A Note on Risky Bond Valuation

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Abstract

This paper develops a corporate bond valuation model that incorporates a default barrier with dynamics depending on stochastic interest rates and variance of the corporate bond function. Since the volatility of the firm value affects the level of leverage over time through the variance of the corporate bond function, more realistic default scenarios can be put into the valuation model. When the firm value touches the barrier, bondholders receive an exogenously specified number of riskless bonds. We derive a closed-form solution of the corporate bond price as a function of firm value and a short-term interest rate, with time-dependent model parameters governing the dynamics of the firm value and interest rate. The numerical results show that the dynamics of the barrier has material impact on the term structures of credit spreads. This model provides new insight for future research on risky corporate bonds analysis and modelling credit risk.

I. INTRODUCTION

In pricing corporate bonds, Black and Cox (1976) assume a bankruptcy-triggering level for the corporate assets whereby default can occur at any time. Longstaff and Schwartz (1995) extend Black-Cox model to allow interest rates to follow the Ornstein-Uhlenbeck process. Upon bankruptcy triggered by touching the barrier, bondholders receive an exogenously given number of riskless bonds. Following Longstaff-Schwartz's model, Briys and de Varenne (1997) and Schöbel (1999) develop pricing models to define the bankruptcy-triggering barrier as a fixed quantity discounted at the riskless rate up to the maturity date of the risky corporate bond. As a result, the model is characterised by a barrier following the stochasticity of the interest rates.

It is obvious to observe that the barrier goes downwards as the time to maturity of the corporate bond increases. Since the barrier denotes the threshold level at which bankruptcy occurs, higher firm value volatility should imply a higher level of leverage over time and thus higher probability of default. The main objective of this paper is to develop a corporate bond valuation model in which the bankruptcy-triggering barrier is defined as a drifted firm value level governed by stochastic risk-free interest rates and instantaneous variance of the corporate bond value. Through the instantaneous variance of the corporate bond value, the firm value volatility is incorporated into the barrier dynamics. There is an additional free parameter b to specify the contribution of the instantaneous variance of the corporate bond to the rate of the drift of the barrier. We derive a closed-form solution of the bond price as a function of firm

volatility, correlation, drift and mean-level of the interest rate.

In the following section we develop the pricing model of discount corporate bonds

of credit spreads. In the last section we shall summarise our investigation.

II.

In the valuation of corporate bonds, we assume a continuos-time framework. The dynamics of the shot-term interest rate r

Vasicek (1977):

$$dr = k(t)[q(t) - r]dt + s_r(t)dz_r \tag{1}$$

where the short-term interest rate is mean-reverting to long-run mean q(t) at speed k(t), and $s_r(t)$ is the volatility of r.

The firm value S is assumed to follow a lognormal diffusion process:

$$dS = m(t)Sdt + s_S(t)Sdz_S \tag{2}$$

where $m(t)$ and $s_S(t)$ are the drift and volatility of the firm value respectively. The Wiener processes dz_S and dz_r are correlated with

$$dz_S dz_r = r dt \tag{3}$$

and the correlation coefficient r is also assumed to be time dependent.

We let the price of a corporate bond be $P(S, r, t)$. Using Ito's lemma and the standard no-arbitrage arguments, the partial differential equation governing the bond is

$$\begin{aligned} \frac{\partial P}{\partial t} &= \frac{1}{2} s_S^2 S^2 \frac{\partial^2 P}{\partial S^2} + \frac{1}{2} s_r^2 \frac{\partial^2 P}{\partial r^2} + r s_S s_r S \frac{\partial^2 P}{\partial S \partial r} + rS \frac{\partial P}{\partial S} + \\ &\quad [k(t)q(t) - k(t)r - l] \frac{\partial P}{\partial r} - rP \end{aligned} \tag{4}$$

where l is the market price of interest rate risk[1]. The value of the corporate bond is obtained by solving equation (4) subject to the final payoff condition and the boundary condition imposed by the default barrier.

