Who Should Buy Long-Term Bonds?

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Abstract

According to conventional wisdom, long-term bonds are appropriate for long-term investors who value stability of income. We develop a model of optimal consumption and portfolio choice for infinitely-lived investors facing stochastic interest rates, solve it using an approximate analytical method, and evaluate the conventional wisdom. We show that the demand for long-term bonds has both a myopic component and an intertemporal hedging component. As risk aversion increases, the myopic component shrinks to zero but the hedging component does not. An infinitely risk-averse investor who is infinitely unwilling to substitute consumption intertemporally should hold a portfolio of long-term indexed bonds that is equivalent to an indexed perpetuity. This portfolio finances a riskless consumption stream and in this sense provides a stable income.

We calibrate our model to postwar US data and compare consumption and portfolio rules with and without bond indexation, portfolio constraints, and the possibility of investment in equities. We find that when indexed bonds are not available, inflation risk leads investors to shorten their bond portfolios and increase their precautionary savings. This has serious welfare costs for conservative investors, who are much better off when they have the opportunity to buy indexed bonds. We also find that the ratio of bonds to equities in the optimal portfolio increases with the coefficient of relative risk aversion, which is consistent with conventional portfolio advice but inconsistent with the mutual fund theorem of static portfolio analysis. Our results illustrate the general point that static portfolio choice models should not be used to study the dynamic problems facing long-term investors.

JEL classification: G12.

Keywords: Indexed bonds, intertemporal portfolio choice, long-term investors, mutual fund theorem, recursive utility, term structure of interest rates.

EXECUTIVE SUMMARY

According to conventional wisdom, long-term bonds are appropriate for long-term investors who value stability of income. We develop a model of optimal consumption and portfolio choice for infinitely-lived investors facing stochastic interest rates, solve it using an approximate analytical method, and evaluate the conventional wisdom.

We show that investors may hold long-term bonds for two reasons. First, if long-term bonds offer a term premium then investors may hold them for speculative purposes, to increase their expected portfolio return even at the cost of some extra short-term risk. This "myopic demand" for long-term bonds can be large when risk aversion is small, because long-term bonds have attractive Sharpe ratios. Second, long-term investors may hold long-term bonds for hedging purposes. Long-term bonds can finance a stable long-run consumption stream even in the face of time-varying short-term interest rates, and this is attractive to risk-averse long-term investors. In the extreme cases where there is no term premium, or where investors are infinitely risk-averse, the myopic demand for long-term bonds is zero and all bond demand is accounted for by the hedging demand.

We show that indexed bonds are particularly suitable for hedging purposes, because they do not impose extraneous inflation risk on long-term investors seeking a stable real consumption path. When long-term indexed bonds are available, an infinitely risk-averse long-term investor with zero intertemporal elasticity of substitution holds a bond portfolio that is equivalent to an indexed perpetuity. The indexed perpetuity is the riskless asset for a long-term investor, since it finances a constant consumption stream forever. When only nominal bonds are available, highly risk-averse investors shorten their bond portfolios in order to reduce their exposure to inflation risk. Less risk-averse investors hold long-term nominal bonds for speculative purposes if there is a positive inflation risk premium.

We extend our approach to solve the intertemporal portfolio choice problem imposing short-sale and borrowing constraints. This is possible because our solution takes the same form as the solution to a static portfolio choice problem for which standard mean-variance analysis is appropriate. Therefore we can solve our constrained problem using methods that have been developed to solve static problems with portfolio constraints.

Our constrained solution enables us to study the welfare effects of bond indexation in a realistic framework. When portfolio constraints are in place, and both nominal and indexed bonds are available to investors, more conservative investors hold in their portfolios relatively more indexed bonds than nominal bonds. These investors benefit substantially from the consumption insurance provided by long-term indexed bonds.

We also study the demand for bonds when equities are available as an alternative investment. We find that the ratio of bonds to stocks in the optimal portfolio increases with risk aversion, very much in line with popular investment advice but contrary to the mutual fund theorem of static portfolio analysis. However the demand for long-term

bonds is only large when these bonds are indexed, or when inflation uncertainty is low as it has been in the Volcker-Greenspan monetary policy regime since 1983.

Our analysis also has interesting implications for the design of pension plans and annuities. Our results suggest that conservative investors should favor indexed defined-benefit plans, while more risk-tolerant investors may be willing to accept some inflation or equity risk in their retirement income in exchange for higher average payments.

1 Introduction

Long-term bonds have been issued for centuries, and they remain extremely common financial instruments. It is natural to suppose that bonds have been popular because they meet the needs of an investor clientele. Investment advisers and financial journalists, for example, often say that bonds are appropriate for long-term investors who seek a stable income.

Curiously, modern financial economics has little to say about the demand for long-term bonds. In the early postwar period Hicks (1946), following Keynes (1930) and Lutz (1940), argued that investors would naturally prefer to hold short-term bonds and would only hold long-term bonds if compensated by a term premium. Modigliani and Sutch (1966) countered that some investors might have a preference for long-term bonds (a long-term "preferred habitat"), and such investors would require a premium to go short, not a premium to go long. However Modigliani and Sutch were vague about the characteristics of investors that would lead to a long-term preferred habitat. They took it as a given that some investors would desire stable wealth at a long rather than a short horizon.[2]

Since the 1960's there has been a vast increase in the sophistication of bond pricing models, but little further progress has been made in understanding the demand for long-term bonds. Recent authors, building on the seminal contributions of Vasicek (1977) and Cox, Ingersoll, and Ross (1985), have related term premia to the covariances of bond returns with an exogenously specified stochastic discount factor, but have not asked what bond portfolios are optimal for different types of investors.[3]

One reason for this gap in the literature may be that it is extremely hard to characterize optimal portfolio strategies for long-term investors. Samuelson (1969) and Merton (1969, 1971) obtained some explicit results under the assumption that real asset returns are independently and identically distributed over time; but this assumption implies that real interest rates are constant, so in the absence of inflation uncertainty—or with full indexation of bond payments to inflation—bond returns are nonrandom and all bonds are perfect substitutes for cash. Fischer (1975), Bodie, Kane, and McDonald (1985), and Viard (1993) have nonetheless used this assump-

[2] They wrote: "Suppose that a person has an n period habitat; that is, he has funds which he will not need for n periods and which, therefore, he intends to keep invested in bonds for n periods. If he invests in n period bonds, he will know exactly the outcome of his investments as measured by the terminal value of his wealth.... If, however, he stays short, his outcome is uncertain.... Thus, if he has risk aversion, he will prefer to stay long" unless compensated by a term premium (pp. 183–184).

[3] Campbell, Lo, and MacKinlay (1997), Dai and Singleton (1997), and Shiller (1990) review the recent bond pricing literature.

tion to study bond demand. In Fischer's model there is one nominal bond with a fixed nominal interest rate, and one indexed bond with a fixed real interest rate. The maturity of these bonds need not be specified, since bonds of all maturities are perfect substitutes for each other.[4] Bodie, Kane, and McDonald use historical data to estimate the variance-covariance matrix of real returns on nominal bonds, assuming that this matrix and mean real bond returns are constant over time. In their model random inflation allows imperfect substitutability among nominal bonds of different maturities, but constant real interest rates imply that long-term and short-term indexed bonds are perfect substitutes. Viard uses the same framework as Bodie, Kane, and McDonald and derives some further analytical results.

Merton (1969, 1971, 1973) studied the intertemporal portfolio choice problem with time-varying investment opportunities, introducing the important concept of intertemporal hedging demand for financial assets, but he did not obtain explicit solutions for portfolio weights. Recently a number of authors such as Balduzzi and Lynch (1997), Barberis (1998), Brandt (1998), and Brennan, Schwartz, and Lagnado (1996, 1997) have used numerical methods to solve particular long-run portfolio choice problems, while Kim and Omberg (1996) and Campbell and Viceira (1999) have derived some analytical results, but these papers generally concentrate on the choice between cash and equities rather than the demand for long-term bonds.

In this paper we study intertemporal portfolio choice in an environment with random real interest rates. We use an approximation technique developed in our earlier papers (Campbell 1993, Campbell and Viceira 1999) to replace the intractable portfolio choice problem with an approximate problem that can be solved using the method of undetermined coefficients. We use the approximate solution to understand the demand for long-term bonds.

We calibrate our model to historical data on the US term structure of interest rates, and report optimal portfolios for investors with a wide range of different attitudes towards risk and intertemporal substitution of consumption. In order to study the effects of inflation risk on optimal bond portfolios and investor welfare, we compare the solutions to our model when only indexed bonds are available with the solutions when only nominal, or both nominal and indexed bonds are available. We also allow

[4]Fischer also considers multiple goods whose relative prices may change; this allows him to introduce multiple indexed bonds, but the bonds are distinguished by the prices to which they are indexed, and not by maturity. Fischer recognizes that his assumptions may be problematic, concluding "It is possible that too little uncertainty about the returns from holding nominal bonds and equity over long periods is reflected in the basic model of the paper and that such uncertainty would result in portfolio holders being willing to pay a substantial premium for a long-term indexed bond" (p. 528). This paper explores Fischer's conjecture.

for borrowing and short-sales constraints, and for the possibility of investment in equities.

