Efficient Monte Carlo Pricing of Basket Options

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Abstract

Montecarlo methods can be used to price derivatives for which closed evaluation formulas are not available or difficult to derive. A drawback of the method can be its high computational cost, especially if applied to basket options, whose payoffs depend on more than one asset.

This article presents two kinds of control variates to reduce variance of estimates, based on unconditional and conditional expectations of assets respectively. We apply the previous variance reduction methods to some basket options (Spread, Dual and Portfolio options), achieving in some case remarkable speed and accuracy in price estimation.

KEYWORDS:Option pricing, MonteCarlo methods, control variates variance reduction, basket options

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1 Introduction

The valuation of complex options produced a vaste literature in last years. Among the methods used when close evaluation formulas are missing or difficult to derive there are tree and partial differential equation (PDE) methods. The former approximate the unknown distribution of payoffs discretizing the jumps in the value of the underlying asset, similarly to the binomial model, [Cox et al., 1979]. The latter solve the numerical partial differential equation satisfied by the price of the option. In many important cases, there is not a close solution (like in the Black–Scholes (BS) paper [Black and Scholes, 1973]) and numerical techniques are employed (see [Wilmott et al., 1995] for a PDE introduction to option pricing).

This paper deals with Monte Carlo pricing methods. [Cox and Ross, 1976] noted that if a riskless hedge can be formed, the option value is the risk-neutral and discounted expectation of its payoff. Hence the price can be estimated by Monte Carlo methods, simulating many independent paths of the underlying assets and taking the discounted mean of the generated payoffs. In principle, this can be done even if complex distributions or payoffs are involved, provided that we know the path generating process of the assets (that are commonly thought to be the realization of a lognormal random walk).

Since the seminalt paper by [Boyle, 1977] on the application of Monte Carlo methods to option pricing, it was realized that refinement of the methods was desiderable, being the accuracy (i.e. standard deviation) of the estimates of the order of $1/\sqrt{N}$. This unfortunately means that to double the precision we have to multiply by four the number of simulations N and the time needed for computation. Hence some variance reduction techniques are introduced. Among the papers dealing with variance reduction of the estimates of options prices there are [Boyle, 1977], [Kemna and Vorst, 1990], [Clewlow and Carverhill, 1993].

This paper aims to propose some variance reduction techniques for Monte Carlo pricing of basket options, whose payoff is a function of more than one asset. In particular, we will define some control variate that can hopefully reduce the number of simulations needed to achieve satisfactory precision. To our knowledge, little work has been done on this subject altough variance reduction methods appears particularly useful when multivariate random variables are generated in simulation, with increased computional cost.

It is implied in some papers that propose tree-based methods that Monte Carlo methods are not suitable for multivariate option pricing. We feel that this is somewhat misleading and that the use of proper variance reduction schemes can greatly enhance the performance of Monte Carlo methods, producing in some cases great speed and accuracy.

In section 2 we present the basic idea of control variates to reduce variance of estimates. Section 3 presents the basic control variates, that are essentially obtained transforming the payoffs to a function of one single asset. This usually allows to evaluate the mean of the control variate with a modification of BS formula. In particular we replace some asset with their unconditional mean in the payoff function. In section 4 we explore the use of conditional mean of one asset, given the values of other assets, to define a different control variates that appear to be more effective especially when the basket option is written on correlated assets. We show the effects in variance reduction of the previously defined control variates in section 5, where prices of spread, dual and portfolio options are calculated. Finally, some concluding remarks are given.

2 The variance reduction idea

We briefly describe in this section the use of control variates to reduce variance of Monte Carlo estimates. A detailed treatment of the subject is in [Ripley, 1987] and [Hammersley and Handscomb, 1967]. We do not discuss the antithetic variates, that could be implemented with little effort.

Suppose we are interested in estimating the expectation of the random variable S, and we are given the independent sample $\{s_1, \ldots, s_N\}$ extracted form the distribution of S. The natural unbiased estimator is the sample mean

$$\hat{M} = \frac{1}{N}\sum_{i=1}^{N} s_i. \tag{2.1}$$

Suppose, moreover, that we can generate from the distribution of Y the independent control variates $\{y_1, \ldots, y_N\}$ simultaneously with the s_i's. Then the estimate

$$\hat{M}_Y = \frac{1}{N}\sum_{i=1}^{N}(s_i - y_i + EY) \tag{2.2}$$

is still unbiased and correct. If we compare the variance of $\hat{M}$ and $\hat{M}_Y$ we get

$$Var(\hat{M}_Y) = Var(\hat{M}) + \frac{1}{N}\left(Var(Y) - 2Covar(S,Y)\right). \tag{2.3}$$

We have that $Var(\hat{M}_Y) \leq Var(\hat{M})$ provided that

$$Covar(S,Y) \geq \frac{Var(Y)}{2}. \tag{2.4}$$

Hence, if the correlation of S and Y is large, the estimator $\hat{S}_Y$ has smaller variance than $\hat{S}$ and is preferable in Monte Carlo simulation.