In order to incorporate the dynamics of the firm value into the dynamics of the default barrier, we propose the barrier $H(r, t)$ to have a drifted dynamics with the form:

$$H(r,t) = S_o Q(r,t) \exp[b c_1(t)] \tag{5}$$

where S_o is the pre-defined asset value of the barrier, $Q(r, t)$ is the riskless bond function according to the Vasicek model with time-dependent parameters, $c_1(t)$ is defined as

$$c_1(t) = \int_0^t dt \left[\frac{1}{2} s_S^2(t) + r(t) s_S(t) s_r(t) c_2(t) + \frac{1}{2} s_r^2(t) c_2^2(t) \right]$$

$$c_2(t) = \exp\left[-\int_0^t dt\, k(t) \right] \int_0^t dt \, \exp\left[\int_0^t dt'\, k(t') \right] \tag{6}$$

and b is a real number parameter to adjust the rate of the drift. It is noted when the parameter b is put to be zero, the barrier follows the dynamics of a riskless bond, i.e. recovering Briys-de Varenne's and Schöbel's models. The function c_1(t) is the integrated instantaneous variance of the corporate bond function over the life of the corporate bond, and the function $c_2^2(t)s_r^2(t)$ is the instantaneous variance of a riskless discount bond price of the Vasicek model with time to maturity *t*. The process of the barrier can therefore be interrupted as a mean drift (adjusted by *b*) arising from the dynamics of *r* and $P(S, r, t)$. The firm value volatility $s_S(t)$ is incorporated into the barrier dynamics through $c_1(t)$.

For a positive *b*, $c_1(t)$ offsets the decreasing effect of the riskless bond value with time to maturity. It makes the decrease in the barrier level with the time to maturity at a slower rate. It means that given an initial S_o as the pre-defined default level, when the variance of the corporate bond value is high, the probability of default to occur increases with the value *b*.

When the firm value breaches the barrier $H(r, t)$, bankruptcy occurs before maturity $t = 0$. The payoffs to bondholders are specified by

$$P(S = H, r, t) = a_1 S_o Q(r, t) \quad t > 0;\ a_1 \le 1 \tag{7}$$

For $b \ge 0$, the payoffs to bondholders at the barrier should be always less than the firm value since $c_1(t)$ is positive definite[2]. On the other hand, if the firm value has never breached the barrier, then the payoffs to bondholders at the bond maturity are:

$$\begin{aligned} &P(S, r, t = 0) = F \qquad S \ge F \\ &P(S, r, t = 0) = a_2 S \qquad S < F\ ; a_2 \le 1 \end{aligned} \tag{8}$$

The solution[3] of equation (4) subject to equation (7) and (8) is

$$\begin{aligned} P = FQ\Bigg\{ & a_2 \frac{l}{Q}\left[N(d_1) - N(d_2)\right] - a_1 \frac{l}{q}\left[N\left(d_1 + \sqrt{2c_1}\right) - N\left(d_1 + \sqrt{2c_1}\right)\right] \\ & - a_2 l q^{-(b+1)} Q^b e^{b(b+1)c_1}\left[N(d_3) - N(d_4)\right] \\ & + a_1 l q^{-b} Q^{b-1} e^{b(b-1)c_1}\left[N\left(d_3 + \sqrt{2c_1}\right) - N\left(d_4 + \sqrt{2c_1}\right)\right] \\ & + \left(1 - \frac{a_1 l}{q}\right) N\left(-d_1 - \sqrt{2c_1}\right) - \left(q^{1-b} - a_1 l q^{-b}\right) Q^{b-1} e^{b(b-1)c_1} N\left(-d_3 - \sqrt{2c_1}\right)\Bigg\} + a_1 S_o Q \end{aligned} \tag{9}$$

where $l = S / F$ is the asset-to-liability ratio, $q = S / S_o$ is an early default ratio, and

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[1] Campbell (1986) shows that a constant l can be justified in a market equilibrium with log-utility investors. l is absorbed into the term $k(t)q(t)$ in the following calculation.

[2] It can be shown by completing square of $c_1(t)$. If the payoff is defined as

$$P(S = H, r, t) = a_1 S_O Q(r, t) \exp\left[b c_1(t)\right],$$

it is less than the firm value at the default barrier for all *b*. However in this paper, we consider the case of $b \ge 0$ to be more realistic.