We begin by specifying a simple two-factor model of the term structure of interest rates, augmented to fit equity as well as bond returns. The two factors are the log real interest rate and the log expected rate of inflation. Each factor follows a normal first-order autoregressive (AR(1)) process with constant variance. This implies that log bond yields are linear in the factors and the model is in the tractable "affine yield" class (Dai and Singleton 1997, Duffie and Kan 1996). The model for the real term structure is a discrete-time version of Vasicek (1977), while the model for the nominal term structure is a discrete-time version of Langetieg (1980). Closely related models are discussed in Campbell, Lo, and MacKinlay (1977), Chapter 11.

Next we consider the portfolio choice problem for an infinitely-lived investor who has only financial wealth and must choose consumption and optimal portfolio weights in each period. Because the investor is infinitely-lived, she does not value stability of wealth at any unique horizon; rather she cares about the long-run properties of her consumption path. We assume that the investor's preferences are of the form suggested by Epstein and Zin (1989, 1991); the investor has constant relative risk aversion and constant intertemporal elasticity of substitution in consumption, but these parameters need not be related to one another. Epstein-Zin preferences nest the traditional power-utility specification in which relative risk aversion is the reciprocal of the intertemporal elasticity of substitution.

We show that the investor's demand for long-term bonds can be decomposed into a "myopic" demand and a "hedging" demand. Myopic demand depends positively on the term premium, and inversely on the variance of long-term bond returns and the investor's risk aversion. As risk aversion increases, myopic demand shrinks to zero. Hedging demand, on the other hand, is proportional to one minus the reciprocal of risk aversion. It is zero when risk aversion is one but accounts for all bond demand when risk aversion is infinitely large. We show that an infinitely risk-averse investor with zero intertemporal elasticity of substitution in consumption will choose an indexed bond portfolio that is equivalent to an indexed perpetuity, that is, a portfolio that delivers a riskless stream of real consumption. In this way we are able to support the commonsense view that long-term bonds are appropriate for long-lived investors who desire stability of income.

Our analysis delivers explicit solutions for portfolio weights, consumption rules, and investor welfare. We can compare investor behavior under alternative assumptions about the available menu of assets. We find that when indexed bonds are not available, inflation risk leads investors to shorten their bond portfolios and increase their precautionary savings. This has serious welfare costs for conservative investors, who are much better off when they have the opportunity to buy indexed bonds.

We also consider optimal portfolios when equities, as well as bonds, are available. We find that the ratio of bonds to equities in the optimal portfolio increases with the coefficient of relative risk aversion. As Canner, Mankiw, and Weil (1997) have pointed out, this is consistent with conventional portfolio advice but inconsistent with static mean-variance analysis. The static mean-variance model with a riskless one-period asset ("cash") predicts that all investors should hold a single mutual fund of risky assets; more conservative investors should increase the ratio of cash to the risky mutual fund, but should not change their relative holdings of risky assets. Our model helps to resolve the asset allocation puzzle identified by Canner, Mankiw, and Weil; more generally it underscores the dangers of using static portfolio choice theory to study the dynamic problems faced by long-term investors.

The organization of the paper is as follows. Section 2 presents the two-factor term structure model, and shows how it can be solved for bond prices at all maturities. Section 3 sets up the investor's intertemporal consumption and portfolio choice problem, explains our approximation to the problem, and discusses the approximate solution in the case where only indexed bonds are available. This section also explains the relation of our solution method to the approach of Cox and Huang (1989). Section 4 asks how things change when only nominal bonds, or both nominal and indexed bonds, are available. This section also shows how to impose borrowing and short-sales constraints. Section 5 considers the consumption and portfolio choice problem in the presence of equities, and section 6 concludes.

2 A Two-Factor Model of the Term Structure of Nominal Interest Rates

2.1 Specification of the model

Our focus in this paper is the microeconomic problem of portfolio choice for an individual investor facing exogenous bond returns. In order to generate empirically reasonable and theoretically well-specified bond returns, however, we start by writing down a general equilibrium bond pricing model. We consider a discrete-time, two-factor homoskedastic model that allows for non-zero correlation between innovations in the short-term real interest rate and innovations in expected inflation.

The real part of the model is determined by the stochastic discount factor (SDF) M_{t+1} that prices all assets in the economy. In a representative-agent framework the SDF can be related to the marginal utility of a representative investor, but here we simply use it as a device to generate a complete set of bond prices. We assume that M_{t+1} has the following lognormal structure, a discrete-time version of Vasicek (1977):

$$\begin{aligned} -m_{t+1} &= x_t + v_{m,t+1}, \\ x_{t+1} &= (1-\phi_x)\,\mu_x + \phi_x x_t + \varepsilon_{x,t+1}, \\ v_{m,t+1} &= \beta_{mx}\varepsilon_{x,t+1} + \varepsilon_{m,t+1}, \end{aligned} \tag{1}$$

where $m_{t+1} = \log(M_{t+1})$ and x_t, the one-period-ahead conditional expectation of m_{t+1}, follows an AR(1) process.

The nominal part of the model is also characterized by a lognormal, conditionally homoskedastic structure:

$$\begin{aligned} \pi_{t+1} &= z_t + v_{\pi,t+1}, \\ z_{t+1} &= (1-\phi_z)\,\mu_z + \phi_z z_t + v_{z,t+1}, \\ v_{z,t+1} &= \beta_{zx}\varepsilon_{x,t+1} + \beta_{zm}\varepsilon_{m,t+1} + \varepsilon_{z,t+1}, \\ v_{\pi,t+1} &= \beta_{\pi x}\varepsilon_{x,t+1} + \beta_{\pi m}\varepsilon_{m,t+1} + \beta_{\pi z}\varepsilon_{z,t+1} + \varepsilon_{\pi,t+1}, \end{aligned} \tag{2}$$

where π_{t+1} is the log inflation rate and z_t is the one-period-ahead conditional expectation of the inflation rate.

The system is subject to four normally distributed, white noise shocks $\varepsilon_{x,t+1}$, $\varepsilon_{m,t+1}$, $\varepsilon_{z,t+1}$, and $\varepsilon_{\pi,t+1}$ that determine the innovations to the log SDF, the log inflation rate, and their conditional means. These shocks are cross-sectionally uncorrelated, with variances σ_x^2, σ_m^2, σ_z^2, and σ_π^2. It is important to note that z_{t+1}, the expected inflation rate, is affected by both a pure expected-inflation shock $\varepsilon_{z,t+1}$ and the shocks to the expected and unexpected log SDF $\varepsilon_{x,t+1}$ and $\varepsilon_{m,t+1}$. That is, innovations to expected inflation can be correlated with innovations in the log SDF, and hence with innovations in the short-term real interest rate. These correlations mean that nominal interest rates need not move one-for-one with expected inflation—that is, the Fisher hypothesis need not hold—and nominal bond prices can include an inflation risk premium as well as a real term premium.

We have written the model with a self-contained real sector (1) and a nominal sector (2) that is affected by shocks to the real sector. But this is merely a matter of modelling convenience. Our model is a reduced form rather than a structural model, so it captures correlations among shocks to real and nominal interest rates but does not have anything to say about the true underlying sources of these shocks.

Campbell, Lo and MacKinlay (1997) note that $\varepsilon_{m,t+1}$ only affects the average level of the real term structure and not its average slope or time-series behavior. Accordingly, we can either drop it or identify its variance with an additional restriction. We follow the second approach and introduce equities in the model. We assume that the unexpected log excess return on equities is affected by shocks to both the expected and unexpected log SDF:

$$r_{e,t+1} - \mathrm{E}_t\, r_{e,t+1} = \beta_{ex}\varepsilon_{x,t+1} + \beta_{em}\varepsilon_{m,t+1}. \tag{3}$$

Campbell (1998) shows that this decomposition of the unexpected log equity return into a linear combination of the shocks to the expected and unexpected log SDF is consistent with a representative-agent endowment model where expected aggregate consumption growth follows an AR(1). From the fundamental pricing equation $1 = \mathrm{E}_t[M_{t+1}R_{t+1}]$ and the lognormal structure of the model it is easy to show that the risk premium on equities, over a one-period riskless return $r_{1,t+1}$, is given by

$$\begin{aligned} \mathrm{E}_t[r_{e,t+1} - r_{1,t+1}] + \frac{1}{2}\mathrm{Var}_t(r_{e,t+1} - r_{1,t+1}) &= \mathrm{Cov}_t(r_{e,t+1} - r_{1,t+1}, -m_{t+1}) \\ &= \beta_{mx}\beta_{ex}\sigma_x^2 + \beta_{em}\sigma_m^2. \end{aligned} \tag{4}$$

The variance term on the left hand side of (4) is a Jensen's Inequality correction that appears because we are working in logs, and the terms on the right hand side relate the risk premium on equities to the covariance of equity returns with innovations in the SDF. This specification implies that the equity premium, like all other risk premia in the model, is constant over time. Thus it ignores the time-variation in the equity premium that is the subject of our earlier paper on long-run portfolio choice (Campbell and Viceira 1999).

2.2 Pricing indexed bonds

Our model can price both indexed bonds and nominal bonds. In this section we show how to price indexed bonds, defined as zero-coupon bonds paying one unit of consumption at maturity and free of default risk.

