

# Good-deal option price bounds with stochastic volatility and stochastic interest rate.

John H. Cochrane\* and Jesús Saá-Requejo<sup>†‡§</sup>

January 4, 1999

## Abstract

Good deal bounds find maximum and minimum option prices in an incomplete market consistent with the absence of arbitrage and a Sharpe ratio limit on all portfolios of hedging assets and the option. We calculate good deal bounds on a call option in a setup with stochastic volatility and a stochastic interest rate. We find that the bounds are quite large in this case, indicating that assumptions about the market prices of interest rate and volatility shocks are, far from innocuous assumptions, central to the results of stochastic volatility models.

---

\*University of Chicago, Federal Reserve Bank of Chicago and NBER.

<sup>†</sup>University of Chicago.

<sup>‡</sup>Address correspondence to John Cochrane, Graduate School of Business, University of Chicago, 1101 E. 58th St., Chicago IL 60637. (773) 702 3059, john.cochrane@gsb.uchicago.edu.

<sup>§</sup>We thank George Constantinides, Kent Daniel, Douglas Diamond, Andrea Eisfeldt, Lars Hansen, Ken Singleton, Steve Ross, many seminar participants and especially Yacine Aït-Sahalia for useful comments. This research was partially supported by the University of Chicago Graduate School of Business and by a grant from the NSF administered by the NBER.

# 1 Introduction

Option pricing techniques are based on replication arguments. In the classic example of Black and Scholes (1973) and Merton (1973b), one finds a portfolio of stocks and bonds that exactly replicates an option's payoff. Then, the option's value must be the same as that of the stock and bond. This conclusion follows with no assumptions about preferences or the structure of aggregate risks beyond the law of one price.

Alas, perfect replication often breaks down. Even when one can trade continuously, state variables such as stochastic volatility or a stochastic (time-varying) interest rate do not correspond to traded assets, or the option may be written on an underlying event that is not a traded asset. In this situation, the pure law of one price says *nothing* about the option's price, and arbitrage bounds are typically too wide to be of much practical use. However, there are often good approximate hedges, and it seems a shame to waste their information by going back to imprecise fully specified models such as the CAPM or consumption-based model.

In this situation Cochrane and Saá-Requejo (1998) advocate a bound on discount factor volatility as a way to slightly strengthen law of one price or absence of arbitrage arguments. A discount factor volatility constraint rules out high Sharpe ratios, which has a long tradition in finance. It also implements the idea that the CAPM is not too bad an approximation: the market portfolio may not be exactly mean-variance efficient, but if it is not too far inside the mean variance frontier then Sharpe ratios of other assets cannot exceed it by too much. The discount factor volatility constraint is equivalent to a constraint that the total (sum of squared) market price of risk of all shocks must lie below a given value.

The resulting option price bounds are not only interesting when they are tight: loose bounds are a sign that imperfect replication is an important problem, and that option pricing results are strongly influenced by plausible assumptions about market prices of risk.

In this paper we calculate good-deal bounds on an index call option, using a model that incorporates stochastic volatility and a stochastic interest rate. The bounds are reasonably tight, but still allow a large range of volatility smiles. We conclude that volatility smile predictions due to stochastic volatility and interest rate are very strongly influenced by what

assumptions one makes about market prices of interest rate and volatility risk.

## 2 Good deal bounds.

Good-deal bounds are derived and motivated at length in Cochrane and Saá-Requejo (1998). We present here a short summary.

Let  $S$  denote the price of the stock, or more generally a vector of basis or hedging assets, which may pay dividends at the rate  $Ddt$ . Let  $V$  denote a vector of additional non-traded state variables, such as stochastic stock volatility and interest rate or a non-traded event on which the option is written. We want to value an option or other derivative asset that pays continuous dividends at rate  $x^c(S, V, t)dt$  and with a terminal payment  $x_T^c(S, V, T)$ . The lower good-deal bound then is generated by choosing a discount factor (marginal utility) process to minimize the option value,

$$\underline{C}_t = \min_{\{\Lambda_s, t < s \leq T\}} E_t \int_{s=t}^T \frac{\Lambda_s}{\Lambda_t} x_s^c ds + E_t \left( \frac{\Lambda_T}{\Lambda_t} x_T^c \right). \quad (1)$$

subject to the constraints that the discount factor  $\Lambda$  must be non-negative, and that it generates the price of the basis assets  $S$  via a similar formula. They show that the bounds satisfy a partial differential equation that though not very pretty can be solved numerically in a straightforward manner. That equation is given as proposition 2 below, and the remaining sections of the paper find solutions.

