)	100	46	36	27	25	73

Table 3

Estimated model parameters and in sample fit-S&P500

	B-S	Merton	Heston	Bates	S-J	SV_SJ	Hyp	V-G
α			0.14	0.15		0.068	22311	
β^*			4.57	8.93		6.5	-22194	
σ_v			0.48	0.22		0.20		
ρ			-0.82	-0.58		-0.48		
λ		1.42		0.39	0.44	0.41		
κ^*		-0.075		-0.11	-0.38	-0.21	0	
δ^2		0.008		0.011	0.15	0.047		
θ					1.53	0.66		-0.045
η^*					32.66	5.06		
σ_λ					4.05	1.069		
$\sqrt{\nu}$	0.14	0.12	0.15	0.15	0.11	0.14		
σ								0.091
v								0.054
SSE (%)	100	53	39	33	35	20	86	93

4.2 Out-of-Sample Forecast

Out-of-sample predictions of different models are presented in Table 4 and Table 5. The entries are average absolute pricing errors. To obtain this, parameters for each model were estimated with options data from each day and used to price options for the next day. The pricing errors for all alternative models (labeled Merton, Heston, Bates, S-J) and the standard Black-Scholes model (labeled B-S1) were generated in the same way, using all traded options to estimate the parameters. Due to the poor in-sample performance of Levy process models, out-of-sample prediction exercise is not performed on them.

The pricing errors in the column labeled B-S2 were produced under a different scheme, estimating a unique volatility for each price/strike ratio — a common empirical practice used by practitioners and traders. Specifically, we grouped the options data according to price/strike ratio, estimated a volatility particular to each of several ranges and used that particular volatility in pricing a similar option on the next day.

The results in Table 4 show that the Black-Scholes model had extreme difficulty in predicting mildly in–the-money IBM calls with price/strike ratio ranging from 1.03 to 1.07 during July 1996. The pricing errors are somewhat smaller with the strike-based volatility scheme, but still amounts to more than 10 cents on average. The Bates model and the S-J model are generally better than the others with the S-J model producing the smallest pricing errors except in the far out-of-money category. The Merton model does a good job in predicting far in–the-money calls, but its limitations are obvious, since it behaves even worse than Black-Scholes for calls out of money.

Results in Table 5 are out of sample pricing errors for S&P 500 options. We can see that the out-of-sample performance for Black and Scholes model is extremely unreliable, averaging around a dollar. The out-of-sample performance for the Merton model, the Heston model and the S-J model are very similar, with the Merton model performs better to predict far in-the-money calls and the Heston model does better over at-the-money options. The Bates model outperforms all other sub-models. The SV_SJ model is the best among all in most categories except for at-the-money options.

Table 4

Out of Sample Predictions-IBM

Price/Strike ratio	B-S1	B-S2	Merton	Heston	Bates	S-J
<0.93	0.052	0.053	0.083	0.051	0.041	0.046
[0.93,0.97]	0.092	0.091	0.077	0.081	0.073	0.071
(0.97,1.03]	0.114	0.112	0.056	0.063	0.062	0.051
(1.03-1.07]	0.410	0.123	0.090	0.058	0.058	0.049
>1.07	0.151	0.095	0.054	0.057	0.053	0.055

Table 5

Out of Sample Predictions-S&P500

Price/Strike ratio	B-S1	Merton	Heston	Bates	S-J	SV_SJ
<0.95	0.61	0.44	0.39	0.31	0.41	0.24
[0.95,0.98]	1.24	0.59	0.58	0.18	0.58	0.29
(0.98,1.03]	1.12	0.80	0.39	0.30	0.52	0.24
(1.03-1.07]	1.33	0.41	0.44	0.22	0.43	0.21

5 Conclusion and Future Research

In this research, we propose a new and general model for option prices that allows for stochastic volatility with jumps at a stochastic mean rate. A computational formula for the option price has been derived. The empirical results with IBM options data and S&P 500 options data show that adding a stochastic jump rate to the traditional jump model, generally reduces the pricing errors in Black-Scholes formula and makes the jump type models a strong alternative to stochastic volatility models. In the case of IBM options, the S-J sub-model outperforms the Bates model both in-sample and out-of-sample, even though it is a more parsimonious model.