[3] The detailed derivation is available upon request.

$$d_1 = \frac{-\ln l + \ln Q - c_1}{\sqrt{2c_1}} \qquad d_2 = \frac{-\ln q + \ln Q - c_1}{\sqrt{2c_1}}$$

$$d_3 = \frac{-\ln l + 2\ln q - \ln Q - (2b+1)c_1}{\sqrt{2c_1}} \qquad d_4 = \frac{\ln q - \ln Q - (2b+1)c_1}{\sqrt{2c_1}}$$

The credit spread C_s of a discount corporate bond price $P(S, r, T)$ with time to maturity T and face value F is given as

$$C_s(S,r,T) = -\frac{1}{T}\ln\frac{P(S,r,T)}{FQ(r,T)} \tag{10}$$

The term structures of credit spreads for a firm with l = 2.5 and q = 2.78 are illustrated in Figure 1 using different b from 0 to 1.5. Other parameters used in the calculations are s_S = 0.3, s_r = 0.02, r = –0.25, r = 4%, q = 6%, k = 0.2 and a_1 = a_2 = 0.8. The credit spreads increase with positive b. The levels of the default barrier with different b imply different early default risk. At the long end, the difference between the credit spreads for b = 0 and b = 1.5 is about 20bp which is significant compared with the credit spread of 42bp for b = 0. The numerical results show similar term structures obtained in previous studies, which match the empirical evidence[4]. The results also show that the variance of the corporate bond which is incorporated into the default barrier's dynamics has material impact on the default probability.

III. SUMMARY

This paper develops a corporate bond valuation model that incorporates a default barrier with dynamics depending on stochastic interest rates and the variance of the corporate bond function. Since the volatility of the firm value affects the level of the default barrier over time through the variance of the corporate bond function, more realistic default scenarios can be put into the valuation model. When the firm value touches the barrier, bondholders receive an exogenously specified number of riskless bonds. We derive a closed-form solution of the corporate bond price as a function of firm value and a short-term interest rate, with time-dependent model parameters governing the dynamics of the firm value and interest rate. The numerical results show that the drifted default barrier has material impact on the term structures of credit spreads.

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[4] See Ogden (1987), and Sarig and Warga (1989).

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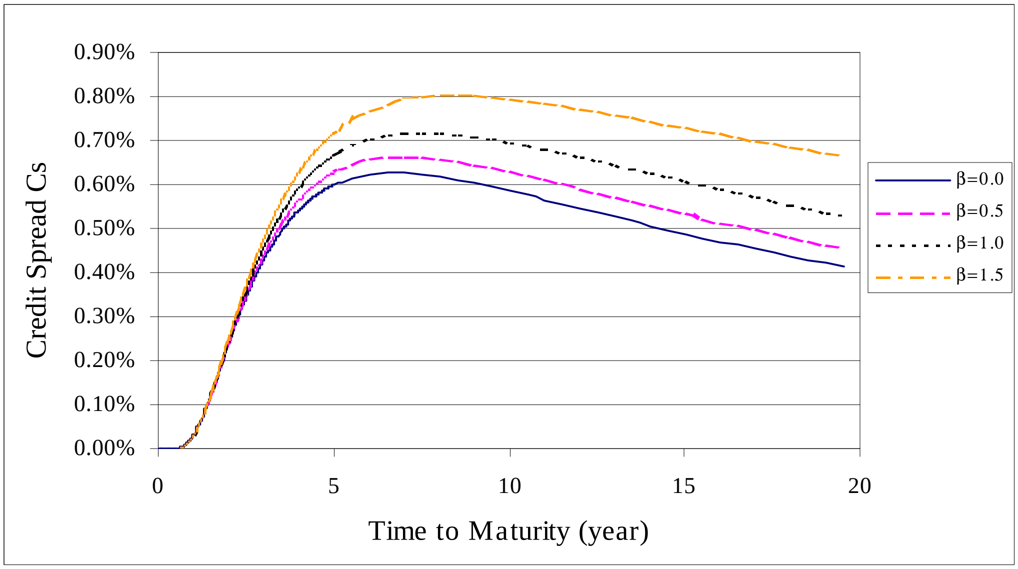

Figure 1. Credit spread as a function of time to maturity with l = 2.5, q = 2.78 and different b. The parameters used are s_S = 0.3, r = 4%, s_r = 0.02, $q = 6\%$, $k = 0.2$, $r = -0.25$ and $a_1 = a_2 = 0.8$.