Valuation of Defaultable Bonds Using Signaling Process – An Extension

C. F. Lo
Physics Department
The Chinese University of Hong Kong
Shatin, Hong Kong
E-mail: cflo@phy.cuhk.edu.hk

C. H. Hui
Banking Policy Department
Hong Kong Monetary Authority
30th Floor, 3, Garden Road, Hong Kong
Tel. (852) 2878 1485
Fax: (852) 2878 2485
Email: Cho-Hoi_Hui@hkma.gov.hk

This paper extends the defaultable bond valuation model developed by Cathcart and El-Jahel [1998]. The extended model incorporates a default barrier with dynamics depending on the volatility and the drift of the signaling variable. The level of the barrier is adjusted by a free parameter. We derive a closed-form solution of the defaultable bond price as a function of a signaling variable and a short-term interest rate, with time-dependent model parameters governing the dynamics of the signaling variable and interest rate. The numerical results calculated from the solution show that the risk adjustable default barrier has material impact on the term structures of credit spreads. The model is capable of producing diverse shape of term structures of credit spreads. It provides new insight for future research on defaultable bonds analysis and credit risk modeling.

I. Introduction

There are generally two approaches to model the valuation of defaultable bonds. The first approach is the structure model which treats default risk equivalent to a European put option on the corporate asset value and the corporate liability is the option strike. Black and Scholes (1973) and Merton (1974) have been the pioneers in this approach. In Merton's framework, default occurs only at bond maturity when the asset value is less than the liabilities due to the bond, and the firm is insolvent. To cope with the possibility of early default before bond maturity, Black and Cox (1976) assume a bankruptcy-triggering level for the corporate assets whereby default can occur at any time. This trigger level is introduced by considering a safety covenant that protects bondholders. Longstaff and Schwartz (1995) extend the risky debt model of Black and Cox to allow interest rate to follow the Ornstein-Uhlenbeck process[1]. Default occurs when the corporate asset value is below a constant or deterministic bankruptcy-triggering barrier. Upon bankruptcy triggered by touching the barrier, bondholders receive an exogenously given number of riskless bonds.

The second approach is the reduced-form models in which default time is a stopping time of some given hazard rate process and the payoff upon default is specified exogenously. This approach has been considered by Artzner and Delbaen [1992], Madan and Unal [1993], Jarrow, Lando, and Turnbull [1994], Jarrow and Turnbull [1995], and Duffie and Singleton [1997].

A middle ground model between the structure model and the reduced-form models is developed by Cathcart and El-Jahel [1998]. In the model, default occurs when some signaling process hits some lower constant default barrier. The model assumes the signaling process for each firm that determines the occurrence of default rather than the value of the assets of the firm. When the signaling variable drops below the default barrier, bondholders receive an exogenously specified number of riskless bonds. The underlying interest rate is assumed to follow a mean-reverting square root process that is uncorrelated with the signaling process. An analytical defaultable bond price solution is derived from the model. However, since the solution is expressed as inverse Laplace transforms, numerical techniques need to be employed to perform the transforms. This may impose some numerical difficulties to obtain numerical results.

The main objective of this paper is to extend Cathcart and El-Jahel's model in which the bankruptcy-triggering barrier is defined as a drifted level governed by the volatility and the drift of the signaling variable. The contribution of the signaling variable's dynamics to the barrier's dynamics is adjusted by a free parameter b. When the parameter b is equal to zero, the model is reduced to the case of a fixed default barrier purposed by Cathcart and El-Jahel. The model in this paper is therefore characterised by a risk adjustable default barrier. More realistic default scenarios can be put into the valuation model through adjusting the parameter b. In addition, we derive a closed-form bond solution in terms of a cumulative normal distribution function. Therefore, no sophisticated numerical technique is needed to compute the solution.

Using the structure model, a moving bankruptcy-triggering barrier has been considered by Briys and de Varenne [1997] and Schöbel [1999] as a fixed quantity discounted at the riskless rate up to the maturity date of a risky corporate bond. When the asset value drops below this barrier, bondholders receive an exogenously specified number of riskless bonds. Therefore, the dynamics of the barrier follows the stochasticity of the interest rate, that is specified as the Ornstein-Uhlenbeck process. As a result, the model is characterised by a stochastic barrier and avoids the limitation of having a constant default boundary as the Longstaff-Schwartz model. However, it is difficult to justify why the barrier just follows the dynamics of interest rates only. It is obvious to observe that the barrier goes downwards as the time to maturity of the corporate bond increases. Since the barrier denotes the threshold level at which bankruptcy occurs, higher firm value volatility could imply a higher level of leverage over time and thus higher probability of default. Hui et al. [1999] develop a corporate bond valuation model in which the bankruptcy-triggering barrier is defined as a drifted firm value level governed by stochastic interest rates and instantaneous variance of the corporate bond value. The firm value volatility affects the level of the barrier over time through the variance of the corporate bond function and its contribution to the barrier's dynamics is adjusted by a free parameter.