This research also shows that the Levy process models couldn't compete with the Brownian motion type alternative models in reducing the Black-Sholes pricing errors. Both Hyperbolic model and variance gamma only manages to do better than the Black-Scholes model. The improvement is far less significant than Brownian type models with comparable complexity, for example, the Merton model and the Heston model. The discontinuity in the stock price process is addressed as discontinuous jumps to a diffusion process rather than as a purely discontinuous process, which is the core assumption of Levy process models.

This study also derives an interesting result. The stochastic jump assumption dramatically improves option pricing behavior, both in-sample and out-of-sample, in the case of an individual stock option. Its effects on the S&P 500 options are less dramatic. We tentatively explain that it may be due to different natures of the options. To understand this perplexing fact more clearly and to further study the merits for stochastic jump rate model, we need to take a closer look at the distribution implications of different models involved in this analysis. We can then examine the effects each parameter has on the behavior of the distribution and its higher moment.

Appendix 1

We determine the characteristic function corresponding to the following model specification:

$$\frac{ds}{s} = (r - d - \lambda\kappa^*) \cdot dt + \sqrt{v} \cdot dZ_t + u \cdot dq$$
$$dv = (\alpha - \beta^* v) \cdot dt + \sigma_v \sqrt{v} \cdot dW_t$$
$$d\lambda = (\theta - \eta^* \lambda) \cdot dt + \sigma_\lambda \sqrt{\lambda} \cdot dM_t$$

First transforming the first equation using Ito's lemma, as:

$$d \ln s = (r - d - \lambda\kappa^* - v/2) \cdot dt + \sqrt{v} \cdot dZ_t + \ln(1+u) \cdot dN_t,$$

we can work out the moment generating function of $\ln s$ as follows, the moment generating function, $M(\xi|s, v, \lambda, T-t)$, must satisfy the partial differentiation equation:

$$-M_{T-t} + (r - d - \lambda\kappa^* - \tfrac{v}{2})M_s + (\alpha - \beta^* v)M_v + \tfrac{1}{2}v(M_{ss} + 2\rho\sigma_v M_{sv} + \sigma_v^2 M_{vv})$$
$$+\lambda E[M(s + \ln(1+u)) - M(s)] + (\theta - \eta^*\lambda)M_\lambda + \tfrac{1}{2}\sigma_\lambda^2 \lambda M_{\lambda\lambda} = 0$$

subject to the following boundary condition:

$$M|_{T-t=0} = e^{\xi s}.$$

Guessing the functional form of M as

$$M(\xi; s, v, \lambda, T-t) = \exp(\xi s + A(T-t,\xi) + B(T-t,\xi)v + C(T-t,\xi)\lambda)$$

and plugging the proposed form into the partial differentiation equation gives

$$-A'_{T-t} - B'_{T-t}v - C'_{T-t}\lambda + (r - d - \lambda\kappa^* - \tfrac{v}{2})\xi + (\alpha - \beta^* v)B + \tfrac{1}{2}v(\xi^2 + 2\rho\sigma_v B + \sigma_v^2 B^2)$$
$$+\lambda[(1+k^*)^{\xi} e^{\delta^2(\xi^2-\xi)/2} - 1] + (\theta - \eta^*\lambda)C + \tfrac{1}{2}\sigma_\lambda^2 \lambda C^2 = 0.$$

The above equation will hold for all values of s, v, λ , so it must satisfy

$$-A'_{T-t} + (r-d)\xi + \alpha B + \theta C = 0 \qquad (1)$$
$$-B'_{T-t} - \frac{\xi}{2} - \beta^* B + \frac{1}{2}\xi^2 + \rho\sigma_v B + \frac{1}{2}\sigma_v^2 B^2 = 0 \qquad (2)$$
$$-C'_{T-t} - \kappa^*\xi + [(1+k^*)^{\xi} e^{\delta^2(\xi^2-\xi)/2} - 1] - \eta^* C + \frac{1}{2}\sigma_\lambda^2 C^2 = 0 \qquad (3)$$