Characterizing the stochastic discount factor is equivalent to characterizing the return on the one-period indexed bond, since $r_{1,t+1} = -\log \mathrm{E}_t[M_{t+1}]$. Because M_{t+1} is lognormal, we have that

$$\begin{aligned} r_{1,t+1} &= \mathrm{E}_t[-m_{t+1}] - \frac{1}{2}\mathrm{Var}_t[m_{t+1}] \\ &= x_t - \frac{1}{2}\left(\beta_{mx}^2\sigma_x^2 + \sigma_m^2\right). \end{aligned} \quad (5)$$

Therefore, our assumptions on m_{t+1} imply that the short term interest rate on indexed bonds is stochastic, though riskless one period in advance. It inherits the stochastic properties of x_{t+1}, and follows an AR(1) process with mean $\mu - (\beta_{mx}^2\sigma_x^2 + \sigma_m^2)/2$ and persistence ϕ.

Campbell, Lo and Mackinlay (1997), following Singleton (1990), Sun (1992), and Backus (1993), show that a lognormal, conditionally homoskedastic stochastic discount factor implies a pricing structure for log indexed bond yields which is affine in x_{t+1}. The log yield on an n-period bond, y_{nt}, times bond maturity n, which equals minus the log price of the bond, p_{nt}, is given by

$$n \cdot y_{nt} = -p_{nt} = A_n + B_n x_t, \quad (6)$$

where A_n and B_n are functions of bond maturity n but not of time t, and satisfy the following recursive equations:

$$\begin{aligned} B_n &= 1 + \phi_x B_{n-1} = \frac{1-\phi_x^n}{1-\phi_x}, \\ A_n - A_{n-1} &= (1-\phi_x)\,\mu_x B_{n-1} - \frac{1}{2}\left[(\beta_{mx} + B_{n-1})^2\sigma_x^2 + \sigma_m^2\right], \end{aligned} \quad (7)$$

and $A_0 = B_0 = 0$. An implication of (6) is that yields on indexed bonds of different maturities are perfectly correlated with each other.

The one-period log return on an n-period bond is by definition $(p_{n-1,t+1} - p_{n,t})$. Combining this expression with (6) and (7), the excess return over the one-period log interest rate is

$$r_{n,t+1} - r_{1,t+1} = -\frac{1}{2}B_{n-1}^2\sigma_x^2 - \beta_{mx}B_{n-1}\sigma_x^2 - B_{n-1}\varepsilon_{x,t+1}, \quad (8)$$

so the n-period bond is risky, with a risk premium given by

$$\begin{aligned} \mathrm{E}_t\left[r_{n,t+1} - r_{1,t+1}\right] + \frac{1}{2}\mathrm{Var}_t(r_{n,t+1} - r_{1,t+1}) &= \mathrm{Cov}_t(r_{n,t+1} - r_{1,t+1}, -m_{t+1}) \\ &= -\beta_{mx} B_{n-1}\sigma_x^2. \end{aligned} \tag{9}$$

The variance term on the left hand side of (9) is a Jensen's Inequality correction that appears because we are working in logs. The conditional covariance of the excess bond return with the log SDF determines the risk premium. In our homoskedastic model the conditional covariance is constant through time but dependent on the bond maturity; thus the expectations hypothesis of the term structure holds for indexed bonds. It is important to realize that constant risk premia do not imply constant investment opportunities because real interest rates are stochastic in our model.

Since $B_{n-1} > 0$, the Jensen's-inequality-corrected risk premium is negative if $\beta_{mx} > 0$, and positive otherwise. With positive β_{mx}, long-term indexed bonds pay off when the marginal utility of consumption for a representative investor is high, that is, when wealth is most desirable. In equilibrium, these bonds must have a negative risk premium. With negative β_{mx}, on the other hand, long-term indexed bonds pay off when the marginal utility of consumption for a representative investor is low, and so in equilibrium they have a positive risk premium.

Equations (8) and (9) imply that the Sharpe ratio for indexed bonds is $-\beta_{mx}\sigma_x$, which is independent of bond maturity. The invariance of the Sharpe ratio to bond maturity follows from the one-factor structure of the real sector of the model. The ratio of the risk premium to the variance of the excess return, which determines a myopic investor's allocation to long-term bonds, is $-\beta_{mx}/B_{n-1}$. This does depend on bond maturity but not on the volatility of the real interest rate.

2.3 Pricing nominal bonds

The pricing of default-free nominal bonds follows the same steps as the pricing of indexed bonds. The relevant stochastic discount factor to price nominal bonds is the nominal SDF $M_{t+1}^{\$}$, whose log is given by:

$$m_{t+1}^{\$} = m_{t+1} - \pi_{t+1}. \tag{10}$$

Since both M_{t+1} and Π_{t+1} are jointly lognormal and homoskedastic, $M_{t+1}^{\$}$ is also lognormal. The log nominal return on a one-period nominal bond is $r_{1,t+1}^{\$} = -\log \mathrm{E}_t[M_{t+1}^{\$}]$, which implies that $r_{1,t+1}^{\$}$ is a linear combination of the expected log real SDF and expected inflation given in the Appendix.

The log price of an n-period nominal bond, $p_{n,t}^{\$}$, also has an affine structure. It is a linear combination of x_t and z_t whose coefficients are time-invariant, though they vary with the maturity of the bond:

$$-p_{n,t}^{\$} = A_n^{\$} + B_{1,n}^{\$} x_t + B_{2,n}^{\$} z_t. \tag{11}$$

The Appendix gives expressions for the coefficients $A_n^{\$}$, $B_{1,n}^{\$}$ and $B_{2,n}^{\$}$.

Since nominal bond prices are driven by shocks to both real interest rates and inflation, they have a two-factor structure rather than the single-factor structure of indexed bond prices. Inflation affects the excess return on an n-period nominal bond over the one-period nominal interest rate, so risk premia in the nominal term structure include compensation for inflation risk. Like all other risk premia in the model, however, the risk premia on nominal bonds are constant over time; thus the expectations hypothesis holds for nominal as well as for real bonds.

2.4 The term structure of interest rates in the US

We estimate the two-factor term structure model using data on US nominal interest rates, equities and inflation. We use nominal zero-coupon yields at maturities 3 months, 1 year, 3 years, and 10 years from McCulloch and Kwon (1993), updated by Gong and Remolona (1996a,b). We take data on equities from the Indices files on the CRSP tapes. We use the value-weighted return, including dividends, on the NYSE, AMEX and NASDAQ markets. We take data on CPI inflation from the SBBI files on the CRSP tapes. Although the raw data are available monthly, we construct a quarterly data set in order to reduce the influence of high-frequency noise in inflation and short-term movements in interest rates.

To avoid the implication of the model that bond returns are driven by only two common factors, so that all bond returns can be perfectly explained by any two bond returns, we assume that bond yields are measured with error. The errors in yields are normally distributed, serially uncorrelated, and uncorrelated across bonds. Then the term structure model becomes a classic state-space model in which unobserved state variables x_t and z_t follow a linear process with normal innovations and we observe linear combinations of them with normal errors. The model can be estimated by maximum likelihood using a Kalman filter to construct the likelihood function (Berardi 1997, Harvey 1989, Pennacchi 1991, Gong and Remolona 1996a,b, Foresi, Penati, and Pennacchi 1997). This is an attractive alternative to the Generalized Method of Moments used to estimate term structure models by Gibbons and Ramaswamy (1993) and others.

In Table 1 we report parameter estimates for the period 1952-96 and the period 1983-96. Interest rates were unusually high and volatile in the 1979-82 period, during which the Federal Reserve Board under Paul Volcker was attempting to reestablish the credibility of anti-inflationary monetary policy and was experimenting with monetarist operating procedures. Many authors have argued that real interest rates and inflation have behaved differently in the monetary policy regime established since 1982 by Federal Reserve chairmen Volcker and Alan Greenspan (see for example Clarida, Gali, and Gertler 1998). Accordingly we report separate estimates for the period starting in 1983 in addition to the full sample period.

In earlier versions of this paper we reported completely unrestricted maximum likelihood estimates of the model. In 1952-96 these estimates fit the data well, but in 1983-96 the unrestricted estimates deliver implausibly low means for short-term nominal and real interest rates. (The model does not necessarily fit the sample means because the same parameters are used to fit both time-series and cross-sectional behavior; thus the model can trade off better fit elsewhere for worse fit of mean short-term interest rates.) Accordingly in this version of the paper we require that the model exactly fit the sample means of nominal interest rates and inflation. This restriction hardly reduces the likelihood at all in 1952-96, and even in 1983-96 it cannot be rejected at conventional significance levels.

The first two columns of Table 1 report parameters and asymptotic standard errors for the period 1952-96. All parameters are in natural units, so they are on a quarterly basis. We estimate a moderately persistent process for the real interest rate; the persistence coefficient ϕ_x is 0.87, implying a half-life for shocks to real interest rates of about 5 quarters. The expected inflation process is much more persistent, with a coefficient ϕ_z of 0.9985 that implies a half-life for expected inflation shocks of almost 115 years! Of course, the model also allows for transitory noise in realized inflation.