In the development of the model in this paper, the dynamics of the bankruptcy-triggering barrier is also influenced by the dynamics of the signaling variable and is risk adjustable through a free parameter. The dynamics of the short-term interest rate is assumed to follow the square root process. When the signaling variable touches the barrier, bondholders receive an exogenously specified number of riskless bonds. A non-enforcement of the strict priority rule upon default can therefore be applied to the payoffs to the bondholders. We derive a closed-form of the bond price as a function of signaling variable and interest rate explicitly. In addition, the model parameters such as volatility, drift and mean-level of the interest rate are time dependent in the derivation. The scheme of this paper is as follows. In the following section we develop the pricing model of discount defaultable bonds with a drifted default barrier, and derive the closed-form pricing formula. Numerical results of the term structures of credit spreads calculated from the pricing formula are shown in section III. In the last section we shall summarise our investigation.

II. Valuation Model of Defaultable Bonds

In the valuation of defaultable bonds, we assume a continuous-time framework, and let the short-term interest rate and the signal process be stochastic variables. The dynamics of the short-term interest rate r is drawn from the term structure model of Cox, Ingersoll, and Ross (CIR) [1985], i.e. the square-root process:

$$dr = k(t)[q(t) - r]dt + s_r(t)\sqrt{r}dz_r \qquad (1)$$

where the short-term interest rate is mean-reverting to long-run mean $q(t)$ at speed $k(t)$, and the stochastic term has a standard deviation proportional to $\sqrt{r}$. The signaling variable S is assumed to follow a lognormal diffusion process:

$$dS = a(t)Sdt + s_S(t)Sdz_S, \qquad (2)$$

where $a(t)$ and $s_S(t)$ are the drift and the volatility of S respectively. The Wiener processes dz_S and dz_r are assumed to be uncorrelated.

The assumption of a signaling process for the occurrence of default is a middle approach between structure and reduced-form models. A signaling process can capture factors that can affect the probability of default. The use of a signaling process is appropriate for entities such as sovereign issuers that issue defaultable debt but do not have an identifiable collection of assets. The time-dependent drift assumption is used instead of the constant drift used by Cathcart and El-Jahel [1998]. The no-correlation assumption between the signaling process and the interest rates is in line with most of the reduced-form models, where the hazard rate of default process is assumed to be uncorrelated with the interest rates. The detailed discussion of the above assumptions is found in Cathcart and El-Jahel [1998].

We let the price of a discount defaultable bond be $P(S, r, t)$. The partial differential equation governing the bond is

$$\frac{\partial P}{\partial t} = \frac{1}{2}\sigma_S^2(t)S^2\frac{\partial^2 P}{\partial S^2} + \frac{1}{2}\sigma_r^2(t)r\frac{\partial^2 P}{\partial r^2} + \alpha(t)S\frac{\partial P}{\partial S} + [\kappa(t)(\theta(t) - r) - \lambda]\frac{\partial P}{\partial r} - rP\,, \qquad (3)$$

where λ is the market price of interest rate risk[2]. The value of a defaultable bond is obtained by solving equation (3) subject to the final payoff condition and the boundary condition imposed by the default barrier.

A constant default barrier is considered by Cathcart and El-Jahel [1998]. However, it is reasonable to assume that the dynamics of the barrier depends on the dynamics of the signaling variable. We propose the barrier $H(t)$ to have a drifted dynamics which is determined by the drift and the volatility of the signaling variable. It is specified as the form:

$$H(t) = S_0 \exp\left[-\beta\left(\alpha - \frac{\sigma_S^2}{2}\right)t\right], \qquad (4)$$

where S_0 is the pre-defined value of the barrier and β is a real number parameter to adjust the rate of the drift. It is noted that when the parameter β is put to be zero, the case of a fixed barrier is obtained, i.e. recovering Cathcart and El-Jahel's model. The movement of the barrier can be interpreted as a mean drift (adjusted by β) arising from the dynamics of S. The barrier levels with different β at different time to maturity are illustrated in Exhibit 1 for σ_S = 20% and α = 1%.