From a practical point of view the definition of suitable control variate Y need some care to give strong positive correlation with S and ease in the evaluation of the mean EY that appears in (2.2). These are often contrasting targets, and it might be difficult to find simultaneously strong correlation with S and close analytical formula for EY.

Example 1. Consider a spread option that pays at time T the (random) sum

$$S = f(S_{1T}, S_{2T}) = \max\{0, S_{2T} - S_{1T} - k\},$$

where S_{1T}, S_{2T} are the values of assets S_1, S_2 at time T and k is the strike price. Intuitively a control variate that can be considered is

$$Y = f(E[S_{1T}], S_{2T}) = \max\{0, S_{2T} - E[S_{1T}] - k\}.$$

Y is obviously correlated with S and the expectation EY can be evaluated in close form, being Y the payoff of a call option on the asset S_2.

In the next sections we define a set of control variates, that appears to be widely applicable to basket options pricing.

3 Unconditional mean variance reduction

Let $S_1, \ldots, S_n$ be assets available on the market, with normally distributed logarithmic returns with means $\tilde{\mu}_1, \ldots, \tilde{\mu}_n$, standard deviations $\sigma_1, \ldots, \sigma_n$ and that pay dividends continously at rate $d_1, \ldots, d_n$ respectively. Let $\mu_i = \tilde{\mu}_i - d_i, i = 1, \ldots, n$ and assume we want to price at time $t = 0$ an european-like asset that pays the sum

$$C_T = f(S_{1T}, \ldots, S_{nT}), \tag{3.5}$$

at time T, where S_{it} denotes the value of i-th asset at time t.

The previous assumptions imply that the prices at time t are lognormally distributed, i.e.

$$S_{it} \sim LN(\mu_i t, \sigma_i^2 t).$$

Under fairly standard assumptions on the the risk neutrality of agents, the price of (3.5) can be estimated by generating many realizations of $\{S_{1T}^{(j)}, \ldots, S_{nT}^{(j)}\}, j = 1, \ldots, N$ and discounting the sample mean of the resulting $\{C_T^{(j)} = f(S_{1T}^{(j)}, \ldots, S_{nT}^{(j)}), j = 1, \ldots, N\}$, providing

$$\hat{C}_T = e^{-rT} \frac{1}{N} \sum_j C_T^{(j)}, \tag{3.6}$$

where r is the risk-free rate of the market.

This might well be a hard computational task, as many vector random variables are to be extracted from a multivariate distribution. Even more importantly, the standard deviation of the estimated price is $O(1/\sqrt{N})$ and hence a huge N might be required to achieve satisfactory precision.

Let us describe a simple implementation of variance reduction scheme based on control variates. Recall from section 2 that a candidate control variate is a random variable possibly correlated with $C_T^{(j)}$ and such that its mean value is available. Consider the following *unconditional mean* control variates $UM_T(i), i = 1, \ldots, n$:

$$UM_T(i) = f(ES_{1T}, \ldots, ES_{i-1,T}, S_{iT}, ES_{i+1,T}, \ldots, ES_{nT}). \tag{3.7}$$

The variate $UM_T(i)$ is obtained from (3.5) replacing S_{jT} with its unconditional mean ES_{jT} if $i \neq j$. It is obvious that $UM_T(i)$ is generally correlated with C_T and $E[UM_T(i)]$ can be easily evaluated in many important cases (i.e. if f assumes a specific functional forms).