Cochrane and Saá-Requejo show that good-deal bounds are recursive, i.e. that we can find the lower bound at date  $t$  as the lowest value of a fictional asset that pays the lower bound at date  $t + \Delta t$ . Thus, we can write the objective for moving one step back in time,

$$\underline{C}_t \Lambda_t = \min_{\{\Lambda_s\}} E_t \int_{s=t}^{t+\Delta t} \Lambda_s x_s^c ds + E_t (\Lambda_{t+\Delta t} \underline{C}_{t+\Delta t}).$$

Letting  $\Delta t \rightarrow 0$ , we can write the objective in differential form,

$$E_t \frac{d\underline{C}}{\underline{C}} + \left( \frac{x_t^c}{\underline{C}} - r^f \right) dt = - \min_{\{d\Lambda\}} E_t \left( \frac{d\Lambda}{\Lambda} \frac{d\underline{C}}{\underline{C}} \right). \quad (2)$$

We assume that there is a riskfree interest rate, and thus  $r_t^f = E_t(d\Lambda/\Lambda)$ . Since the second and third terms on the left hand side are fixed, this condition sensibly tells us to find the lowest price  $\underline{C}$  by maximizing the drift  $E_t d\underline{C}$  at each date. As usual, a discrete-time recursion

in which we infer  $\underline{C}_t$  from knowledge of  $\underline{C}_{t+\Delta t}$  becomes in continuous time a condition in which we infer the drift terms from knowledge of the volatility terms of the  $d\underline{C}$  process.

The constraints on the discount factor increment  $d\Lambda$  are 1) the discount factor must price the basis assets  $S$  at each moment,

$$E_t \frac{dS}{S} + \left( \frac{D}{S} - r \right) dt = -E_t \left( \frac{d\Lambda}{\Lambda} \frac{dS}{S} \right). \quad (3)$$

and 2) the instantaneous volatility of the discount factor is limited,

$$\frac{1}{dt} E_t \left( \frac{d\Lambda^2}{\Lambda^2} \right) \leq A^2. \quad (4)$$

This constraint imposes a Sharpe ratio limit. Since covariance is less than the product of standard deviations, any asset or portfolio that obeys (3) satisfies

$$\frac{(\tilde{\mu}_S - r)^2}{\sigma_S^2} \leq \frac{1}{dt} E_t \left( \frac{d\Lambda^2}{\Lambda^2} \right)$$

where  $\tilde{\mu}_S \equiv E_t(dS/S) + D/S dt$  is the conditional expected return and  $\sigma_S^2 \equiv E_t(dS^2/S^2)$  is the conditional variance of return.

To proceed, we need a statistical model of the state variables. We specify diffusion models,<sup>1</sup>

$$\frac{dS}{S} = \mu_S(S, V, t)dt + \sigma_S(S, V, t)dz \quad (5)$$

$$dV = \mu_V(S, V, t)dt + \sigma_{Vz}(S, V, t)dz + \sigma_{Vw}(S, V, t)dw \quad (6)$$

where  $dz$  and  $dw$  are orthogonal Brownian increments,  $E_t \left( \begin{bmatrix} dz & dw \end{bmatrix}' \begin{bmatrix} dz & dw \end{bmatrix} \right) = I$ . The instantaneous riskfree rate  $r(S, V, t)$  is an element of  $V$ . This and all statistical models specify the *true*, not a risk neutral probability measure.

In order to solve the problem 2 subject to constraints (3) and (4), it is convenient to use the fact that a discount factor  $\Lambda_t$  prices the basis assets  $S, r$  if and only if it can be represented as

$$\frac{d\Lambda}{\Lambda} = \frac{d\Lambda^*}{\Lambda^*} - vdw \quad (7)$$

---

<sup>1</sup>Rather than complicate the notation, understand division to operate element by element on vectors, e.g.,  $dS/S = [ dS_1/S_1 \quad dS_2/S_2 \quad \dots ]$ . When explicit enumeration of arguments is not necessary, we write  $S$  for  $S(t)$  and  $\mu_S$  and  $\sigma_S$  for  $\mu_S(S, V, t)$  and  $\sigma_S(S, V, t)$ . We assume that all diffusion parameters,  $\mu_S(S, V, t)$ ,  $\sigma_S(S, V, t)$ ,  $\mu_V(S, V, t)$ ,  $\sigma_V(S, V, t)$ , etc. are continuous in all their arguments. We assume that all variance-covariance matrices such as  $\sigma_S(S, V, t)\sigma_S(S, V, t)'$ ,  $\sigma_V(S, V, t)\sigma_V(S, V, t)'$  are non-singular for all  $S \in \mathcal{R}^{n_S}$ ,  $V \in \mathcal{R}^{n_V}$ ,  $t \in [0, T]$ .

where

$$\begin{aligned}\frac{d\Lambda^*}{\Lambda^*} &\equiv -r dt - \tilde{\mu}'_S \Sigma_S^{-1} \sigma_S dz; \\ \tilde{\mu}_S &\equiv \mu_S + \frac{D}{S} - r; \quad \Sigma_S = \sigma_S \sigma'_S.\end{aligned}$$

and  $v$  is an arbitrary<sup>2</sup>  $1 \times n_V$  matrix.  $d\Lambda^*/\Lambda^*$  is a discount factor, constructed from the interest rate and the shocks to the basis assets alone that prices the basis assets by construction. Then, the discount factor can load arbitrarily on the orthogonal shocks  $dw$  with no effect on its ability to price the basis assets. (The discount factor can load on other orthogonal shocks as well, but we will soon put such loading to zero in order to satisfy the volatility constraint.)