For the given parameters where the term $(\alpha - \sigma_S^2/2)$ is less than zero, Exhibit 1 shows that the barrier level increases with the time to maturity for a positive β. On the other hand, given a negative β, the barrier level decreases with the time to maturity. It means that given an initial S_0 as the pre-defined default level, the probability of default increases with the value β when $\sigma_S^2/2$ is higher than the drift α. Given the same β, the barrier moves away from S_0 with time to maturity at a faster rate when σ_S is higher. This demonstrates the effect of σ_S on early default risk of a defaultable bond. For β = 1, the barrier basically moves with the mean drift of the signaling variable. The barrier dynamics incorporating the adjustable mean drift of the signaling variable is more realistic than the constant barrier specified in the Cathcart and El-Jahel model.

When S breaches the barrier $H(t)$, bankruptcy occurs before maturity at t = 0. The payoffs to bondholders are specified by

$$P(S = H, r, t) = WFQ(r, t) \qquad t > 0\,; W \leq 1, \tag{5}$$

where $Q(r, t)$ is the default-free bond function according to the CIR model and F is the bond face value. On the other hand, if S has never breached the barrier, the payoff to bondholders at the bond maturity is

$$P(S, r, t = 0) = F \qquad S > H(t). \tag{6}$$

The parameter W lets the payoffs upon default deviate from the absolute priority rule. Therefore, for W = 1, the strict priority rule is enforced and shareholders receive nothing. For W being between zero and one, it implies the non-enforcement of the strict priority rule.

The solution of Equation (3) subject to the boundary condition and the final payoff condition of Equation (4), (5) and (6) is

$$P(S,r,t) = FQ\left\{W + (1-W)\left[N(d_1) - \left(\frac{S}{S_o}\right)^{-2(1-b)\left(a - s_S^2/2\right)/s_S^2} e^{-2b(1-b)\left(a - s_S^2/2\right)^2 t/s_S^2} N(d_2)\right]\right\}, \tag{7}$$

where

$$d_1 = \frac{\ln(S/S_o) + \left(a - s_S^2/2\right)t}{s_S\sqrt{t}},$$

$$d_2 = \frac{-\ln(S/S_o) - (2b-1)\left(a - s_S^2/2\right)t}{s_S\sqrt{t}},$$

and N is a cumulative normal distribution function. The detailed derivation of the solution in Equation (7) is given in the Appendix. It is easy to show from Equation (7) that the defaultable bond price is equal to the recovery value $WFQ(r,t)$ when S breaches the barrier. The credit spreads of defaultable bonds are calculated from Equation (7) and illustrated in the following section.

III. Credit Spread Analysis

The credit spread C_S of a defaultable discount bond price $P(S, r, T)$ with time to maturity T and face value F is given as

$$C_s(S,r,T) = -\frac{1}{T}\ln\frac{P(S,r,T)}{FQ(r,T)} . \tag{8}$$

The term structures of credit spreads for a low risk defaultable bond, with S/S_0 = 2.5 are illustrated in Exhibit 2 using different b from -1.0 to 2.0. Other parameters used in the calculations are s_S = 0.2, $s^2{}_r$ = 0.078, a = 0.01, r = 4%, $q = 9\%$, $k = 0.5$ and W = 0.75. Given these parameters, $\left(a - s_S^2/2\right)$ is less than zero. Exhibit 2 shows that the credit spreads increase with positive b since the barrier level increases with time to maturity. This demonstrates that the levels of the default barrier with different b imply different early default risk. At the long end, the difference between the credit spreads for b = -1.0 and b = 2.0 is about 20bp which is significant compared with the average credit spread of 47bp between ten to twenty years time to maturity for b = -1.0. The credit spreads of the low risk defaultable bonds calculated here correspond with empirical evidence found in Caouette et al. [1998] that reports an average yield spread on AAA-rated bonds of 55bp with standard deviation of 22bp for the years 1985 -1996. Although the credit spreads are different for different b, the shape of their term structures is similar.