The bottom of Table 1 reports the implications of the estimated parameters for the means and standard deviations of real interest rates, nominal interest rates, and inflation, measured in percent per year. The implied mean log yield on an indexed three-month bill is 0.85% for the 1952-96 sample period. Taken together with the mean log yield on a nominal three-month bill of 5.31% and the mean log inflation rate of 3.99% (both restricted to equal the sample means over this period), and adjusting for Jensen's Inequality using one-half the conditional variance of log inflation, the implied inflation risk premium in a three-month nominal Treasury bill is 49 basis points. This fairly substantial risk premium is explained by the significant positive coefficient $\beta_{\pi x}$ and the significant negative coefficient $\beta_{\pi m}$ in Table 1.[5]

Risk premia on long-term indexed bonds, relative to a three-month indexed bill, are determined by the parameter β_{mx}. This is negative and highly significant, implying positive risk premia on long-term indexed bonds and an upward sloping term structure of real interest rates. Risk premia on nominal bonds, relative to indexed bonds, are determined by the inflation-risk parameters β_{zx} and β_{zm}. The former is positive but statistically insignificant, while the latter is negative and significant. Both point estimates imply positive inflation risk premia on nominal bonds relative to indexed bonds.

Table 2 explores the term-structure implications of our estimates in greater detail. The table compares implied and sample moments of term structure variables, measured in percent per year. Panel A of Table 2 reports sample moments for returns and yields on nominal bonds, together with the moments implied by our estimated model; panel B shows comparable implied moments for indexed bonds, and panel C reports sample and implied moments for equities. Row 1 of the table gives Jensen's-Inequality-corrected average excess returns on n-period nominal bonds over 1-period nominal bonds, while row 2 gives the standard deviations of these excess returns. Row 3 reports annualized Sharpe ratios for nominal bonds, the ratio of row 1 to row 2. Row 4 reports mean nominal yield spreads, row 5 reports the standard deviations of nominal yield spreads, and row 6 reports the standard deviations of changes in nominal yields. Rows 7 through 12 repeat these moments for indexed bonds. Note that the reported risk premia and Sharpe ratios for nominal and indexed bonds are not directly comparable because they are measured relative to different short-term assets, nominal and indexed respectively.

A comparison of the model implications in rows 1 and 7 shows that 10-year nominal bonds have a risk premium over three-month nominal bills of 2.06% per year, while 10-year indexed bonds have a risk premium over three-month indexed bills of 1.62% per year. These numbers, together with the 49-basis-point risk premium on three-month nominal bills over three-month indexed bills, imply a 10-year inflation risk premium (the risk premium on 10-year nominal bonds over 10-year indexed bonds) of slightly less than 1%. This estimate is consistent with the rough calculations in Campbell and Shiller (1996).

Rows 2 and 8 show that nominal bonds are much more volatile than indexed bonds; the difference in volatility increases with maturity, so that 10-year nominal bonds have

[5]This result is somewhat sensitive to the sample period. An earlier version of this paper found a smaller inflation risk premium at the short end of the term structure over the period 1952–79, consistent with the results of Foresi, Penati, and Pennacchi (1997).

a standard deviation three times greater than 10-year indexed bonds. This difference in volatility makes the Sharpe ratio for indexed bonds in row 9 considerably higher than the Sharpe ratio for nominal bonds in row 3. Since indexed bond returns are generated by a single-factor model, the Sharpe ratio for indexed bonds is independent of maturity at 0.46. The Sharpe ratio for nominal bonds declines with maturity; short-term nominal bonds have a ratio close to that for indexed bonds, but the Sharpe ratio for 10-year nominal bonds is only 0.20. These numbers imply that in our portfolio analysis, investors with low risk aversion will have a strong myopic demand for indexed bonds.

Table 2 can also be used to evaluate the empirical fit of the model. A comparison of the model's implied moments with the sample moments for nominal bonds shows that the model fits the volatility of excess nominal bond returns and changes in yields extremely well. It somewhat overstates the average excess nominal bond return and the nominal Sharpe ratio, but this can be attributed in part to the upward drift in interest rates over the 1952-96 sample period which biases downward these sample means. Overall the model appears to provide a good description of the nominal US term structure considering its parsimony and the fact that we have forced it to fit both time-series and cross-sectional features of the data.

Rows 13, 14 and 15 give comparable figures for equities: the annualized Jensen's-Inequality-corrected average excess returns on equities relative to nominal bills, the standard deviation of these excess returns, and their Sharpe ratio. The model fits the standard deviation of equities extremely well but overpredicts the equity premium and the Sharpe ratio for equities. With an implied Sharpe ratio of 0.55, investors with low risk aversion will have an extremely large myopic demand for equities.

The right hand sides of Tables 1 and 2 repeat these estimates for the Volcker-Greenspan period 1983–96. Many of the parameter estimates are quite similar; however we find that in this period real interest rates are much more persistent, with $\phi_x = 0.986$ and an implied half-life for real interest rate shocks of about 12 years. The expected inflation process now mean-reverts much more rapidly, with $\phi_z = 0.866$ implying a half-life for expected inflation shocks of about 5 quarters. These results are consistent with the notion that since the early 1980's the Federal Reserve has more aggressively controlled inflation at the cost of greater long-term variation in the real interest rate (Clarida, Gali, and Gertler 1998). The increase in real-interest-rate persistence increases the risk premia on indexed and nominal bonds, but it also greatly increases the volatility of indexed bond returns so the Sharpe ratio for indexed bonds is lower at 0.15. In the remainder of the paper we present portfolio choice results based on our full-sample estimates for the period 1952–96, but we also discuss results for the 1983–96 period where they are importantly different.

3 The Demand for Indexed Bonds

3.1 Assumptions on investor preferences

The investor is infinitely-lived, lives off her financial wealth and faces the investment environment described above. We assume that her preferences are described by the recursive utility proposed by Epstein and Zin (1989) and Weil (1989):

$$U\left(C_t, \mathrm{E}_t\, U_{t+1}\right) = \left\{(1-\delta)\, C_t^{\frac{1-\gamma}{\theta}} + \delta\left(\mathrm{E}_t\, U_{t+1}^{1-\gamma}\right)^{\frac{1}{\theta}}\right\}^{\frac{\theta}{1-\gamma}}, \tag{12}$$

where $\delta < 1$ is the discount factor, $\gamma > 0$ is the coefficient of relative risk aversion, $\psi > 0$ is the elasticity of intertemporal substitution, and $\theta \equiv (1-\gamma)/(1-1/\psi)$. The recursive utility function (12) reduces to the standard time-separable, power utility function with relative risk aversion γ when $\psi = \gamma^{-1}$, which implies $\theta = 1$.

Recursive preferences are useful because they allow us to separate the investor's attitude towards risk from her attitude towards intertemporal substitution of consumption over time. This separation is particularly important in our framework, where the short-term interest rate moves over time giving the investor an incentive to change her planned consumption growth rate.

3.2 Euler equations and the value function

The investor maximizes (12) subject to the intertemporal budget constraint

$$W_{t+1} = R_{p,t+1}\left(W_t - C_t\right), \tag{13}$$

where $R_{p,t+1}$ is the gross return at time $t+1$ on her portfolio at time t.

Epstein and Zin (1989, 1991) have shown that when the budget constraint is given by (13), the optimal portfolio and consumption policies must satisfy the following Euler equation for any asset i:

$$1 = \mathrm{E}_t\left[\left\{\delta\left(\frac{C_{t+1}}{C_t}\right)^{-\frac{1}{\psi}}\right\}^{\theta} R_{p,t+1}^{-(1-\theta)} R_{i,t+1}\right]. \tag{14}$$

When $i = p$, (14) reduces to:

$$1 = \mathrm{E}_t\left[\left\{\delta\left(\frac{C_{t+1}}{C_t}\right)^{-\frac{1}{\psi}} R_{p,t+1}\right\}^{\theta}\right]. \tag{15}$$

Dividing (12) by W_t and using the budget constraint we obtain the following expression for utility per unit of wealth:

$$V_t = \left\{ (1-\delta) \left(\frac{C_t}{W_t} \right)^{1-\frac{1}{\psi}} + \delta \left(1 - \frac{C_t}{W_t} \right)^{1-\frac{1}{\psi}} \left(\mathrm{E}_t \left[V_{t+1}^{1-\gamma} R_{p,t+1}^{1-\gamma} \right] \right)^{\frac{1}{\theta}} \right\}^{\frac{1}{1-\psi}}, \qquad (16)$$

where $V_t = U_t / W_t$. Epstein and Zin (1989, 1991) show that the value function per unit of wealth can be written as a power function of $(1-\delta)$ and the consumption-wealth ratio:

$$V_t = (1-\delta)^{-\frac{\psi}{1-\psi}} \left(\frac{C_t}{W_t} \right)^{\frac{1}{1-\psi}}. \qquad (17)$$

Two special cases are worth noting. First, as ψ approaches one, the exponents in (17) increase without limit. The value function has a finite limit, however, because the ratio C_t/W_t approaches $(1-\delta)$ as shown by Giovannini and Weil (1989). Second, as ψ approaches zero, V_t approaches C_t/W_t. A consumer who is extremely reluctant to substitute intertemporally consumes the annuity value of wealth each period, and this consumer's utility per dollar is the annuity value of the dollar.

