

Apparent multifractality in financial time series

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Abstract. We present an exactly soluble model for financial time series that mimics the long range volatility correlations known to be present in financial data. Although our model is asymptotically 'monofractal' by construction, it shows apparent multiscaling as a result of a slow crossover phenomenon on finite time scales. Our results suggest that it might be hard to distinguish apparent and true multifractal behavior in financial data. Our model also leads to a new family of stable laws for sums of correlated random variables.

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Many time series exhibit interesting *scaling* properties. This means that if $x(t)$ denotes the time series, the probability distribution of the variations $\delta_T x = x(t+T) - x(t)$, rescaled by a lag-dependent factor $\xi(T)$, can be written as:

$$P(\delta_T x, T) = \frac{1}{\xi(T)} \mathcal{F} \left(\frac{\delta_T x}{\xi(T)} \right), \quad (1)$$

where $\mathcal{F}(u)$ is a time independent scaling function. For example, if $x(t)$ is constructed by summing independent identically distributed random variables with finite variance σ^2 , one has $\xi(T) = \sigma\sqrt{T}$ and $\mathcal{F}(u) = \exp(-u^2/2)/\sqrt{2\pi}$ for large T . Note that often the 'time' is actually a space coordinate, as it is the case in the analysis of turbulent velocity fields (where x is the fluid velocity) [1], or fracture surfaces (where x is the height of the profile) [2]. Equation (1) implies that all moments of $\delta_T x$ that are finite, scale similarly:

$$m_q \equiv \langle |\delta_T x|^q \rangle = A_q \xi(T)^q, \quad (2)$$

where A_q is a q -dependent number. Very often, $\xi(T)$ behaves as a simple power-law: $\xi(T) \propto T^\zeta$. In this case of a *monofractal* process, one therefore has, $m_q \propto T^{q\zeta}$, with $\zeta_q \equiv q\zeta$.

This is however not the only possibility, and in some cases, one can observe *multifractal* scaling, in the sense that $m_q \propto T^{\zeta_q}$, with $\zeta_q \neq q\zeta$. Such a possibility has been advocated for turbulent velocity fields [3–6] and, more recently, for financial time series [7–10], see also [11–13]. In the case of turbulence, there is strong theoretical evidence

in favor of such a multifractal behavior [1,4]. One can actually analytically derive a non trivial function ζ_q within a simple ('passive scalar') model, which is thought to retain some essential features of real turbulence [14]. The situation is much less clear in the case of financial markets, where the only evidence is based on the empirical analysis of the moments of several time series (typically currencies or stock indices). The idea of *multiplicative cascades*, which is at the heart of the arguments in favor of multiscaling in turbulence, is not easily applicable to price time series (see, however, [15]).

In this note, we study to which extent empirical studies on multiscaling behavior in finance are sensitive to crossover behavior that results in *apparent* multiscaling, even though the studied process is a monofractal. To that end we present a soluble model that is based on the study of financial time series. In the model, the 'volatility' (or the variance) of the elementary price increments is a random variable with long range correlations, which have been shown to be present in financial time series [15–19]. The model is an exact monofractal, but nevertheless it leads to an apparent multiscaling behavior [20,21]. As we argue below, one finds effective exponents $\tilde{\zeta}_q \neq \zeta_q$ due to a very long crossover effect, which leads to a systematic negative correction to the true asymptotic behavior $\zeta_q = q\zeta$. The correction grows with q and results thereby in a nontrivial functional form of $\tilde{\zeta}_q$. The numerical simulation of such a model, which mimics quite well the observed behavior of real prices, accurately reproduces the published data in favor of multiscaling in financial markets.

The model that we propose is also interesting in its own right. As a function of the strength of the correlations, we find a transition between a simple Gaussian behavior for

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the scaling function \mathcal{F} (together with the usual \sqrt{T} scaling for $\xi(T)$) for weak correlations, to a *new* family of stable laws (with a non trivial scaling of $\xi(T)$) for strong enough correlations. This adds to the very few cases where the limit distribution for sums of correlated random variables is exactly known.