Example 2. Assume we have to price a spread options, with payoff

$$f(S_{1T}, S_{2T}) = \max\{0, S_{2T} - S_{1T} - k\}.$$

Then the two control variates $UM_T(i), i = 1, 2$ are given by

$$UM_T(1) = \max\{0, ES_{2T} - S_{1T} - k\}, \tag{3.8}$$

and

$$UM_T(2) = \max\{0, S_{2T} - ES_{1T} - k\}. \tag{3.9}$$

The previous assumptions on the distribution of logarithmic returns of assets S_{1T}, S_{2T} yields

$$ES_{1T} = S_{10} \exp\left(\mu_1 T + \frac{\sigma_1^2 T}{2}\right), \quad ES_{2T} = S_{20} \exp\left(\mu_2 T + \frac{\sigma_2^2 T}{2}\right). \tag{3.10}$$

The means of the control variates $UM_T(i), i = 1, 2$ are readily evaluated. Note, for example, that (3.9) is the payoff of an european call option on the asset S_2 and strike price $K = ES_{1T} - k$, hence the expected value is given by the Black–Scholes formula

$$E\left[UM_T(1)\right] = e^{rT}\left[S_{20}\exp(-d_2 T)N(p_1) - (ES_{1T} + k)e^{-rT}N(p_2)\right], \tag{3.11}$$

where

$$p_1 = \frac{\log S_{20}/(ES_{1T} + k) + (r - d_2 + \sigma_2^2/2)T}{\sigma_2\sqrt{T}}, \quad p_2 = p_1 - \sigma_2\sqrt{t}.$$

and $N(\cdot)$ is the cumulative normal distribution.

In the same way, observing that (3.8) is the payoff of a put option on asset S_1 with strike price $ES_{2T} - K$, we can easily evaluate $E\left[UM_T(2)\right]$ by the put BS pricing formula.

The control variates (3.7) allow us to obtain a set of Monte Carlo estimates $\hat{C}_T^i$, $i = 1, \ldots, n$ of the unknown price $\hat{C}_T$:

$$\hat{C}_T^i = e^{-rT} \frac{1}{N} \sum_{j=1}^{N} C_T^{(j)} - UM_T^{(j)}(i) + E\left[UM_T(i)\right], \tag{3.12}$$

where $UM_t^{(j)}(i)$ denotes the $j-$th realization of the control variate $UM_t(i)$.

Obviously, in order to use any control variate, it is necessary to evaluate its mean. Often this is done by an analytic formula, exactly as in the previous example. In section 5, it is shown how other basket options give rise to control variates (3.7) whose means are evaluated by BS formula.

4 Conditional mean variance reduction

The key observation in section 3 was that the replacement of S_{iT} with its unconditional mean produces the payoff function of a standard european option which is easily priced. It is interesting to speculate on the use of *conditional* means to define others control variates.

To avoid complicate notations we restrict our analysis to options on two assets S_1 and S_2. Let x_i be the (random) return of the i-th asset from time 0 to T:

$$x_i = \log S_{iT} - \log S_{i0} \sim N(\mu_i T; \sigma_i^2 T), \quad i = 1, 2. \tag{4.13}$$

As x_1, x_2 are jointly normal with correlation ρ, the random variable $x_i | x_j, i \neq j$ is still normally distributed and standard normal theory is applicable[1].

Quite naturally, we can try to exploit control variates of the form $f(S_{1T}, E\left[S_{2T}|S_{1T}\right])$. Hovewer, some reflection makes clear that, being $E\left[S_{2T}|S_{1T}\right]$ a function of S_{1T}, we could find difficulties in evaluating analitically the control variate under the usual assumptions of lognormality of assets S_1 and S_2. Infact, the mean of a transformation of lognormal variates might not be available in close form and indeed this is essentially the reason for the lack of close pricing formulas for some basket options, like the ones presented in section 5.

In order to avoid such problems, we use the following approximation

$$\begin{aligned} f(S_{1T}, E\left[S_{2T}|S_{1T}\right]) &\sim f(S_{10} \exp x_1, S_{20} \exp\left(E\left[x_2|x_1\right]\right)) \sim & (4.15)\\ &\sim f(S_{10}(1 + x_1), S_{20}(1 + E\left[x_2|x_1\right])), & (4.16) \end{aligned}$$

where a Taylor approximation around 0 has been used from (4.15) to (4.16). The previous expression has the advantage to be a function of the *normal* variate x_1 only.