From (2), we can solve for the functional form of $B(T-t,\xi)$ as

$$B(T-t,\xi) = \frac{-\xi - \xi^2}{\rho\sigma_v\xi - \beta^* + \rho\sigma_v + \gamma_1 \frac{1+e^{\gamma_1(T-t)}}{1-e^{\gamma_1(T-t)}}},$$

where

$$\gamma_2 = \sqrt{(\rho\sigma_v - \beta^*)^2 + \sigma_v^2(\xi - \xi^2)}.$$

From (3), we can solve for the functional form of $C(T-t,\xi)$ as

$$C(T-t,\xi) = -\frac{2[(1+k^*)^{\xi} e^{\frac{\delta^2}{2}(\xi+\xi^2)} - 1 - \kappa^*\xi]}{-\eta^* + \gamma_2' \frac{1+e^{\gamma_2'(T-t)}}{1-e^{\gamma_2'(T-t)}}},$$

where

$$\gamma_2' = \sqrt{\eta^{*2} - 2\sigma_\lambda^2[(1+\kappa^*)^{\xi} e^{\frac{\delta^2(\xi^2\iota\xi)}{2}} - 1 - \kappa^*\xi]} \; .$$

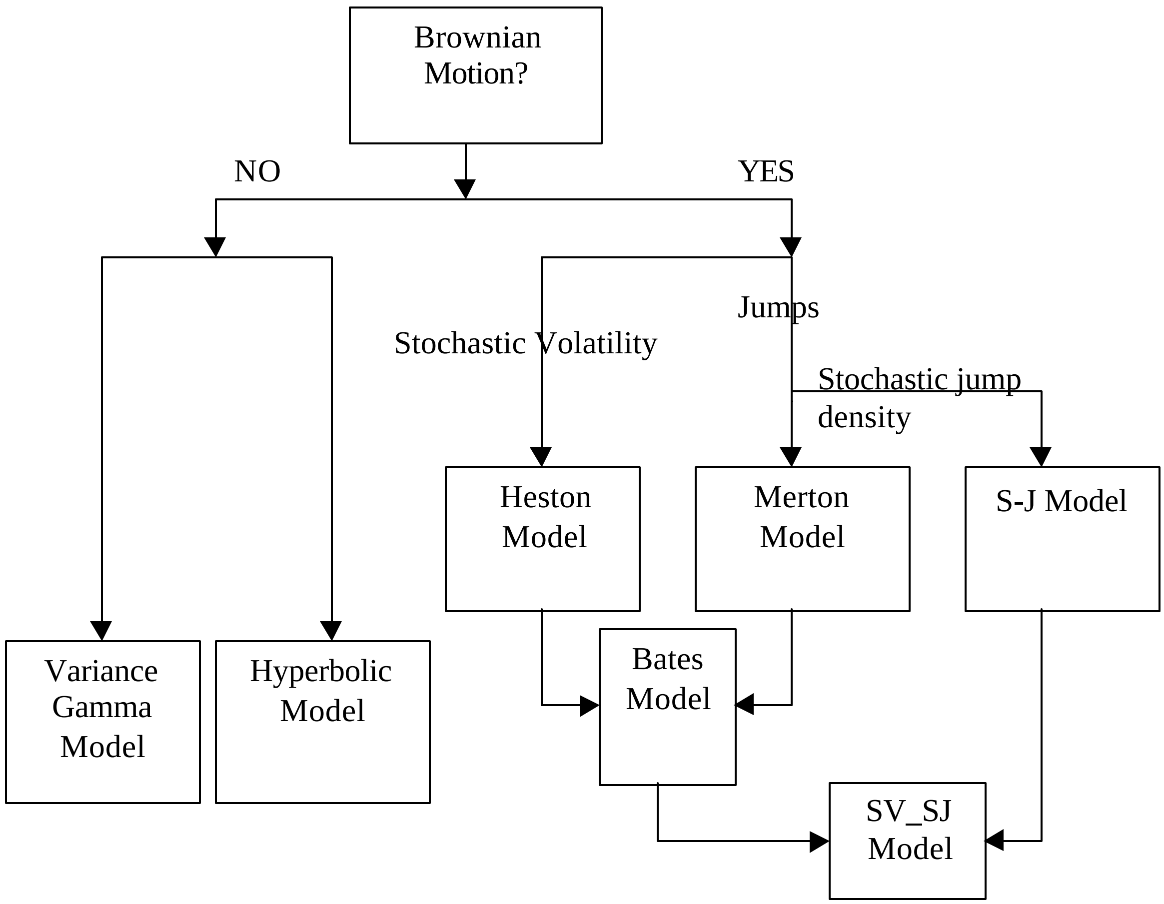
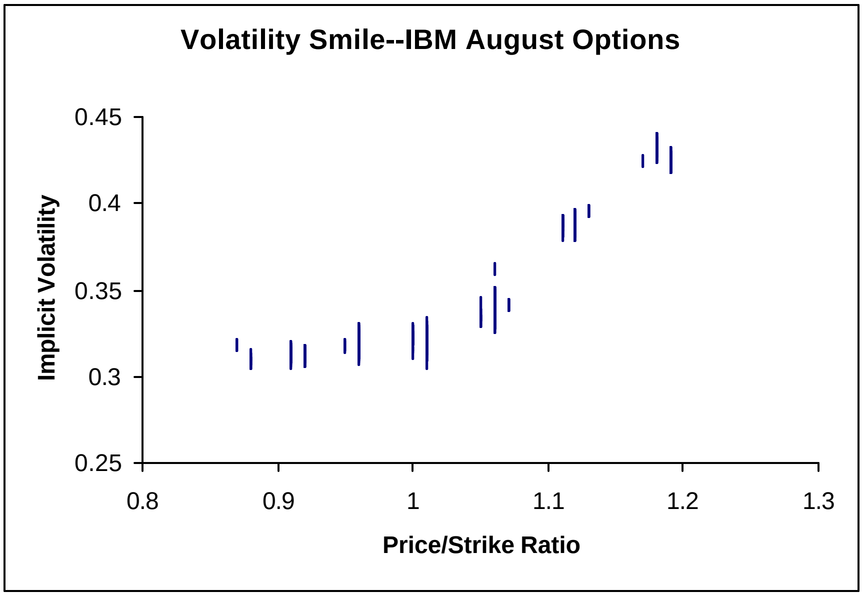