Our model is the following: we consider that the time series is built by summing random variables:

$$x(t) = \sum_{k=1}^N \delta x_k \quad N = \frac{t}{\tau}, \quad (3)$$

where τ is a microscopic time scale. The elementary increments δx_k are assumed to be given by the product of two independent random variables, a ‘sign’ ϵ_k and an amplitude σ_k : $\delta x_k = |\sigma_k| \epsilon_k$. The ϵ_k are furthermore assumed to be independent Gaussian random variables of variance unity. σ_k^2 therefore is the (random) variance of the elementary increments. We choose the σ_k ’s to be Gaussian random variables of zero mean [22], with a correlation function given by $\langle \sigma_k \sigma_{k+\ell} \rangle = C(|\ell|)$. All moments m_q of δx_k are therefore finite, and the even ones given by $m_q = [(q-1)!!]^2 C^{q/2}(0)$, where $(q-1)!! \equiv 1 \cdot 3 \cdot \dots \cdot (q-1)$.

The correlation function $C(|\ell|)$ will be chosen to be a power law for large arguments: $C(|\ell|) \simeq \gamma |\ell|^{-\nu}$. From several studies of financial markets, one knows that the variance of the price increments is indeed also random, with a very slowly decaying time correlation function:

$$\langle \sigma_i^2 \sigma_{i+\ell}^2 \rangle - \langle \sigma_i^2 \rangle^2 = 2C^2(|\ell|) \simeq_{\ell \rightarrow \infty} \frac{2\gamma^2}{|\ell|^{2\nu}}, \quad (4)$$

where the exponent ν is found to be on the order of $\nu = 0.1 - 0.3$ for different markets [16–19, 15]. The important point here is that $\nu < 1/2$.

More precisely, we will use the following explicit representation of the σ ’s:

$$\sigma_k = \frac{s_0}{\sqrt{N}} \sum_{m=1}^{N/2} \left(\frac{2\pi m}{N} \right)^{\frac{\nu-1}{2}} \left(z_m e^{\frac{2i\pi m k}{N}} + z_m^* e^{-\frac{2i\pi m k}{N}} \right). \quad (5)$$

where the z_m ’s are independent complex Gaussian variables of unit variance. In the large N limit, the resulting correlation function $C(\ell)$ is well defined and decays, for large ℓ , as a power-law with $\gamma = s_0^2 \Gamma(\nu) \cos(\pi\nu/2)/\pi$ while $C(0)$ tends to $s_0^2 \pi^{\nu-1}/\nu$ [23].

We now turn to the calculation of the cumulants c_q of $x(t)$, as given by equation (3). We will show that these cumulants scale anomalously with N as soon as $n \geq 2$ (for $0 < \nu < 1/2$).

After making a gauge transform $\epsilon_i \rightarrow \text{sign}(\sigma_i) \epsilon_i$, one finds [24]:

$$\begin{aligned} e^{-G_N(z)} &\equiv \int dx P_N(x) e^{-izx} \\ &= \int \prod_{j=1}^N \left(\frac{d\epsilon_j d\sigma_j}{2\pi} \right) \frac{1}{\sqrt{\det \mathbf{C}}} \\ &\times \exp \left[iz \sum_{j=1}^N \sigma_j \epsilon_j - \sum_{j=1}^N \frac{\epsilon_j^2}{2} - \sum_{j,k=1}^N \frac{\sigma_j (\mathbf{C}^{-1})_{jk} \sigma_k}{2} \right]. \end{aligned} \quad (6)$$

The Gaussian integrals can be easily performed, and leads to the following expression for the characteristic function $G(z)$:

$$G_N(z) = \frac{1}{2} \text{Tr} \log(\mathbf{1} + z^2 \mathbf{C}) = \sum_{m=1}^{N/2} \log(1 + z^2 \tilde{C}(m)), \quad (7)$$

where the bold characters is used for matrices, and where $\tilde{C}(m)$ are the eigenvalues of the matrix \mathbf{C} . From the very construction of the σ ’s, one finds that $\tilde{C}(m) = s_0^2 (2\pi m/N)^{\nu-1}$, each of which is twofold degenerate.