3.3 Loglinear approximation of the model

At this point, to simplify the analysis we assume that there are only two bonds available to the investor, a one-period indexed bond and an n-period indexed bond. Given the one-factor structure of our model for indexed bonds, this is equivalent to providing the investor with a complete indexed term structure. Under this assumption, $R_{p,t+1}$ is equal to

$$R_{p,t+1} = \alpha_{n,t} \left(R_{n,t+1} - R_{1,t+1} \right) + R_{1,t+1}, \qquad (18)$$

where $\alpha_{n,t}$ is the fraction of the investor's savings allocated to the n-period indexed bond at time t.

In order to find optimal savings and the optimal allocations to the two bonds, we adopt an approximate analytical solution method. The first step is to characterize $\alpha_{n,t}$, the optimal allocation to the n-period bond, by combining a second-order loglinear approximation to the Euler equation with a first-order approximation to the intertemporal budget constraint. We then guess a form for the optimal consumption and portfolio policies and show that policies of this form satisfy the approximate Euler equation and budget constraint. Finally we use the method of undetermined coefficients to identify the coefficients of the optimal policies from the primitive parameters of the model. By using a second-order expansion of the log Euler equation we account for second-moment effects in the model.

Following Campbell (1993, 1996), Campbell and Viceira (1996), and Restoy (1992), we first log-linearize the Euler equation (14) for $i = n$ and $i = 1$, where asset 1 is the short-term riskless asset. Subtracting the log-linearized Euler equation for the riskless asset from the log-linearized equation for asset n, we find:

$$\begin{aligned} \mathrm{E}_t\left[r_{n,t+1} - r_{1,t+1}\right] + \frac{1}{2}\mathrm{Var}_t\left(r_{n,t+1}\right) &= -\left(\frac{1-\gamma}{1-\psi}\right)\mathrm{Cov}_t\left(\Delta c_{t+1}, r_{n,t+1}\right) \\ &\quad + \left(\frac{1-\psi\gamma}{1-\psi}\right)\mathrm{Cov}_t\left(r_{p,t+1}, r_{n,t+1}\right), \end{aligned} \tag{19}$$

where lowercase letters denote variables in logs and Δ is the first-difference operator. This expression obtains from (14) by using both a second-order Taylor approximation around the conditional mean of $\{r_{p,t+1}, \Delta c_{t+1}\}$ and the approximation $\log(1+x) \approx x$ for small x. It holds exactly if consumption growth and the return on wealth have a joint conditional lognormal distribution. We show later that this is indeed the case along the optimal path: the approximate optimal policies imply that the log return on wealth and log consumption growth are jointly normal.

We can log-linearize (15) in a similar fashion. After reordering terms, we obtain the well-known equilibrium linear relationship between expected log consumption growth and the expected log return on wealth:

$$\mathrm{E}_t\Delta c_{t+1} = \psi \log\delta + v_{p,t} + \psi \mathrm{E}_t r_{p,t+1}, \tag{20}$$

where the term $v_{p,t}$ is an intercept proportional to the conditional variance of log consumption growth in relation to log portfolio returns:

$$v_{p,t} = -\frac{1}{2}\left(\frac{1-\gamma}{1-\psi}\right)\mathrm{Var}_t\left(\Delta c_{t+1} - \psi r_{p,t+1}\right). \tag{21}$$

In general this intercept is time-varying, but in our model it becomes a constant. These equations, like (19), hold exactly if consumption and asset returns are jointly conditionally lognormal.

Taking the return on wealth as given, we can also log-linearize the intertemporal budget constraint (13) around the mean log consumption-wealth ratio:

$$\Delta w_{t+1} \approx r_{p,t+1} + \left(1 - \frac{1}{\rho}\right)(c_t - w_t) + k, \tag{22}$$

where $k = \log(\rho) + (1-\rho)\log(1-\rho)/\rho$, and $\rho = 1 - \exp\{\mathrm{E}(c_t - w_t)\}$ is a log-linearization parameter. Note that ρ is endogenous in that it depends on the average log consumption-wealth ratio which is unknown until the model has been solved. Campbell (1993) and Campbell and Koo (1997) have shown that the approximation (22) holds exactly if the consumption-wealth ratio is constant over time.

Finally, equation (18) allows us to approximate $r_{p,t+1}$, the log return on wealth, as follows:

$$r_{p,t+1} = \alpha_{n,t}\left(r_{n,t+1} - r_{1,t+1}\right) + r_{1,t+1} + \frac{1}{2}\alpha_{n,t}\left(1 - \alpha_{n,t}\right)\mathrm{Var}_t\left(r_{n,t+1}\right), \tag{23}$$

which is a discrete-time version of the log return on wealth in continuous time, where Ito's Lemma can be applied to equation (18).

Combining the trivial equality

$$\Delta c_{t+1} = \left(c_{t+1} - w_{t+1}\right) - \left(c_t - w_t\right) + \Delta w_{t+1} \tag{24}$$

with equations (19), (22), and (23) and the definition of θ we find that

$$\begin{aligned} \alpha_{n,t} &= \frac{1}{\gamma}\frac{\mathrm{E}_t\left[r_{n,t+1} - r_{1,t+1}\right] + \frac{1}{2}\mathrm{Var}_t\left(r_{n,t+1}\right)}{\mathrm{Var}_t\left(r_{n,t+1}\right)} \\ &\quad - \left(\frac{1}{1-\psi}\right)\left(\frac{\gamma-1}{\gamma}\right)\frac{\mathrm{Cov}_t\left(r_{n,t+1}, c_{t+1} - w_{t+1}\right)}{\mathrm{Var}_t\left(r_{n,t+1}\right)}. \end{aligned} \tag{25}$$

This equation was first derived by Restoy (1992). The first term is the myopic component of asset demand; it is proportional to the risk premium on the n-period bond and the reciprocal of the coefficient of relative risk aversion. The second term is Merton's (1969, 1971, 1973) intertemporal hedging demand. It reflects the strategic behavior of the investor who wishes to hedge against future adverse changes in investment opportunities, as summarized by the consumption-wealth ratio. In our setup the investment opportunity set is time-varying because interest rates are time-varying (although expected excess returns are constant); accordingly the investor may want to hedge her consumption against adverse changes in interest rates. Intertemporal hedging demand is zero when risk aversion $\gamma = 1$, but as γ increases myopic demand shrinks to zero and hedging demand does not. In the limit as γ becomes arbitrarily large, hedging demand accounts for all the demand for the risky asset.

An important special case arises when the elasticity of intertemporal substitution is unity. As $\psi \to 1$, the log consumption-wealth ratio becomes constant so the covariance of asset returns with this ratio approaches zero. However the covariance is divided by $1 - \psi$, which also approaches zero. Giovannini and Weil (1989), by taking appropriate limits, have shown that portfolio choice is not myopic in this case even though the consumption-wealth ratio is constant. The solution presented in this paper is exact for the case $\psi = 1$.

3.4 An explicit solution

Equation (25) is recursive in the sense that it relates current portfolio decisions to future consumption and portfolio decisions. In order to get a complete solution to the model we need to derive consumption and portfolio rules that depend only on current state variables. We do this by guessing that the consumption function takes the form

$$c_t - w_t = b_0 + b_1 x_t. \tag{26}$$

Calculations summarized in the Appendix verify this guess and show that the coefficients are given by

$$\begin{aligned} b_0 &= \frac{\rho}{1-\rho}\left[\frac{(1-\psi)(1-\gamma)}{2\gamma}\left(\beta_{mx} + \frac{\rho}{1-\rho\phi_x}\right)^2 \sigma_x^2 - \frac{1}{2}(1-\psi)\,\sigma_m^2\right. \\ &\qquad \left. -\psi\log\delta + k + \mu(1-\phi)\frac{\rho(1-\psi)}{1-\rho\phi_x}\right] \end{aligned} \tag{27}$$

and

$$b_1 = (1-\psi)\frac{\rho}{1-\rho\phi_x}. \tag{28}$$

In addition, the optimal portfolio share in the risky asset is constant over time and can be written as

$$\alpha_{nt} = \alpha_n = \frac{-1}{\gamma B_{n-1}}\left[\beta_{mx} + (1-\gamma)\frac{\rho}{1-\rho\phi_x}\right]. \tag{29}$$

These solutions are analytical, given the log-linearization parameter ρ. But ρ itself is a nonlinear function of the coefficients b_0 and b_1, since $\rho = 1 - \exp\{\mathrm{E}[c_t - w_t]\} = 1 - \exp\{b_0 + b_1\mu_x\}$. Equations (27), (28), and the expression for ρ define implicitly a nonlinear mapping of ρ onto itself which has an analytical solution only in the case $\psi = 1$, when $\rho = \delta$. In all other cases we solve for ρ numerically using a simple recursive algorithm. We set ρ to some initial value (typically $\rho = \delta$) and compute the coefficients of the optimal policies; given these coefficients we compute a new value for ρ, from which we obtain a new set of coefficients, and so forth. We continue until the difference between two consecutive values of ρ is less than 10^{-4}. This recursion converges very rapidly to a number between zero and one in cases where the value function of the model is finite; there are some cases, however, in which ρ is driven to zero or one because the value function is infinitely positive or negative and the infinite-horizon optimization problem is not well defined. We discuss these cases further below.

Equations (26) and (28) show that the log consumption-wealth ratio is linear in the short-term real interest rate (since x_t is linearly related to $r_{1,t+1}$). The response of consumption to the interest rate depends on the investor's elasticity of intertemporal substitution, but does not depend directly on her relative risk aversion. The risk aversion coefficient affects the dynamic behavior of consumption only indirectly through its effect on the log-linearization parameter ρ. Below we show that this indirect effect is quantitatively negligible.