Using this fact, we can impose the form (7) and reduce the problem to finding an  $n_V$  dimensional vector  $v$  at each date; this imposes the pricing constraint and substantially simplifies the minimization problem. Substituting (7) into (4) the volatility constraint can now be expressed as a constraint on the sum of squared elements of  $v$ :

$$vv' \leq A^2 - \frac{1}{dt} E_t \frac{d\Lambda^{*2}}{\Lambda^{*2}} = A^2 - \tilde{\mu}'_S \Sigma_S^{-1} \tilde{\mu}_S. \quad (8)$$

Rather than carry around first and second moments of  $d\underline{C}/\underline{C}$ , it is convenient to also assume that the bound  $\underline{C}$  follows a diffusion process,

$$\frac{d\underline{C}}{\underline{C}} = \mu_{\underline{C}}(S, V, t) dt + \sigma_{\underline{C}z}(S, V, t) dz + \sigma_{\underline{C}w}(S, V, t) dw. \quad (9)$$

Now we can characterize the solution of the good-deal bound problem:

**Proposition 1** *The lower bound discount factor  $\underline{\Lambda}_t$  follows*

$$\frac{d\underline{\Lambda}}{\underline{\Lambda}} = \frac{d\Lambda^*}{\Lambda^*} - \underline{v} dw \quad (10)$$

and  $\mu_{\underline{C}}, \sigma_{\underline{C}z}$  and  $\sigma_{\underline{C}w}$  satisfy the restriction

$$\mu_{\underline{C}} + \frac{x^c}{\underline{C}} - r^f = -\frac{1}{dt} E_t \left( \frac{d\Lambda^*}{\Lambda^*} \sigma_{\underline{C}z} dz \right) + \sigma_{\underline{C}w} \underline{v}' \quad (11)$$

where

$$\underline{v} = \sqrt{A^2 - \frac{1}{dt} E_t \frac{d\Lambda^{*2}}{\Lambda^{*2}}} \frac{\sigma_{\underline{C}w}}{\sqrt{\sigma_{\underline{C}w} \sigma'_{\underline{C}w}}} \quad (12)$$

The upper bound process  $\overline{C}_t$  and discount factor  $\overline{\Lambda}_t$  have the same representation with  $\overline{v} = -\underline{v}$ .

---

<sup>2</sup>We require  $E \left[ \exp \left( \frac{1}{2} \int_0^T |\tilde{\mu}'_S \Sigma_S^{-1} \sigma_S|^2 dt \right) \right] < \infty$  and  $E \left[ \exp \left( \frac{1}{2} \int_0^T |v|^2 dt \right) \right] < \infty$ , to ensure that the stochastic integrals that describe the dynamics of  $\Lambda$  are well-defined.

Figure 1:

Substituting (7) into (2), and using the diffusion representation (9) for the differentials  $d\underline{C}$ , the objective can be written as

$$\mu_{\underline{C}} + \left( \frac{x_t^c}{\underline{C}} - r^f \right) = \tilde{\mu}'_S \Sigma_S^{-1} \sigma_S \sigma'_{\underline{C}z} + \max_{\{v\}} \sigma_{\underline{C}w} v'.$$

Thus, the we have a linear objective  $\sigma_{\underline{C}w} v'$  subject to a quadratic constraint  $vv' \leq A^2 - \tilde{\mu}'_S \Sigma_S^{-1} \tilde{\mu}_S$ . The situation is graphed in figure ?? for a two-dimensional  $V$ . As the figure shows, the minimization or maximization puts  $v$  parallel to  $\sigma_{\underline{C}w}$ , at the largest values allowed by the quadratic constraint. That is exactly what proposition 1 says in equations.

An equivalent interpretation is that the good-deal bound chooses the unobserved market prices of the non-traded risks  $dw$  at each moment so as to minimize the option value, subject to the constraint that the total (sum of squared) market price of risk is less than  $A^2$ . From 3, any security with diffusion  $dz$  should have an expected excess return equal to

$$-\frac{1}{dt} E_t \left( \frac{d\Lambda}{\Lambda} dz \right).$$

This predicted excess return is the “market price” of the risk  $dz$ . Applying this definition to (7),  $v$  is the vector of market prices of risks of the  $dw$  shocks:

$$-\frac{1}{dt}E\left(\frac{d\Lambda}{\Lambda}dw\right) = v.$$

The objective (2) told us to maximize the lower bound’s drift at each date; restriction (11) tells us what that maximized drift rate is.

We can plug in the definition of  $\Lambda^*$  to obtain explicit, if less intuitive, expressions for the optimal discount factor and the resulting lower bound,

$$\frac{d\Lambda}{\Lambda} = -rdt + \tilde{\mu}'_S \Sigma_S^{-1} \sigma_S dz - \sqrt{A^2 - \tilde{\mu}'_S \Sigma_S^{-1} \tilde{\mu}_S} \frac{\sigma_{\underline{C}w}}{\sqrt{\sigma_{\underline{C}w} \sigma'_{\underline{C}w}}} dw \quad (13)$$

$$\mu_{\underline{C}} + \frac{x^c}{\underline{C}} - r = \tilde{\mu}'_S \Sigma_S^{-1} \sigma_S \sigma_{\underline{C}z} + \sqrt{A^2 - \tilde{\mu}'_S \Sigma_S^{-1} \tilde{\mu}_S} \sqrt{\sigma_{\underline{C}w} \sigma'_{\underline{C}w}}. \quad (14)$$