Expanding $G_N(z)$ in powers of z leads to the cumulants $c_q(N)$ of $P_N(x)$. All odd order cumulants are zero, while even order cumulants are given by:

$$c_q(N) = 2(q-1)! \sum_{m=1}^{N/2} \tilde{C}(m)^{q/2}. \quad (8)$$

Let us first analyze the case $0 < \nu < 1/2$. In the large N limit, the sum over m is convergent when $q \geq 4$ and leads to cumulants which do not scale as N :

$$c_q \simeq \left(\frac{N}{2\pi} \right)^{(1-\nu)q/2} s_0^q (q-1)! \sum_{m=1}^{\infty} m^{(\nu-1)q/2}, \quad (9)$$

while for $q = 2$ one finds exactly $c_2 = \text{Tr} \mathbf{C} = NC(0)$. The normalized cumulants $c_q/c_2^{q/2}$ therefore behave as $N^{-q\nu/2}$ for $q \geq 4$ and vanish for $N \rightarrow \infty$. This means that the distribution of x/\sqrt{N} indeed tends to a Gaussian for large N . However, the approach to the Gaussian is slower than for sums of independent random variables. In particular, the kurtosis c_4/c_2^2 of $P_N(x)$ decays anomalously, as $N^{-2\nu}$, for $0 < \nu < 1/2$. For larger values of ν (*i.e.* when the volatility correlations are weaker), one recovers the usual scaling of the kurtosis as $1/N$ that holds for independent increments. Such an anomalous decay of the kurtosis with time was first reported for financial time series in [21, 18].

The important outcome of the above calculation is that the *moments* m_q of the distribution $P_N(x)$ are not simple power-laws, but sums of power laws with similar exponents. For example:

$$m_4 = A_{4,0} N^{2-2\nu} + A_{2,2} N^2, \quad (10)$$

$$m_6 = A_{6,0} N^{3-3\nu} + A_{4,2} N^{3-2\nu} + A_{2,2,2} N^3, \quad (11)$$

where the A 's are some coefficients. If ν is small, these sums of power-laws can be fitted on many decades with an effective exponent $\tilde{\zeta}_q$ such that $m_q \propto N^{\tilde{\zeta}_q}$. The exponent $\tilde{\zeta}_q$ is less than $q/2$, and more and more so as q increases. However, the true asymptotic behavior predicted by our model is $\zeta_q = q/2$. In fact, equations (10, 11) show that our model (and possibly also real financial data, or even turbulence data) is better characterized by the cumulants than by the moments.

In order to illustrate this point numerically, we have generated a surrogate time series in a way closely related to the above model. Instead of writing $\delta x_i = |\sigma_i| \epsilon_i$, we have chosen to take $\delta x_i = \exp(\sigma_i) \epsilon_i$. This leads to a more realistic time series as compared with real data from financial markets, without changing the crucial feature of the above model, *i.e.* the very slow decay of the volatility correlations. In particular, the distribution of the volatility has a positively skewed, log-normal shape. The length of our surrogate time series was taken to be comparable to those analyzed previously. The moments $m_q = \langle |\delta_T x|^q \rangle$ are plotted as a function of T for different q 's, for the choice $\nu = 0.2$ (see Fig. 1). The interval of T was chosen to be $\tau = 1 \leq T \leq 6000$, again comparable to the region investigated in previous studies [7–10]. The power law fits are extremely good, and lead to a function $\tilde{\zeta}_q$ bending downwards as q increases, shown in Figure 2. For our choice of parameters, the numerical values of $\tilde{\zeta}_q$ actually match precisely those reported in [10]. We have also checked that the same model, but without volatility correlations, leads very precisely to $\tilde{\zeta}_q = \zeta_q = q/2$.

Note that, by construction, all the moments of our surrogate time series are finite, although the distribution of price differences has fat tails comparable to those observed on empirical data. It has however been argued recently (see [25,27]) that these empirical tails are in fact power-laws with an exponent around 4, such that the moments $q \geq 3$ are ill-defined. One could generalize our model to allow for such power-law tails. For empirical studies on finite-size data sets, even stronger transient effects would be observed on $\tilde{\zeta}_q$, mimicking again multifractality.