The log consumption-wealth ratio is constant only when $\psi = 1$. In this case $c_t - w_t$ equals $\log(1 - \delta)$. For this reason investors with unit elasticity of intertemporal substitution are called "myopic consumers." Since $0 < \rho < 1$ and $|\phi| < 1$, the consumption-wealth ratio increases with the interest rate if $\psi < 1$ and falls with the interest rate otherwise. An increase in the short-term real interest rate is equivalent to an improvement in the investment opportunity set, and it has both income and substitution effects. An investor with low ψ is reluctant to substitute intertemporally, and for her the income effect dominates, leading her to increase her consumption relative to her wealth. This increase in consumption is larger, the more persistent is the improvement in investment opportunities—the closer is ϕ to one. Conversely, the substitution effect dominates for an investor with high $\psi > 1$. This investor will reduce present consumption when the interest rate increases, and will do so more aggressively when the interest rate process is persistent.

Equation (29) shows that the optimal portfolio allocation to the long-term bond is constant over time and independent of the level of the short-term interest rate. The portfolio allocation depends on the bond maturity, on the persistence of the short-term interest rate, and on the investor's relative risk aversion, but does not depend directly on her elasticity of intertemporal substitution. The elasticity of intertemporal substitution affects portfolio choice only indirectly through its effect on the log-linearization parameter ρ, and we show below that this indirect effect is quantitatively negligible.

The first term inside the brackets in (29) represents the myopic demand for long-term bonds, while the second term inside the brackets represents the intertemporal hedging demand. The myopic demand depends on the parameter β_{mx} which determines the term premium; it is zero if $\beta_{mx} = 0$, and it shrinks as risk aversion γ increases. The intertemporal hedging demand for bonds is zero when $\gamma = 1$. That is, the long-term bond demand of investors with unit relative risk aversion coefficient is driven exclusively by the risk premium. For this reason they are called "myopic investors."

Hedging demand is negative for investors with $\gamma < 1$; these investors prefer to hold assets that pay off when investment opportunities are good, so they "reverse hedge" the risk of adverse shifts in investment opportunities. As risk aversion γ increases, the hedging demand increases and becomes positive when $\gamma > 1$. Meanwhile the myopic demand for bonds shrinks, so the hedging demand for bonds increases relative to the myopic demand; in section 3.7 we discuss what happens in the limit as the investor becomes infinitely risk averse.

3.5 Implications of complete markets

We have allowed the investor to form a portfolio from only two assets, a short-term indexed bond and a single long-term indexed bond. Even with only two assets, however, markets are complete with respect to real-interest-rate risk because our real term-structure model has only one factor. This fact has several interesting implications.

First, with complete markets the investor can combine short- and long-term bonds so that the return on her bond portfolio is independent of the maturity of the long-term bond traded in the market. That is, she can synthesize her own optimal long-term bond, with the maturity optimal for her given her risk preferences. The return on the optimal bond portfolio is given by

$$\begin{aligned} r_{p,t+1} &= -\left(\frac{1}{2}\left(\alpha_n B_{n-1}\right)^2 + \alpha_n B_{n-1}\beta_{mx}\right)\sigma_x^2 - \frac{1}{2}\left(\beta_{mx}^2\sigma_x^2 + \sigma_m^2\right) \\ &\quad + x_t - \alpha_n B_{n-1}\varepsilon_{x,t+1}, \end{aligned} \tag{30}$$

and only the product $\alpha_n B_{n-1}$ enters this expression. Our portfolio solution (29) implies that $\alpha_n B_{n-1}$ does not depend on n.

Second, if real-interest-rate variation is the only source of risk, then markets are complete with respect to all sources of risk. We can explore this case by setting $\sigma_m^2 = 0$ so that $\varepsilon_{m,t+1}$ drops from the definition of m_{t+1} in (1). In this case the SDF is unique. Since the intertemporal marginal rate of substitution (IMRS) of any investor can be used as a valid SDF, it follows that all investors must have the same IMRS which must equal the SDF we specified exogenously for our term structure model. This provides a check on the internal consistency of our solution. Using (14) to express the investor's IMRS as a function of consumption growth and the portfolio return, we must have

$$IMRS_{t+1} = \left\{\delta\left(\frac{C_{t+1}}{C_t}\right)^{-\frac{1}{\psi}}\right\}^{\theta} R_{p,t+1}^{-(1-\theta)} = M_{t+1}. \tag{31}$$

Taking logs and using our solution, it is straightforward to show that

$$\begin{aligned}\log\left(IMRS_{t+1}\right) &= \theta\log\delta - \frac{\theta}{\psi}\Delta c_{t+1} - (1-\theta)\, r_{p,t+1} \\ &= -x_t - \beta_{mx}\varepsilon_{x,t+1} = m_{t+1},\end{aligned} \tag{32}$$

which is the required result.

Third, Cox and Huang (1989) have proposed an alternative solution method for intertemporal consumption and portfolio choice problems with complete markets. They work in continuous time and show that with complete markets, optimally invested wealth must satisfy a partial differential equation (PDE). Unfortunately this PDE does not generally have a closed-form solution. We now show that our solution methodology is equivalent to a discrete-time version of the Cox-Huang approach; our loglinear approximation allows us to solve the discrete-time equivalent of the Cox-Huang PDE in closed form.

To keep the analysis simple, we will specialize the discussion to the power utility case ($\psi = 1/\gamma$). The extension to recursive utility is discussed in the Appendix. We start by defining a new variable $W_t^* = W_t - C_t$—invested wealth—and note that from the budget constraint (13), the portfolio return equals $R_{p,t+1} = (W_{t+1}^* + C_{t+1})/W_t^*$. Then the Euler equation (15) implies that the ratio W_t^*/C_t must satisfy the following equation:

$$\frac{W_t^*}{C_t} = \mathrm{E}_t\left[\left(1 + \frac{W_{t+1}^*}{C_{t+1}}\right)\left(\frac{C_{t+1}}{C_t}\right)\delta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\right], \tag{33}$$

where $\delta(C_{t+1}/C_t)^{-\gamma}$ is the investor's IMRS.

Equation (33) defines an expectational difference equation for W_t^*/C_t that depends on endogenous consumption growth. However, if markets are complete, the equality of IMRS and SDF implies that $(C_{t+1}/C_t) = \delta^{1/\gamma}M_{t+1}^{-1/\gamma}$. Substituting into (33), we can derive an expectational difference equation for W_t^*/C_t that contains only the exogenous SDF and not endogenous consumption growth:

$$\frac{W_t^*}{C_t} = \mathrm{E}_t\left[\left(1 + \frac{W_{t+1}^*}{C_{t+1}}\right)\delta^{\frac{1}{\gamma}}M_{t+1}^{1-\frac{1}{\gamma}}\right]. \tag{34}$$

This nonlinear expectational difference equation is the discrete-time equivalent of the Cox-Huang PDE.

Equation (34) does not generally have a closed form solution, so it must be solved numerically or using an analytical approximation method. We can apply the same approximation that we have already used. Taking logs on both sides of (34) and using the same approximation around the mean log consumption-wealth ratio that we use to loglinearize the budget constraint[6], we can write (34) in log form as

$$\begin{aligned} c_t - w_t &= \rho[k - \frac{1}{\gamma}\log\delta - \left(1-\frac{1}{\gamma}\right)\mathrm{E}_t\, m_{t+1} + \mathrm{E}_t\,(c_{t+1} - w_{t+1}) \\ &\quad -\frac{1}{2}\,\mathrm{Var}_t\left(\left(1-\frac{1}{\gamma}\right)m_{t+1} + (c_{t+1} - w_{t+1})\right)] \end{aligned} \tag{35}$$

which is linear. It is trivial to show that this equation has the same solution that we have already derived.

3.6 Empirical properties of the solution

Tables 3–5 explore the properties of our solution using the bond-pricing parameters estimated in Table 1 for the period 1952–96. We compute optimal portfolio and consumption rules for investors with the same time discount rate (4%) but different coefficients of relative risk aversion and elasticities of intertemporal substitution. We consider risk aversion coefficients of 0.75, 1, 2, 5, 10, and 5000 (effectively almost infinite), and elasticities of intertemporal substitution that are the reciprocals of these values. The tables are organized so that very risk-averse investors are at the bottom, investors who are very reluctant to substitute intertemporally are at the right, and power-utility investors (for whom the elasticity of intertemporal substitution is the reciprocal of risk aversion) are along the main diagonal. The top panel of each table assumes that the bonds available to investors are one-quarter and ten-year zero-coupon indexed bonds.