If we knew that  $\sigma_{\underline{C}w}$  were constant over time – equivalently that the market prices of interest rate and volatility risk were constant over time – then we could find the option value by finding its discounted value using the discount factor (13). Unfortunately, in a stochastic volatility-stochastic interest rate context, the bound may be generated by prices of risk that vary over time. Hence, we have to solve a two-dimensional partial differential equation. To state that equation, we hypothesize a solution  $\underline{C}(S, V, t)$ ; using Ito’s lemma we express  $\mu_{\underline{C}}$  and  $\sigma_{\underline{C}z}, \sigma_{\underline{C}w}$  in terms of the partial derivatives of  $\underline{C}(S, V, t)$ , and then we rewrite restriction (14). The result is ugly, but straightforward to evaluate numerically, and analytically in special cases. It expresses the time derivative  $\partial \underline{C} / \partial t$  in terms of derivatives with respect to state variables, and thus can be used to work back from a terminal period.

**Proposition 2** *The price bound  $\underline{C}(S, V, t)$  is the solution to the partial differential equation*

$$\begin{aligned} & x^c - r\underline{C} + \frac{\partial \underline{C}}{\partial t} + \\ & + \frac{1}{2} \sum_{i,j} \frac{\partial^2 \underline{C}}{\partial S_i \partial S_j} S_i S_j \sigma_{S_i} \sigma'_{S_j} + \frac{1}{2} \sum_{i,j} \frac{\partial^2 \underline{C}}{\partial V_i \partial V_j} (\sigma_{Vz_i} \sigma'_{Vz_j} + \sigma_{Vw_j} \sigma'_{Vw_j}) + \sum_{i,j} \frac{\partial^2 \underline{C}}{\partial S_i \partial V_j} S_i \sigma_{S_i} \sigma'_{Vz_j} = \\ & = \left(\frac{D}{S} - r\right)' (S\underline{C}_S) + \left(\tilde{\mu}'_S \Sigma_S^{-1} \sigma_S \sigma'_{Vz} - \mu'_V\right) \underline{C}_V + \sqrt{A^2 - \tilde{\mu}'_S \Sigma_S^{-1} \tilde{\mu}_S} \sqrt{\underline{C}'_V \sigma_{Vw} \sigma'_{Vw} \underline{C}_V} \end{aligned}$$

*subject to the boundary conditions provided by the focus asset payoff  $x^c_T$ .  $\underline{C}_V$  denotes the vector with typical element  $\partial \underline{C} / \partial V_j$  and  $(S\underline{C}_S)$  denotes the vector with typical element  $S_i \partial \underline{C} / \partial S_i$ .*

Replacing  $+$  with  $-$  before the square root gives the partial differential equation satisfied by the upper bound.

## 2.1 Stochastic volatility and interest rates

Since volatility and interest rate are the only ingredients in the Black-Scholes formula, it is a natural generalization of the Black-Scholes approach to allow volatility and interest rates to vary over time. At a minimum, this approach allows one to recognize that option prices (implied volatilities) do change over time in an internally consistent way. In addition, stochastic volatility models can produce a rich variety of pricing effects compared with Black-Scholes model. For example, stochastic volatility of the underlying asset increases the kurtosis of underlying returns, which can generate a volatility smile. Some examples of this literature are Hull and White (1987), and Bates (1995).

However, pure arbitrage pricing is impossible in a stochastic volatility or stochastic interest rate model. Models in the literature follow two equivalent paths: they assume values for the market price of risk for the nontraded processes (often risk-neutrality), or they assume prices of unobserved securities driven by these shocks, and then use arbitrage pricing techniques.

We posit a standard stochastic volatility model, similar to those of Heston (1994) and Stein and Stein (1991). There is stochastic stock volatility, a stochastic interest rate and potentially correlated shocks:

$$\begin{aligned}\frac{dS}{S} &= \mu_S(S, V, r)dt + \sqrt{V}dz_S \\ dV &= \alpha_V(\bar{V} - V)dt + \beta_S\sqrt{V}dz_S + \beta_V\sqrt{V}dz_V \\ dr &= \alpha_r(\bar{r} - r)dt + \sigma_S\sqrt{r}dz_S + \sigma_V\sqrt{r}dz_V + \sigma_r\sqrt{r}dz_r\end{aligned}$$

where  $dz_S$ ,  $dz_V$  and  $dz_r$  are independent Brownian increments. By including varying interest rates, we show how the technique can handle multiple sources of nontradeable uncertainty.

We use the following parameter values :

$$\begin{aligned}\frac{dS}{S} &= (0.5\sqrt{V} + r) dt + \sqrt{V}dz_S \\ dV &= 1.4 \times (\ln(1.16)^2 - V)dt + 0.5\sqrt{V}dz_V \\ dr &= 1.214 \times (\ln(1.05) - r)dt + 0.18\sqrt{r}dz_r\end{aligned}$$



We do not prolong the paper with a formal estimation, but instead just try to pick reasonable values. Our values are roughly consistent with the stochastic volatility literature, such as Stein and Stein (1991), and Heston (1994). We present calculations for the case  $\beta_S = \sigma_V = \sigma_S = 0$ , or no correlation between the shocks.  $\beta_V$  then drives the volatility of volatility, we use Stein and Stein's (1991) estimate. We pick  $\bar{r}$  and  $\bar{V}$  as the same 5% interest rate and 16% stock volatility we used above. The  $\alpha$  describes the speed of mean-reversion in volatility and the short rate. Our values give a little less than a year half-life to volatility and the short rate.