We are ready to define the *conditional mean* control variates as

$$\begin{aligned} CM_T(1) &= f(S_{10}(1 + x_1), S_{20}(1 + E\left[x_2|x_1\right])) & (4.17)\\ CM_T(2) &= f(S_{10}(1 + E\left[x_1|x_2\right]), S_{20}(1 + x_2)) & (4.18) \end{aligned}$$

[1] If $X \sim N(\mu_X; \sigma_X^2), Y \sim N(\mu_Y; \sigma_Y^2)$ are jointly normal with correlation ρ, then [Casella and Berger, 1990]

$$X|Y \sim N\left(\mu_X + \rho \frac{\sigma_X}{\sigma_Y}(Y - \mu_Y); \sigma_X^2(1 - \rho^2)\right). \tag{4.14}$$

The mean of $CM_T(i)$ can be evaluated in some interesting cases, using a "normal" version of BS formula, as can be seen in the following example.
Example 3. Let us consider again the case of a spread option on two assets, whose returns have correlation ρ. The control variate $CM_T(2)$ is

$$\begin{aligned} CM_T(2) &= \max\{0, S_{20}(1+x_2) - S_{10}(1+E[x_1|x_2]) - k\} = \\ &= \max\{0, S_{20}(1+x_2) - S_{10}(1+\mu_1 T + \rho\frac{\sigma_1}{\sigma_2}(x_2-\mu_2 T) + \frac{1}{2}\sigma_1^2 T(1-\rho^2)) - k)\} \\ &= \max\{0, (S_{20} - S_{10}\rho\frac{\sigma_1}{\sigma_2})x_2 + S_{20} - S_{10}(1+\mu_1 T - \rho\frac{\sigma_1}{\sigma_2}\mu_2 T + \frac{1}{2}\sigma_1^2 T(1-\rho^2)) - k)\}. \end{aligned}$$

The mean of the previous expression can be evaluated noting that it is of the form

$$E[\max\{0, X-K\}] \tag{4.19}$$

where X is normally distributed[2]. Note that we are approximating a lognormal stock price with a normal random variable and this might, at first glance, appear an economical nonsense due to possible negative values. Hovewer, control variates are simple technical devices used to reduce the variance of estimated price and there is no strict need of positivity. Moreover, in practical applications, the means of the density of stock prices are many standard deviations away from zero and the probability of negative values is ususally absolutely negligible. In any case, we never found such an istance in the many simulations we have performed with conditional mean control variates.

Reduced variance estimates of the price of the options are given by

$$\hat{C}_T^{(1)} = e^{-rT}\frac{1}{N}\sum_{i=1}^{N} S_{2T} - S_{1T} - k - CM_T(1) + E[CM_T(1)] \tag{4.22}$$

and

$$\hat{C}_T^{(2)} = e^{-rT}\frac{1}{N}\sum_{i=1}^{N} S_{2T} - S_{1T} - k - CM_T(2) + E[CM_T(2)]\,. \tag{4.23}$$

5 Applications

In this section we apply the variance reduction methods described in the previous sections to some basket options, for which there is no close pricing formula. In these case the use of a Monte Carlo method provides an estimate of the value of the option, together with the sample standard deviation to assess the precision of the result.

As a benchmark, we first consider an exchange option [Margrabe, 1978] whose payoff is

$$f(S_{1T}, S_{2T}) = \max\{0, S_{2T} - S_{1T}\}, \tag{5.24}$$

[2] Some calculations show that, if $X \sim N(\mu, \sigma^2)$ then

$$E[\max\{0, X-K\}] = \sigma n\left(\frac{\mu-K}{\sigma}\right) + \frac{1}{2}(\mu-K)\left[1 + erf\left(\frac{\mu-K}{\sqrt{2}\sigma}\right)\right], \tag{4.20}$$

where $n(x)$ is the pdf of a standard normal and

$$erf(x) = \int_0^x \exp(-t^2/2)dt. \tag{4.21}$$

and for which the following analyitc pricing formula is available

$$C_T = S_{20} e^{-d_2 T} N(p) - S_{10} e^{-d_1 T} N(p - \Sigma\sqrt{T}), \tag{5.25}$$

where $N(\cdot)$ is the cumulative normal distribution and

$$p = \frac{\log(\frac{S_{20} d_2^{-T}}{S_{10} d_1^{-T}})}{\Sigma\sqrt{T}} + \frac{1}{2}\Sigma\sqrt{T}, \quad \Sigma^2 = \sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2. \tag{5.26}$$

Setting $S_{10} = S_{20} = 100, r = \log(1.1), d_1 = d_2 = \log(1.05), \sigma_1 = 0.3, \sigma_2 = 0.2, \rho = -0.5$ and $T = 0.95$ we get the price 16.0606. Table 1 shows some estimates obtained by plain Monte Carlo and unconditional mean variance reduction Monte Carlo with relative standard deviation for different sample sizes N. It is apparent that variance reduction techniques produce an error 4 to 5 times smaller than plain Monte Carlo methods. This means that, given a predetermined precision, the evaluation of the price can be obtained 16 to 25 times faster. Note also that the variance reduced estimate with $N = 1000$ is preferable to the result with $N = 10000$ naive simulations. Figure 1 depicts the estimated prices against N and the true price. A look at the plot shows that the reduced variance estimates are smoothly converging to the true price, while the plain Monte Carlo fluctuate widely around the proper price.