The top panel of Table 3 reports the percentage portfolio share of a ten-year zero-coupon indexed bond. Since indexed bonds have attractive Sharpe ratios, we find that investors with low risk aversion have a very large myopic demand for long-term indexed bonds; they want to invest many times their total wealth in these bonds and borrow at the short-term riskless interest rate. As risk aversion increases, the demand for indexed bonds gradually declines, but it does not go to zero because highly

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[6] The approximation is

$$\begin{aligned} \log\left(\frac{W_t^*}{C_t}\right) &= \log\left(\frac{W_t}{C_t}\left(1-\frac{C_t}{W_t}\right)\right) = -\left(c_t - w_t\right) + \log\left(1-\frac{C_t}{W_t}\right) \\ &\approx -\left(c_t - w_t\right) + k + \left(1-\frac{1}{\rho}\right)\left(c_t - w_t\right) \; = k - \frac{1}{\rho}\left(c_t - w_t\right), \end{aligned}$$

where ρ and k have been defined in (22).

risk averse investors have a positive intertemporal hedging demand for long-term indexed bonds. Table 4 clarifies this point by reporting the share of intertemporal hedging demand in the total demand for long-term bonds. This share rises from zero when $\gamma = 1$ to 99.7% when $\gamma = 5000$. Note also that different columns of these tables, corresponding to different elasticities of intertemporal substitution, are almost identical. This confirms our theoretical claim that the elasticity of intertemporal substitution, operating only indirectly through the log-linearization parameter ρ, has a negligible effect on portfolio allocation.

Table 5 summarizes the optimal consumption behavior that is associated with these portfolio rules. The left hand side of the table shows the average consumption-wealth ratio, while the right hand side shows the standard deviation of optimal consumption growth. To understand the patterns of average consumption-wealth ratios, recall that an investor with zero elasticity of intertemporal substitution consumes the annuity value of wealth each period, so the average consumption-wealth ratio for this investor is just the average expected return on the portfolio. This average return declines with risk aversion, and so the average consumption-wealth ratio also declines with risk aversion as shown in the 1/5000 column. Investors with higher elasticities of consumption, shown to the left of the 1/5000 column, are willing to substitute intertemporally in response to incentives. The direction of the substitution depends on the average return on the portfolio in relation to the time discount rate and the risk of the portfolio. Investors with low risk aversion (at the top of the panel) have high average portfolio returns so they substitute by reducing present consumption, while investors with high risk aversion (at the bottom of the panel) have low average portfolio returns so they substitute by increasing present consumption. The magnitude of these effects is such that all investors with unit elasticity of substitution have the same average consumption-wealth ratio of $(1-\delta)$, regardless of their risk aversion.

The accuracy of our loglinear approximation to the intertemporal budget constraint depends on the volatility of the log consumption-wealth ratio. We do not report this volatility in Table 5, but it is zero for $\psi = 1$ (the case where our approximation holds exactly) and is roughly proportional to $(\psi - 1)$. (It would be exactly proportional if the log-linearization parameter ρ were fixed.) The maximum standard deviation of the log consumption-wealth ratio is about 3% for ψ close to zero at the far right of the table. These numbers suggest that our approximation should be extremely accurate for a term-structure model of the sort we have estimated in 1952–96. Campbell and Koo (1997) use numerical methods to solve a model with an exogenous portfolio return that follows an AR(1) process like the endogenous portfolio return in our model; they find that approximation error is very small whenever the standard deviation of the log consumption-wealth ratio is 5% or below.

The right hand part of Table 5 illustrates some interesting patterns in the conditional volatility of consumption growth. Investors with low risk aversion hold leveraged bond portfolios that given them highly volatile consumption, regardless of their intertemporal elasticity of substitution in consumption. Conservative investors hold indexed bonds for hedging purposes. Investors who are both highly risk-averse and highly reluctant to substitute consumption intertemporally reduce the conditional volatility of their consumption growth to zero; highly risk-averse investors who are willing to substitute intertemporally, however, respond to interest rate movements by adjusting their consumption, so their conditional consumption volatility is positive. We now explore in more detail the behavior of highly risk-averse investors.

3.7 On the economic definition of the riskless asset

In financial economics a one-period indexed bond is usually thought of as riskless. Over one period, a nominal bond is a good substitute for an indexed bond (Viard 1993), and thus by extension the riskless asset is often identified with a short-term nominal asset such as a Treasury bill. In a world with time-varying interest rates, however, only the current short-term real interest rate is riskless; future short-term real interest rates are uncertain. This makes a one-period bond risky from the perspective of long-horizon investors. For such investors, a more natural definition of a riskless asset might be a real perpetuity, since this asset pays a fixed coupon of one unit of consumption per period forever.

We now show that in our model an individual who is infinitely risk-averse and infinitely reluctant to substitute consumption intertemporally chooses a portfolio of indexed bonds that is equivalent to a real perpetuity. That is, if a real perpetuity were available, the portfolio would be fully invested in that bond. To see this, we first note that, from (30) and (29), the interest-rate sensitivity of the the optimal portfolio for an infinitely risk-averse individual is given by

$$\lim_{\gamma\to\infty}\frac{\partial r_{p,t+1}}{\partial u_{x,t+1}}=\lim_{\gamma\to\infty}\left(-\alpha_n B_{n-1}\right)=-\frac{\rho}{1-\rho\phi_x}. \tag{36}$$

A real perpetuity pays a fixed coupon of one unit of consumption per period forever. The log coupon on the bond is therefore $d_{c,t}=0\ \forall t$. Campbell, Lo and MacKinlay (1997, p. 408), following Shiller (1979), show that a log-linear approximation to the log yield on a real perpetuity is

$$y_{c,t}\approx(1-\rho_c)\,\mathrm{E}_t\sum_{j=0}^{\infty}\rho_c^j r_{c,t+1+j}, \tag{37}$$

where $y_{c,t}$ is the log yield on the real perpetuity, $r_{c,t+1}$ is the log return on the perpetuity, and ρ_c is a log-linearization constant equal to $\rho = 1 - \exp\{\mathrm{E}(-p_{c,t})\}$, where $p_{c,t}$ is the log "cum-dividend" price of the perpetuity including its current coupon.[7]

The return on the real perpetuity must verify the pricing relation $1 = \mathrm{E}_t[M_{t+1}R_{c,t+1}]$. Assuming that $R_{c,t+1}$ is lognormally distributed, we must have that

$$\begin{aligned} \mathrm{E}_t\left[r_{c,t+1}\right] &= x_t - \frac{1}{2}\mathrm{Var}_t\left(m_{t+1}\right) - \frac{1}{2}\mathrm{Var}_t\left(r_{c,t+1}\right) + \mathrm{Cov}_t\left(m_{t+1}, r_{c,t+1}\right) \\ &= x_t + \omega_c^2, \end{aligned} \tag{38}$$

where ω_c^2 is a positive constant. Equations (37) and (38) imply that

$$y_{c,t} \approx \omega_c^2 + \left(1 - \frac{1-\rho_c}{1-\rho_c\phi_x}\right)\mu_x + \left(\frac{1-\rho_c}{1-\rho_c\phi_x}\right)x_t. \tag{39}$$

But Campbell, Lo and MacKinlay (1997, p. 408, eq. 10.1.19) also show that

$$\begin{aligned} r_{c,t+1} &\approx \left(\frac{1}{1-\rho_c}\right)y_{c,t} - \left(\frac{\rho_c}{1-\rho_c}\right)y_{c,t+1} \qquad (40) \\ &= \omega_c^2 + \left(\frac{\rho_c}{1-\rho_c\phi_x}\right)\left(1-\phi_x\right)\mu_x + \left(\frac{1}{1-\rho_c\phi_x}\right)x_t - \left(\frac{\rho_c}{1-\rho_c\phi_x}\right)x_{t+1}, \end{aligned}$$

which in turn implies that the interest-rate sensitivity of a real perpetuity is given by

$$\frac{\partial r_{c,t+1}}{\partial u_{x,t+1}} = -\frac{\rho_c}{1-\rho_c\phi_x}. \tag{41}$$

Equations (36) and (41) differ only by the log-linearization constants ρ and ρ_c. These two constants are the same for an individual who is infinitely reluctant to substitute consumption intertemporally ($\psi = 0$). Such an individual consumes the annuity value of wealth, the consumption stream that can be sustained indefinitely by the initial level of wealth. But the annuity value of a real perpetuity is just its dividend of one. Thus for this investor $C/W = 1/P_c$, which implies $\mathrm{E}[c - w] = \mathrm{E}[-p_c]$, and thus, from the definitions of the log-linearization parameters, $\rho = \rho_c$. The infinitely risk-averse investor who is infinitely reluctant to substitute intertemporally holds a

[7] Campbell, Lo, and MacKinlay give an alternative definition of ρ_c in relation to the "ex-dividend" price of the consol excluding its current coupon. This is more natural in a bond pricing context, but less convenient here because the form of the budget constraint (13) implies that we are measuring wealth inclusive of current consumption, that is, on a "cum-dividend" basis.

real perpetuity that finances a riskless consumption stream over the infinite future. In this sense a real perpetuity rather than a one-period indexed bond is a riskless asset for a long-horizon investor.

Our empirical results in Table 3 illustrate these findings. Given the parameters of the real term-structure model we have estimated, a 10-year indexed zero-coupon bond has a very slightly greater interest-rate sensitivity than an indexed perpetuity. Accordingly an investor with risk aversion of 5000 (effectively infinite) and elasticity of intertemporal substitution of 1/5000 (effectively zero) holds 96% of her wealth in 10-year indexed zero-coupon bonds, and 4% in indexed bills, creating a portfolio that is equivalent to an indexed perpetuity.