The term structures of credit spreads for a medium risk defaultable bond, with S/S_0 = 2.0 and W = 0.5, are illustrated in Exhibit 3. The difference between the credit spreads for b = -1.0 and b = 2.0 is on the average at 44bp. Again this difference is material compared with the credit spread of 164bp for b = -1.0 at ten years time to maturity. For time to maturity between five to twenty years, the credit spreads range from 165bp to 210bp for b = 2.0, and from 142bp to 165bp for b = -1.0. These spreads correspond with BBB-rated bonds, which are reported by Caouette et al. [1998] to have a credit spread of 140bp on the average with standard deviation of 37bp for the years 1985 - 1996.

For a high risk defaultable bond, with S/S_0 = 1.5 and W = 0.5, the credit spreads of different b are illustrated in Exhibit 4. The difference between the credit spreads for b = -1.0 and b = 2.0 is on the average at 111bp. The difference is more material in absolute terms compared with the previous two cases. However, the impact on the credit spreads from b = -1.0 to b = 2.0 in percentage terms is about 19% to 26% (relative to the credit spread for b = 2.0 at ten years time to maturity) in different S/S_0 ratios. This reflects that the dynamics of the default barrier gives almost the same relative impact on different risky bonds'default probabilities.

To study the impact of the volatility of the signaling variable on the credit spreads, different figures of volatility, s_S = 0.2, 0.25 and 0.3, are used to calculate the model credit spreads with the following parameters: S/S_o = 2.0, $s^2{}_r$ = 0.078, a = 0.01, r = 4%, $q = 9\%$, $k = 0.5$ and W = 0.5. The credit spreads are illustrated in Exhibits 5 and 6 for b = 0.0 and b = 1.0 respectively. The results show that the volatility has significant impact on the term structures. The term structures change from upward-sloping shape to humped shape with higher volatility. The hump-shaped term structures are usually observed in higher risk defaultable bonds. Exhibits 5 and 6 also show that the differences between the credit spreads with different volatility for b = 1.0 are more significant than that for b = 0.0. The average difference between s_S = 0.2 and 0.3 for b = 1.0 is 247bp, while the corresponding average difference for b = 0.0 is 179bp. This finding is consistent with the barrier structure presented in Equation 4, which moves above S_0 at higher rates when s_S is higher and b is positive. The dynamics of the barrier increases the default probability and hence increases the credit spreads. For b = 0.0, the default barrier is constant and the default probability only depends on the dynamics of the signaling variable.

In summary, the numerical results show that the dynamics of the default barrier incorporating the volatility and the drift of the signaling variable has material impact on credit spreads of defaultable bonds. For $\left(a - s_S^2/2\right) < 0$, the credit spreads increase with positive b and the magnitude of the increases is sensitive to the volatility. The term structures of credit spreads derived from the model are similar to the term structures obtained in previous studies by Cathcart and El-Jahel [1998], which match the empirical evidence[3].

IV. Summary

This paper extends the defaultable bond valuation model developed by Cathcart and El-Jahel [1998]. The extended model incorporates a default barrier with dynamics depending on the volatility and the drift of the signaling variable. Since the volatility of the signaling variable affects the level of the default barrier over time, more realistic default scenarios can be put into the valuation model through adjusting the barriers' dynamics. We derive a closed-form solution of the defaultable bond price as a function of a signaling variable and a short-term interest rate, with time-dependent model parameters governing the dynamics of the signaling variable and interest rate. The numerical results calculated from the solution show that the risk adjustable default barrier has material impact on the term structures of credit spreads. Given $\left(a - s_S^2 / 2\right) < 0$, the credit spreads increase with positive *b* and the magnitude of the increases is sensitive to the volatility of the signaling variable. This demonstrates that the default barrier with different *b* imply different early default risk. The model incorporating the risk adjustable default barrier, deviations from the absolute priority rule, and time-dependent model parameters is capable of producing diverse shape of term structures of credit spreads. It provides new insight for future research on defaultable bonds analysis and credit risk modelling.