Other pricing experiments on exchange options with different parameters show the same qualitative behaviour and are not reported.

Table 1: Estimated prices and relative standard deviations for plain and reduced variance Monte Carlo for an exchange option.

	Plain MC		Red. MC	
N	$\hat{C}$	std	$\hat{C}$	std
1000	15.39	0.73	16.11	0.15
2000	15.28	0.47	16.04	0.11
3000	16.13	0.42	16.14	0.08
4000	15.55	0.34	15.99	0.08
5000	16.49	0.32	16.10	0.07
6000	15.74	0.29	15.99	0.06
7000	15.74	0.26	16.03	0.06
8000	15.95	0.25	16.11	0.05
9000	16.06	0.24	16.09	0.05
10000	15.80	0.22	16.09	0.05
true	16.0606			

Next, we examine spread, dual and portfolio call options on two assets. There is no known close formula to evaluate such assets. The payoffs are as follows.

- *Spread option.* The payoff at time T is given by

$$f(S_{1T}, S_{2T}) = \max\{0, S_{2T} - S_{1T} - k\}. \tag{5.27}$$

- *Dual option.* Given two strike prices k_1, k_2 for asset S_1 and S_2 respectively, the final payoff is

$$f(S_{1T}, S_{2T}) = \max\{0, S_{1T} - k_1, S_{2T} - k_2\}. \tag{5.28}$$

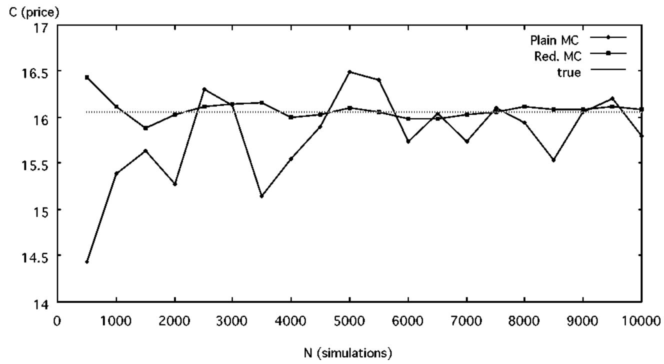

Figure 1: Estimates of exchange option price with plain and variance reduction Monte Carlo.

- *Portfolio option.* It is an european option on a portfolio made of n_1 units of asset S_1 and n_2 units of S_2. The payoff at time T is

$$f(S_{1T}, S_{2T}) = \max\{0, n_1 S_{1T} + n_2 S_{2T} - k\}. \tag{5.29}$$

An inspection of (5.27), (5.28), (5.29) just given show that the expectations of UM control variates can be evaluated, being the resulting payoff equal to that of an ordinary european option on one asset. Hence the use of BS formula enables easy evaluation of resulting put or call options, exactly as shown in example 2.

Dual Options			
$\sigma_1 \backslash \rho$	0.5	0.0	-0.5
0.1	0.030	0.030	0.029
	0.002	0.005	0.006
	0.029	0.029	0.030
	0.004	0.003	0.002
0.2	0.031	0.032	0.031
	0.011	0.017	0.019
	0.025	0.028	0.029
	0.013	0.008	0.005
0.3	0.040	0.040	0.039
	0.024	0.032	0.035
	0.023	0.027	0.029
	0.018	0.012	0.006

Portfolio Options			
$\sigma_1 \backslash \rho$	0.5	0.0	-0.5
0.1	0.044	0.038	0.031
	0.033	0.035	0.037
	0.016	0.017	0.017
	0.009	0.010	0.011
0.2	0.055	0.046	0.034
	0.032	0.034	0.035
	0.032	0.034	0.034
	0.012	0.015	0.018
0.3	0.069	0.058	0.044
	0.032	0.033	0.033
	0.048	0.050	0.051
	0.013	0.018	0.023

Table 2: Standard deviation of estimated prices with unconditional variance reduction. For each value of σ_1 and ρ the four figures are the standard deviations of (5.30) to (5.33) respectively. For SPREAD options we set $S_{01} = S_{02} = 100, k = 6, T = 0.95, r = \log(1.1), d_1 = d_2 = \log(1.05), \sigma_2 = 0.2, N = 100000$; DUAL: $k_1 = 110, k_2 = 100, T = 0.5$ and other parameters as before; PORTFOLIO: $S_{01} = S_{02} = 100, k = 200, n_1 = n_2 = 1, T = 0.5, d_2 = 0$ and other parameters as before.