The stock drift,  $\mu_S(\cdot) = 0.5\sqrt{\bar{V}} + r$ , is unusual, noteworthy, and results from an important lesson we learned in producing this and other applications. Standard statistical models for option pricing focus on estimating volatilities. Many such models do a bad job of estimating conditional means, and thus lead to wildly implausible estimates of conditional Sharpe ratios. Spuriously high conditional Sharpe ratio estimates are a particular danger in multiple time series models, as ex-post mean-variance frontiers are notorious for 20/20 hindsight. Now that means and Sharpe ratios matter, one must be careful to specify statistical models that produce reasonable Sharpe ratios.

If we use for example  $(\mu + r)dt$ , then the Sharpe ratio  $\mu/\sqrt{V}$  varies as volatility  $V$  varies. For low enough realizations of  $V$ , the conditional Sharpe ratio will be higher than any constant bound. To even compute bounds, one must either specify a bound higher than the maximum conditional Sharpe ratio over all states of nature – infinity in this case – or one must specify a Sharpe ratio bound that varies with the conditional Sharpe ratio of the basis assets, such as  $h_t =$  twice the time  $t$  market Sharpe ratio. But are such assumptions reasonable? Does one really believe that Sharpe ratios vary so strongly over time? In this context, we took the simplest approach. Our drift process gives a constant instantaneous Sharpe ratio for the basis assets of  $(\mu_S - r)/\sqrt{\bar{V}} = 0.5$ . Specifying reasonable and believable variation in conditional Sharpe ratios is an important extension.

To find price bounds, we solve the partial differential equation of proposition 2 numeri-

cally, using an explicit method<sup>3</sup>, subject to the usual boundary conditions,

$$C(S_T, V, r, T) = \max(S_T - K, 0); \quad C(0, V, r, t) = 0.$$

We set up a grid of values for  $S, V, r$ . Starting with values for  $\underline{C}(S, V, r, t)$  on the grid at time  $t$ , we take numerical partial derivatives with respect to  $S, V, r$ . The partial differential equation of proposition 2 can then be solved for  $\partial \underline{C} / \partial t$  at each value of the  $S, V, r$  grid. We work backward to obtain values one instant earlier,  $\underline{C}(S, V, r, t - \Delta t) = \underline{C}(S, V, r, t) - \partial \underline{C}(S, V, r, t) / \partial t \times \Delta t$ . We use a very fine time grid  $\Delta t = 0.00005$  year, a stock price grid from \$10 to \$500 spaced \$5, a volatility (standard deviation units) grid from 0.1 to 0.3 spaced 0.01, and an interest rate grid from 0.01 to 0.11 spaced 0.02. We smooth second derivatives by averaging the values for three adjacent grid points.

Figures 2 and 3 presents price bounds for call options with one year to expiration. We calculate price bounds with a target Sharpe ratio twice that of the market, or  $h = 1.0$ . Option prices depend on the two state variables  $V$  and  $r$  in addition to the stock price  $S$ . The figure presents option prices at the steady state values of the state variables,  $V = \bar{V}$ ,  $r = \bar{r}$ . For clarity, we plot the *difference* between the option price bound and the Black-Scholes price, using  $\bar{V}$  and  $\bar{r}$  in the Black-Scholes formula. The figures also includes the option prices (difference relative to Black-Scholes) calculated assuming zero market price of volatility and interest rate risk.

Figure 2 presents price bounds for one year call options with a constant interest rate equal to  $\bar{r}$ . If we assume a zero market price of volatility risk, we reproduce the standard volatility smile prediction in the near-money region. However, the option price bounds are fairly loose around the Black Scholes price, especially for near-money options. The volatility smile all lies within our bounds! This finding suggests that the volatility smile is not a robust feature of the stochastic volatility model, but depends crucially on the assumed value of the market price of volatility risk.

Figure 3 adds a stochastic interest rate as well as stochastic volatility. Option price

---

<sup>3</sup>The more common and elegant *implicit* method for solving a partial differential equation simultaneously searches for the new value  $C(S, V, r, t - \Delta t)$  and its partial derivatives with respect to  $S, V, r$  that generate the value of  $C(S, V, r, t)$ . See Brennan and Schwartz (1978). The non-linear nature of our partial differential equation makes it difficult to apply a fully implicit method. The nonlinearity means that the simultaneous solution of the value of  $C(S, V, r, t - \Delta t)$  and its partial derivatives that generate  $C(S, V, r, t)$  is a set of nonlinear equations, where usually this step just requires a matrix inversion.