Appendix

In the model of Cathcart and El-Jahel, the price P of a defaultable bond, which is a function of the value S of a signaling variable determining the occurrence of default, the short-term interest rate r and the time to maturity t, is governed by the partial differential equation

$$\frac{\partial P}{\partial t} = \frac{1}{2} s_S^2(t) S^2 \frac{\partial^2 P}{\partial S^2} + \frac{1}{2} s_r^2(t) r \frac{\partial^2 P}{\partial r^2} + a(t) S \frac{\partial P}{\partial S} + [k(t)(q(t) - r) - l] \frac{\partial P}{\partial r} - rP \qquad \text{(A.1)}$$

To solve this partial differential equation, we first rewrite it in terms of the variable x = InS as follows[2]:

$$\frac{\partial P(x,r,t)}{\partial t} = \frac{1}{2} s_S^2(t) \frac{\partial^2 P(x,r,t)}{\partial x^2} + \frac{1}{2} s_r^2(t) r \frac{\partial^2 P(x,r,t)}{\partial r^2}$$
$$\left[a(t) - \frac{1}{2} s_S^2(t)\right] \frac{\partial P(x,r,t)}{\partial x} + k(t)[q(t) - r] \frac{\partial P(x,r,t)}{\partial r} - rP(x,r,t) \qquad \text{(A.2)}$$

Since the variables x and r are separable, and the boundary conditions for r are: (a) $P(x,r,t)$ is finite as $r \to 0$, and (b) $P(x, r \to \infty, t) = 0$, the price function $P(x,r,t)$ can be expressed as the product $Q(r,t)F(x,t)$, where $Q(r,t)$ is the price of a riskless bond function of the CIR model with explicitly time-dependent parameters, and $F(x,t)$ satisfies the equation

$$\frac{\partial F(x,t)}{\partial t} = \frac{1}{2} s_S^2(t) \frac{\partial^2 F(x,t)}{\partial x^2} + \left[a(t) - \frac{1}{2} s_S^2(t)\right] \frac{\partial F(x,t)}{\partial x}. \qquad \text{(A.3)}$$

Assuming the natural boundary conditions for S, the solution of Equation (A.3) is found to be

$$F(x,t) = \int_{-\infty}^{\infty} dy K_0(x-y,t) F(y,0) \qquad \text{(A.4)}$$

where the kernel $K_0(x-y,t)$ is given by

$$K_0(x-y,t) = \frac{1}{\sqrt{2pB(t)}} \exp\left\{-\frac{1}{2B(t)}[x - y + C(t)]^2\right\}$$

$$C(t) = \int_0^t dt \left[a(t) - \frac{1}{2} s_S^2(t)\right] \qquad \text{(A.5)}$$

$$B(t) = \int_0^t dt\, s_S^2(t)..$$

For the case of constant a and s_S, using an approach based upon the *method of images,* we can straightforwardly incorporate an absorbing barrier with a drifted dynamics of the form $H(t) = S_o \exp[-b(a - s_S^2/2)t]$ into the model, where S_0 is the pre-defined signal value of the barrier and b is a real adjustable parameter. The corresponding bond price $P(y, r, t)$ is then given by

$$P(y,r,t) = \int_0^\infty dy' G(y, y', t; r) \exp\left[(b-1)\left(\frac{a}{s_S^2} - \frac{1}{2}\right)(y - y')\right] P(y', r, 0)$$

$$G(y, y', t; r) = K(y - y', t; r) - K(y + y', t)$$

$$K(y - y', t) = \frac{\Psi(r,t)}{\sqrt{2ps_S^2 t}} \exp\left\{-\frac{1}{2s_S^2 t}\left[y - y' + b\left(a - \frac{1}{2}s_S^2 t\right)\right]^2\right\}$$

$$\Psi(r,t) = Q(r,t) \exp\left[-\frac{1-b^2}{2s_S^2}\left(a - \frac{1}{2}s_S^2\right)^2 t\right] \qquad \text{(A.7)}$$

where $y = \ln(S/S_0)$ and $y' = \ln(S'/S_0)$. It should be noted that this solution vanishes at the barrier; that is, it is the solution associated with the homogeneous boundary condition only. Nevertheless, it is an easy task to extend the solution to satisfy the inhomogeneous boundary condition: $P(S, r, t) = WS_0 Q(r, t)$ at $S = H(t)$, by simply adding the trivial solution $WS_0 Q(r, t)$ of the pricing equation in Equation (A.1). Then, by requiring that the solution associated with the inhomogeneous boundary condition obeys the prescribed final payoff condition, we can readily obtain the desired defaultable bond function.

Endnotes

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[1] Interest rates following the Ornstein-Uhlenbeck process are studied by Shimko et al. [1993] in valuation of corporate bonds without any default barrier.

[2] Campbell [1986] shows that a constant market price of risk l can be justified in a market equilibrium with log-utility investors. In the derivation, l is absorbed into the term $k(t)q(t)$.

[3] See Jones et al. [1984] and Sarig and Warga [1989].

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