Figure 1: Relationship of Alternative Option Pricing Models



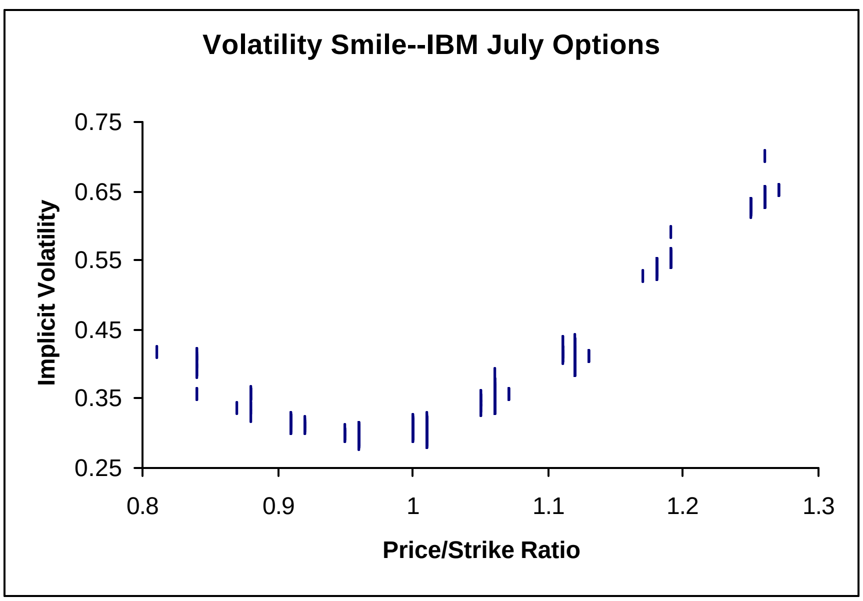
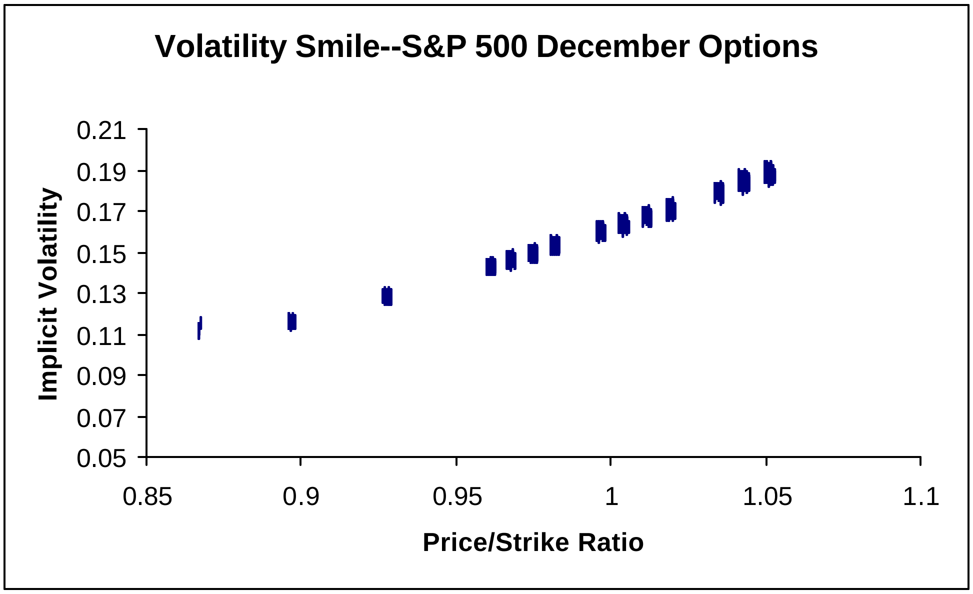


Figure 2: Volatility Smile of IBM Options



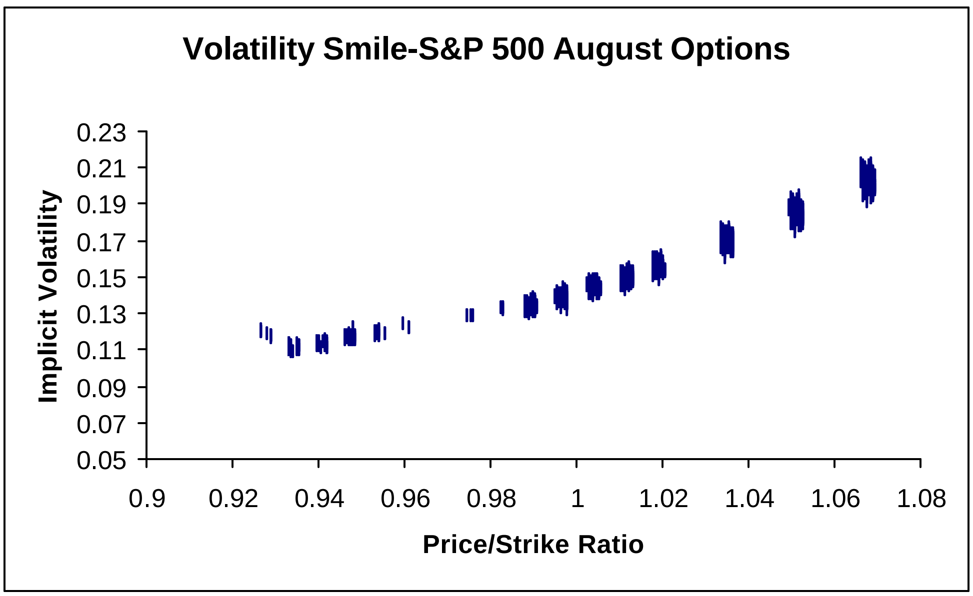


Figure 3: Volatility Smile of S&P 500 Options

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