### Price bounds with stochastic V

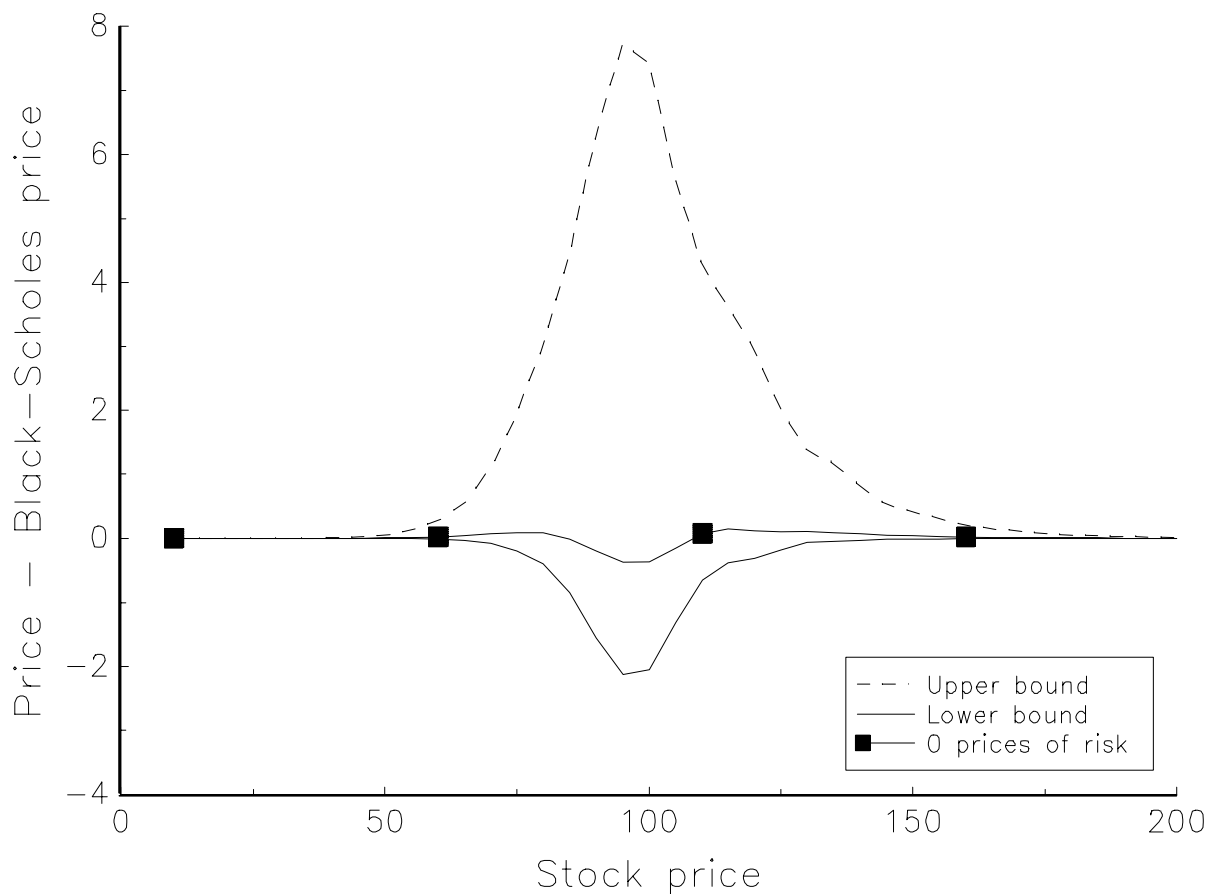


Figure 2: Difference between option price bounds and Black-Scholes price, using stochastic volatility model with constant interest rate. The line in the middle gives the difference between the option price calculated with a zero price of volatility risk and the Black-Scholes price.

bounds are again loose around the Black Scholes price, and the volatility smile is not “significant.” However, now the bounds are large for far-in-the money options at the right end of the graph as well as for nearly at the money options as in Figure 2. At-the money options