Table 2 shows the results of application of the Monte Carlo pricing techniques when UM control variates are used. For each value of parameters (notably σ_1 and ρ) we provide 4 figures, namely the standard deviation of the price obtained with no variance reduction, with control variates $UM_T(1)$ and $UM_T(2)$, and with both $UM_T(1), UM_T(2)$ at the same time, i.e. the standard deviations of

$$\hat{C}_T = e^{-rT}\frac{1}{N}\sum_{i=1}^{N} f(S_{1T}, S_{2T}), \tag{5.30}$$

$$\hat{C}_T^{(1)} = e^{-rT}\frac{1}{N}\sum_{i=1}^{N} f(S_{1T}, S_{2T}) - UM_T(1) + E\left[UM_T(1)\right], \tag{5.31}$$

$$\hat{C}_T^{(2)} = e^{-rT}\frac{1}{N}\sum_{i=1}^{N} f(S_{1T}, S_{2T}) - UM_T(2) + E\left[UM_T(2)\right], \tag{5.32}$$

$$\hat{C}_T^{(1,2)} = e^{-rT}\frac{1}{N}\sum_{i=1}^{N} f(S_{1T}, S_{2T}) - UM_T(1) - UM_T(2) + E\left[UM_T(1)\right] + E\left[UM_T(2)\right]. \tag{5.33}$$

The results obtained in valuation of spread options with unconditional and conditional mean control variates are shown in Table 3. We list the standard deviations of the estimates (5.30) to (5.33) where UM deviates are replaced by the corresponding CM's. For example, the price estimate using $CM_T(2)$ is

$$\hat{C}_T^{(2)} = e^{-rT}\frac{1}{N}\sum_{i=1}^{N} f(S_{1T}, S_{2T}) - CM_T(2) + E\left[CM_T(2)\right].$$

Spread Options			
$\sigma_1 \backslash \rho$	0.5	0.0	-0.5
0.1	0.027	0.036	0.043
	0.017	0.016	0.016
	0.030	0.035	0.037
	0.017	0.013	0.010
0.2	0.029	0.043	0.054
	0.030	0.030	0.031
	0.033	0.038	0.039
	0.027	0.021	0.014
0.3	0.036	0.053	0.065
	0.041	0.042	0.043
	0.036	0.040	0.041
	0.033	0.025	0.016

Spread Options (cond.)			
$\sigma_1 \backslash \rho$	0.5	0.0	-0.5
0.1	0.027	0.036	0.043
	0.015	0.017	0.016
	0.027	0.035	0.034
	0.015	0.015	0.018
0.2	0.029	0.043	0.054
	0.026	0.031	0.027
	0.029	0.038	0.036
	0.025	0.023	0.027
0.3	0.036	0.053	0.065
	0.036	0.043	0.037
	0.033	0.042	0.039
	0.033	0.029	0.039

Table 3: Standard deviation of estimated prices for spread options. Same meaning and parameters as in Table 2.

Some conclusive remarks are the following. In general, the use of control variates appears to reduce considerably the standard deviation of estimated prices. Given a fixed accuracy, computations run on average from 10 to 50 times faster than plain Monte Carlo methods, with exception of some spread options where improvement is smaller. The best results are obtained pricing certain dual options where reduction in computer time is almost 200.

The conditional mean control variates CM's are expected to have stronger correlations with pay-offs, at least when $\rho \neq 0$. This reflects in smaller standard deviations produced by a single application of CM with respect to corresponding UM control variate. When $\rho = 0$ then the conditional mean is equal to the unconditional mean, but using the latter is preferable as no error by the Taylor expansion is introduced.

The use of both control variates (in the last row of each block of the tables) is very often extremely effective for UM but not for CM variates. Hence the best results are almost always given by using the two UM's.

Finally, it should be straightforward to use these control variates in bigger dimension, i.e. when payoffs depend on more than 2 assets. A possible bonus in this case is the fact that each new dimension 'brings' a new control variate.

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