We now turn to the case $\nu < 0$. This corresponds to a non stationary process for the volatility, which typically *grows* with ℓ as $\ell^{-\nu/2}$. More precisely, from equation (5), one can show that $\langle (\sigma_k - \sigma_{k+\ell})^2 \rangle \propto \ell^{-\nu}$. In this case, after changing variables to $x = \hat{x} \sqrt{N^{1-\nu}}$ and $z = \hat{z} \sqrt{(N/2\pi)^{\nu-1}}$, one finds that the asymptotic distribution of \hat{x} has a characteristic function given by:

$$G_\nu(\hat{z}) = \sum_{m=1}^{\infty} \log(1 + \hat{z}^2 m^{\nu-1}) \quad (12)$$

(we have set $s_0 = 1$, which amounts to a change of scale in x).

The above result means that after rescaling by a factor $\sqrt{N^{1-\nu}}$, the sum of (strongly) correlated random variables converges to a non-Gaussian distribution $\mathcal{F}_\nu(\hat{x})$, obtained as the Fourier transform of the exponential of $-G_\nu(\hat{z})$ given by equation (12). Since the expansion of $G_\nu(\hat{z})$ is

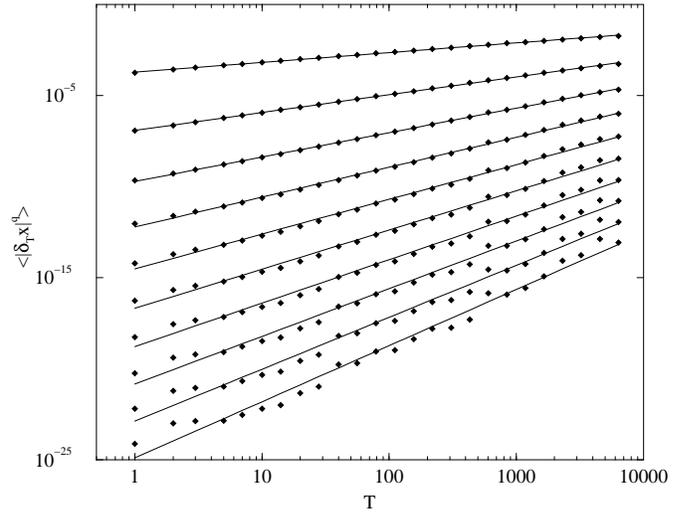


Fig. 1. Simulation of our surrogate financial time series and determination of the scaling of the moments as a function of window size T . From top to bottom $\langle |\delta_T x|^q \rangle$ for $q = 1-10$. Each moment scales relatively well with T as indicated by the 10 linear (log-log) fit lines. In this simulation the log-volatility follows a correlation Gaussian process as defined in the text with $\nu = 0.2$. The variance of this process has been set so that the kurtosis at the unit time scale be $\kappa_0 = 65$. This is a typical value for high frequency financial data. The simulated data set contained 500 000 points.

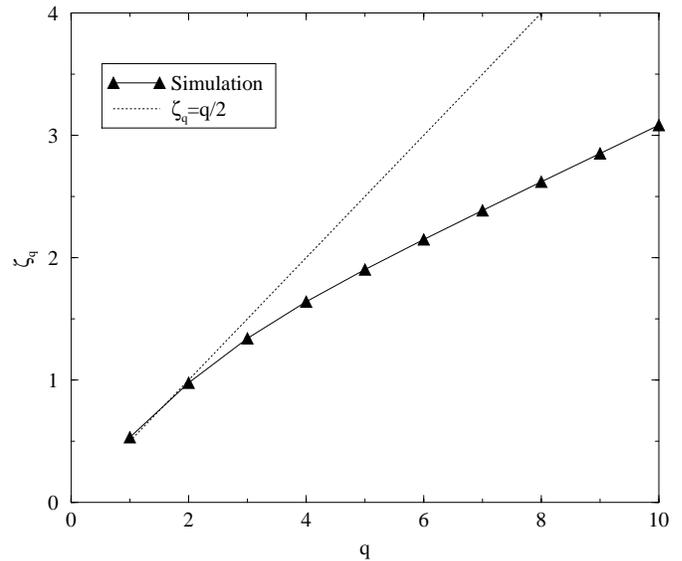


Fig. 2. Slope ζ_q of the fitted data of Figure 1. The deviations from the true asymptotic scaling $\zeta_q = q/2$ are quite clear. The value of ζ_q precisely match those reported in [10].