Price bounds with stochastic  $V$  and  $r$

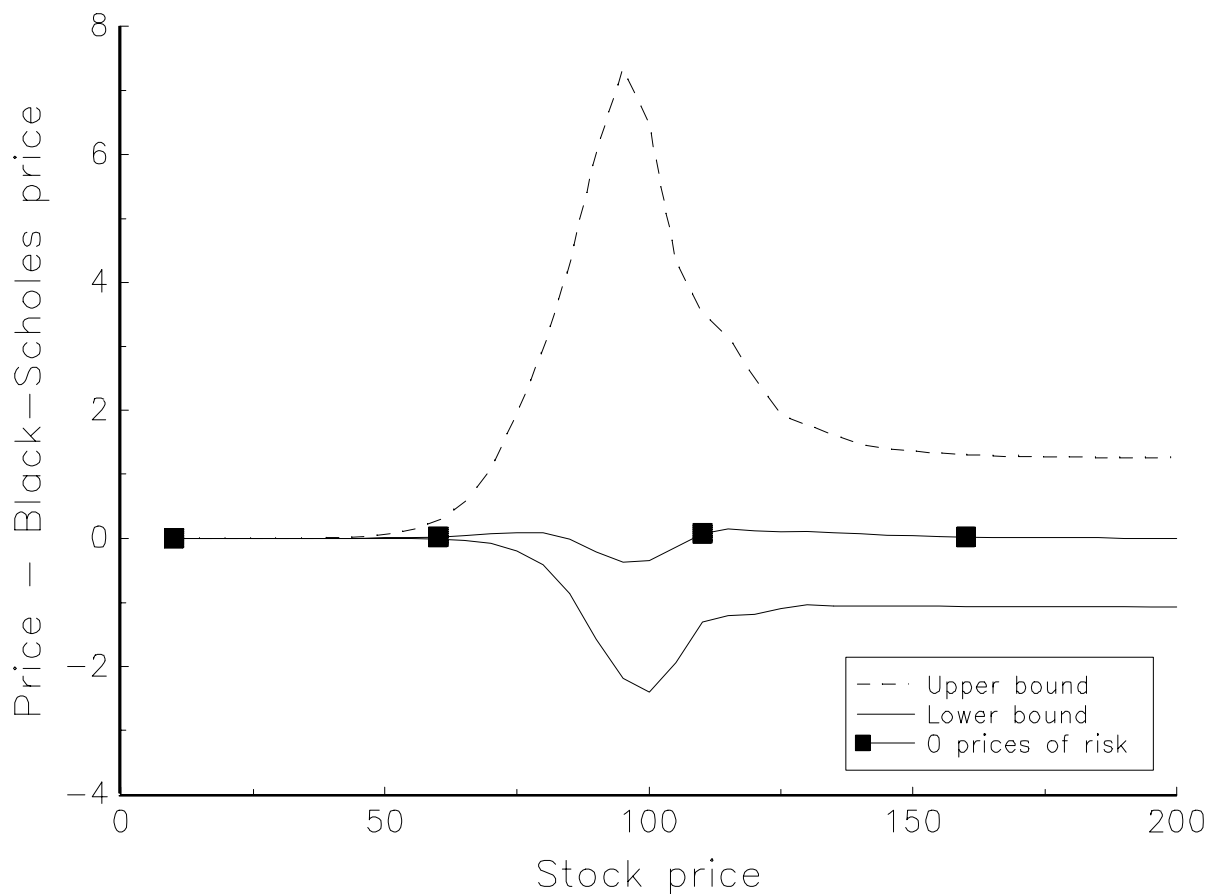


Figure 3: Option price bounds with stochastic volatility and stochastic interest rates. The middle line assumes zero price of volatility and interest rate risk. All lines graph the difference between the price and the Black-Scholes price.

are very sensitive to changes in volatility, because in this area the curvature of the payoff as a function of underlying asset price is highest. Far in the money options are much more sensitive to interest rates, since their payoff is essentially replicated by holding the underlying asset. Therefore, our bounds assign most of the total price of risk to interest rate risk if we

are analyzing far-in-the money options and to volatility risk if we are analyzing around-the money options.

When we assume a zero market price of volatility and interest rate risk, we reproduce the standard volatility smile prediction in the near-money region: at the money prices are lower than Black-Scholes and somewhat out of the money prices are higher. However, the option price bounds are loose around the Black Scholes price, especially for near-money options. The predicted volatility smile all lies within the bounds. This finding suggests that the volatility smile prediction of a stochastic volatility model depends crucially on the assumed value of the market price of volatility and to a lesser extent interest rate risk.

### 3 References

- Black, Fischer and Myron Scholes, 1973, "The Valuation of Options and Corporate Liabilities," *Journal of Political Economy*, 81, 637-654.
- Brennan, Michael and Eduardo Schwartz, 1978, "Finite Difference Methods and Jump Processes Arising in the Pricing of Contingent Claims: A Synthesis," *Journal of Financial and Quantitative Analysis*, 13, 462-474.
- Cochrane, John H. and Jesús Saá-Requejo, 1998, "Beyond Arbitrage: Good Deal Asset Price Bounds in Incomplete Markets," forthcoming *Journal of Political Economy* available at <http://www-gsb.uchicago.edu/fac/john.cochrane/>
- He, Hua and Neil Pearson, 1992, "Consumption and Portfolio Policies with Incomplete Markets: The Infinite Dimensional Case," *Journal of Economic Theory*, 54, 259-305.
- Heston, Steven ,1994, "A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options," *The Review of Financial Studies*, 6, 327-343.
- Hull, John C. and Allan White, 1987, "The Pricing of Options on Assets with Stochastic Volatilities," *Journal of Finance*, 42, 281-300.
- Merton, Robert C., 1973b, "Theory of Rational Option Pricing," *Bell Journal of Economics and Management Science*," 4, 141-83.

Stein, Elias M. and Jeremy C. Stein, 1991, "Stock Price Distributions with Stochastic Volatility: An Analytic Approach," *The Review of Financial Studies*, 4, 727-752.

## 4 Appendix

### 4.1 Partial differential equation

We guess that the bound is a twice-differentiable function of the state variables,

$$\underline{C} = \underline{C}(S, V, t).$$

Then, we can use Ito's lemma to relate the terms  $\underline{\mu}_C, \underline{\sigma}_{Cz}, \underline{\sigma}_{Cw}$  in the law of motion for  $\underline{C}$ ,

$$\frac{d\underline{C}}{\underline{C}} = \underline{\mu}_C dt + \underline{\sigma}_{Cz} dz + \underline{\sigma}_{Cw} dw$$

to partial derivatives of the function  $\underline{C}(S, V, t)$ :

$$\begin{aligned} d\underline{C} &= \underline{C}_t dt + \underline{C}'_S dS + \underline{C}'_V dV \\ &+ \frac{1}{2} \sum_{i,j} \underline{C}_{S_i S_j} dS_i dS_j + \frac{1}{2} \sum_{i,j} \underline{C}_{V_i V_j} dV_i dV_j + \sum_{i,j} \underline{C}_{S_i V_j} dS_i dV_j \end{aligned}$$

where  $C_x$  denotes  $\partial \underline{C}(S, V, t) / \partial x$ . Substituting for the  $S$  and  $V$  processes from equations (5) and (6),

$$\begin{aligned} d\underline{C} &= \underline{C}_t dt + (S\underline{C}_S)'(\mu_S dt + \sigma_S dz) + \underline{C}'_V(\mu_V dt + \sigma_{Vz} dz + \sigma_{Vw} dw) + \\ &+ \left[ \frac{1}{2} \sum_{i,j=1}^{n_S} \underline{C}_{S_i S_j} S_i S_j \sigma_{S_i} \sigma'_{S_j} + \frac{1}{2} \sum_{i,j=1}^{n_V} \underline{C}_{V_i V_j} (\sigma_{Vz_i} \sigma'_{Vz_j} + \sigma_{Vw_j} \sigma'_{Vw_j}) \right. \\ &\left. + \sum_{i=1}^{n_S} \sum_{j=1}^{n_V} \underline{C}_{S_i V_j} S_i \sigma_{S_i} \sigma'_{Vz_j} \right] dt, \end{aligned}$$

where  $(S\underline{C}_S)$  indicates element by element multiplication, and  $\sigma_{Vz_j}$  denotes the  $j$ th row of  $\sigma_{Vz}$ , etc. Therefore, we can write  $\underline{\mu}_C$  and  $\underline{\sigma}_{Cz}, \underline{\sigma}_{Cw}$  in terms of derivatives of the  $\underline{C}(\cdot)$  function,

$$\begin{aligned} \underline{C}\underline{\mu}_C &= \underline{C}_t + (S\underline{C}_S)' \mu_S + \underline{C}'_V \mu_V + \frac{1}{2} \sum \underline{C}_{S_i S_j} S_i S_j \sigma_{S_i} \sigma'_{S_j} \\ &+ \frac{1}{2} \sum \underline{C}_{V_i V_j} (\sigma_{Vz_i} \sigma'_{Vz_j} + \sigma_{Vw_i} \sigma'_{Vw_j}) + \sum \underline{C}_{S_i V_j} S_i \sigma_{S_i} \sigma'_{Vz_j} \end{aligned}$$

and

$$\begin{aligned} \underline{C}\underline{\sigma}_{Cz} &= (S\underline{C}_S)' \sigma_S + \underline{C}'_V \sigma_{Vz} \\ \underline{C}\underline{\sigma}_{Cw} &= \underline{C}'_V \sigma_{Vw} \end{aligned}$$

We recover the terms needed in equation (14)

$$\underline{C}^2 \sigma_{\underline{C}w} \sigma'_{\underline{C}w} = \underline{C}'_V \sigma_{Vw} \sigma'_{Vw} \underline{C}_V$$

and

$$\begin{aligned} \underline{C} \sigma_{\underline{C}z} \sigma'_S \Sigma_S^{-1} \tilde{\mu}_S &= [(S\underline{C}_S)' \sigma_S + \underline{C}'_V \sigma_{Vz}] \sigma'_S \Sigma_S^{-1} \tilde{\mu}_S \\ &= (S\underline{C}_S)' \tilde{\mu}_S + \underline{C}'_V \sigma_{Vz} \sigma'_S \Sigma_S^{-1} \tilde{\mu}_S. \end{aligned}$$

Plugging these expressions in restriction (14) and simplifying, we obtain the partial differential equation given in proposition (2).

## 4.2 Stochastic volatility

We assume for simplicity that the stock pays no dividends. Specializing proposition 2 to this stochastic volatility model and simplifying slightly, the partial differential equation is

$$\begin{aligned} &C_t + C_S S r + C_V [\alpha_V (\bar{V} - V) - \beta_S (\mu_S - r)] + C_r [\alpha_r (\bar{r} - r) - (\mu_S - r) \sigma_S \sqrt{\frac{r}{V}}] + \\ &+ \frac{1}{2} [C_{SS} S^2 V + C_{VV} (\beta_S^2 + \beta_V^2) V + C_{rr} (\sigma_S^2 + \sigma_V^2 + \sigma_r^2) r] \\ &+ C_{SV} \beta_S S V + C_{Sr} \sigma_S S \sqrt{rV} + C_{Vr} (\beta_S \sigma_S + \beta_V \sigma_V) \sqrt{rV} - rC \\ = &\pm \sqrt{A^2 - \frac{(\mu_S - r)^2}{V}} \sqrt{(C_V \beta_V \sqrt{V} + C_r \sigma_V \sqrt{r})^2 + C_r^2 \sigma_r^2 r}. \end{aligned}$$

The plus sign in front of the square root generates the lower bound and the minus sign generates the upper bound.