regular for $\hat{z} \rightarrow 0$, all the moments of \mathcal{F}_ν are finite. From the leading singularity of $G_\nu(\hat{z})$ around $\hat{z} = \pm i$, one obtains the asymptotic behavior of $\mathcal{F}_\nu(\hat{x})$ for large arguments as:

$$\mathcal{F}_\nu(\hat{x}) \simeq_{\hat{x} \rightarrow \infty} e^{-|\hat{x}|}. \quad (13)$$

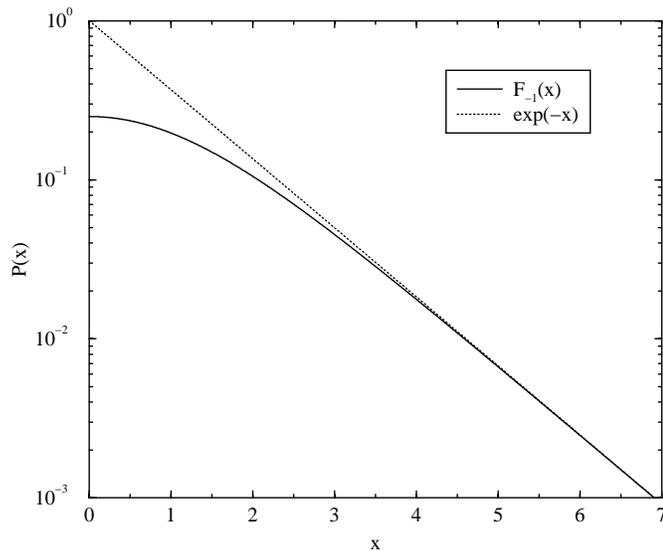


Fig. 3. Graph of the stable distribution $P(x) = \mathcal{F}_{-1}(x)$. The asymptotic behavior $\mathcal{F}_{-1}(x) \sim \exp(-|x|)$ is shown as the dotted curve.

In the special case where $\nu = -1$, corresponding to a ‘volatility random walk’, the sum in (12) can be explicitly performed, and leads to [28]:

$$G_{-1}(\hat{z}) = \log \left(\frac{\sinh(\pi \hat{z})}{\pi \hat{z}} \right). \quad (14)$$

This distribution is shown in Figure 3, together with the predicted asymptotic behavior (dotted line). The kurtosis of this distribution is equal to $6/5$. Interestingly, \mathcal{F}_ν has a shape similar to hyperbolic distributions [29] with exponential tails which have been proposed in a financial context (see [21]). The appearance of such laws might thus be related to the existence of long-ranged correlations in the volatility.

In summary, the purpose of this paper was to show, on an exactly soluble ‘stochastic volatility’ model, that an apparent multiscaling behavior can appear as a result of very long transient effects, induced by the long range nature of the volatility correlations. This model is inspired by real price time series, and leads to an effective exponent spectrum $\tilde{\zeta}_q$ in close correspondence that reported in recent papers on the subject. We therefore suspect that indications of multifractal behavior found in financial data might be misleading, as they could be caused by crossover effects that do not correspond to the true asymptotic behavior. To check more carefully for crossover effects, it might be helpful to analyze not only the moments but also the cumulants in empirical studies. We also have, *en passant*, found a new family of stable laws for sums of *correlated* random variables in the case where the volatility correlation is growing with time. It would be very interesting to characterize the attraction basin of these new stable laws.

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22. One could also choose a non zero mean value for σ and not take the absolute value. This would still lead to an soluble model.
23. The (time-domain) correlation function defined by this construction explicitly depends on the number of variables in the sum N . This is not a problem when $\nu > 0$, since when N is large, this correlation function is equal to a well-defined N -independent limit plus sub-leading correction which can be safely neglected. Therefore, any subpart of the series of size n such that $1 \ll n \ll N$ will behave, as a function of n , as the whole sum.
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