4 The Demand for Nominal Bonds

4.1 Unconstrained demand

The results in the previous section can easily be generalized to the case where the investor can hold only nominal bonds, or both nominal and indexed bonds. We can assume that the short-term asset is indexed or nominal, or allow both types of short-term asset. For realism, however, and since inflation risk is modest at the short end of the term structure, we now assume that the one-period asset is nominal.

Even in the presence of nominal assets, the log consumption-wealth ratio still depends only on the state variable x_t, and not on expected inflation z_t. Furthermore this ratio is still a linear function of x_t, and the slope coefficient is still given by $b_1 = \rho(1-\psi)/(1-\rho\phi)$ as in equation (28). The menu of available assets affects this coefficient only indirectly by affecting the intercept b_0 of the consumption function, which in turn determines the log-linearization parameter ρ.

We write the vector of allocations to long-term (nominal and indexed) bonds as $\boldsymbol{\alpha}$. Then we have

$$\boldsymbol{\alpha} = \frac{1}{\gamma}\boldsymbol{\Sigma}^{-1}\mathbf{a}, \tag{42}$$

where $\boldsymbol{\Sigma}$ is the variance-covariance matrix of excess bond returns over the short-term asset and

$$\mathbf{a} = \mathbf{m} + \gamma\mathbf{p} + \frac{1-\gamma}{1-\psi}\mathbf{h}. \tag{43}$$

Here $\mathbf{m}$ is a vector of Jensen's-Inequality-corrected mean excess returns whose ith element is

$$m_i = \mathrm{E}_t[r_{i,t+1} - (r^{\$}_{1,t+1} - \pi_{t+1})] + \frac{1}{2}\mathrm{Var}_t[r_{i,t+1} - (r^{\$}_{1,t+1} - \pi_{t+1})], \tag{44}$$

$\mathbf{p}$ is a vector of conditional covariances with the real return on the short-term bond whose ith element is

$$p_i = \mathrm{Cov}_t[r_{i,t+1} - (r^{\$}_{1,t+1} - \pi_{t+1}), r^{\$}_{1,t+1} - \pi_{t+1}], \tag{45}$$

and $\mathbf{h}$ is a vector of conditional covariances with the consumption-wealth ratio whose ith element is

$$\begin{aligned} h_i &= \mathrm{Cov}_t[r_{i,t+1} - (r^{\$}_{1,t+1} - \pi_{t+1}), c_{t+1} - w_{t+1}] \\ &= \frac{\rho}{1-\rho\phi}(1-\psi)\mathrm{Cov}_t[r_{i,t+1} - (r^{\$}_{1,t+1} - \pi_{t+1}), x_{t+1}]. \end{aligned} \tag{46}$$

The vector **m** gives the standard one-period mean-variance analysis, while the vector **p** appears because we have assumed that the short-term asset is nominal so that it is risky in real terms. In practice the elements of **p** are all extremely small and have little impact on the portfolio allocation. The vector **h** represents the intertemporal hedging component of bond demand.

The solution of the model is analytical given the loglinearization parameter ρ. We find ρ using the same recursive procedure as before; we assume a value for ρ, solve the model, get a new value for ρ, and so on until convergence.

The second panel in Tables 3–5 reports the optimal portfolio and consumption rules implied by the nominal term structure model estimated over the period 1952–96, assuming that the only assets available to investors are one-quarter and ten-year nominal zero-coupon bonds. Nominal bonds have slightly higher average returns than indexed bonds, but are subject to inflation risk. Table 3 shows that when investors are forced to bear this risk they shorten the maturities of their bond portfolios.

The third panel in Tables 3–5 reports the solution to a model in which both indexed and nominal bonds are available. We allow investors to hold three-month nominal, ten-year nominal, and ten-year indexed bonds. Investors with low risk aversion hold a mix of both indexed and nominal bonds, seeking to earn both the real term premium and the inflation risk premium, and exploiting the imperfect correlations between the real and nominal sources of risk. More conservative investors concentrate their portfolios on indexed bonds.

The bottom panel in Tables 3-5 reports the optimal portfolio allocation and consumption choice when the assets available to investors are three-month, three-year, and ten-year nominal bonds. Investors hold highly leveraged portfolios, with long positions in the three-year nominal bond and short positions in the ten-year nominal bond. Risk-tolerant investors do this because they are attracted by the high Sharpe ratio of the three-year nominal bond relative to the Sharpe ratio of the ten-year nominal bond, and they short ten-year bonds to reduce their portfolio risk. Conservative investors exploit the fact that three-year bonds have a greater real-interest-rate sensitivity than ten-year bonds to create bond portfolios with similar properties to long-term indexed bonds, even though indexed bonds are not directly available in the marketplace. They are not able to avoid all inflation risk, however, because there are three sources of risk in the model—shocks to real interest rates, expected inflation, and unexpected inflation—and only three assets are available. Thus markets are not quite complete.

When only nominal bonds are available, our solution procedure fails to converge for investors with risk aversion $\gamma = 5000$ and elasticity of intertemporal substitution $\psi \neq 1$. The explanation is that these almost infinitely risk-averse investors are forced to bear inflation risk. They respond by increasing their precautionary saving (when $\psi < 1$) or decreasing it (when $\psi > 1$) so that the consumption-wealth ratio is zero or one.

4.2 Constrained demand

The unconstrained portfolio allocations reported in Table 3 are often highly leveraged, and this may not be realistic. We now analyze the optimal allocations to indexed and nominal bonds when investors' portfolio choice is limited by borrowing and short-sales constraints. Because the unconstrained optimal porfolio policy is constant over time, we can do this using results in Teplá (1997). Following Cvitanić and Karatzas (1993), Teplá (1997) shows that standard results in static portfolio choice with borrowing and short-sales constraints extend to intertemporal models whose unconstrained optimal portfolio policies are constant over time. The optimal portfolio allocations under borrowing constraints are the unconstrained allocations with a higher short-term interest rate, and the optimal portfolio allocations under short-sales constraints are found by reducing the dimensionality of the asset space until the optimal unconstrained allocations imply no short sales.

Table 6 reports the optimal allocations under borrowing and short-sales constraints for the same cases as in Table 3. The optimal constrained allocations in the upper two panels of the table, where only a short-term bond and a single long-term bond are available, are trivially zero for those cases in which the unconstrained allocation implies short sales of the long-term bond, and 100% for those cases in which the unconstrained allocation implies borrowing. The lower two panels are more interesting, because they consider scenarios where there are two long-term bonds available to the investor in addition to the short-term bond.

The third panel of Table 6 reports the constrained portfolio allocations when the only assets available to the investor are a three-month nominal, a ten-year indexed, and a ten-year nominal bond. The constrained demand for long-term indexed bonds relative to long-term nominal bonds increases with the coefficient of relative risk aversion. Investors with low risk aversion hold predominantly nominal bonds, despite their lower Sharpe ratios, as a way to increase their risk and expected return without using leverage. Conservative investors hold predominantly indexed bonds because of their consumption hedging properties.

The bottom panel of Table 6 reports the constrained portfolio allocations in the model with two long-term nominal bonds, a three-year bond and a ten-year bond. Investors with low risk aversion hold some ten-year bonds, despite their lower Sharpe ratios, as a way to increase their risk and expected return without using leverage.

Investors with moderate risk aversion hold only three-year bonds, while extremely conservative investors hold three-year bonds and cash.

4.3 A welfare analysis of bond indexation

We now consider the welfare effects of switching from a world where only nominal bonds are available to a world where all debt instruments are indexed, or to a world where both nominal and indexed bonds are available. We study this issue using equation (17), which gives the value function per unit wealth as a monotonic transformation of the optimal consumption-wealth ratio. Hence, we can compute an approximate value function without any need for further approximations just by substituting into this expression our approximate consumption-wealth ratio. For very low ψ, the value function per unit wealth will actually equal the consumption-wealth ratio.[8]

The log value function is a linear function of the short-term interest rate, so it is time-varying. Table 7, which has the same structure as Tables 3–6, reports the mean value function per unit of wealth implied by our solution method for the unconstrained allocations on the left, and for the constrained allocations on the right. Because the recursive utility function (12) is normalized to be homogeneous of degree one in wealth, we can take the ratios of these numbers across panels, for investors with identical preferences, as representing the wealth ratios that would be required to compensate investors for changes in the available assets and investment constraints.

Comparing the first two panels of Table 7, we see that bond indexation can have substantial benefits to investors. Investors with low risk aversion benefit because indexed bonds have higher Sharpe ratios than nominal bonds, while more conservative investors benefit because indexation eliminates an unwelcome source of risk. The effects on welfare can be substantial; when their portfolio choice is unconstrained, for many investors the value function is more than twice as high in a fully indexed environment than in a purely nominal environment. Such investors would be willing to pay more than half their wealth to enjoy the benefits of indexation. The only investors who lose from indexation are investors with low risk aversion who are subject to borrowing and short-sales constraints. These investors prefer to hold nominal bonds, despite their low Sharpe ratios, as a way to increase risk and expected return without using leverage.

These findings are in strong contrast with the claim of Viard (1993) that indexation

[8] We handle the case $\psi = 1$ by taking appropriate limits in (17). Campbell and Viceira (1999) provide a more detailed discussion in a related model.

has only minor welfare effects. Viard models indexation as elimination of the inflation risk in a one-period asset, and studies the benefits to one-period investors. Since there is little risk in inflation over one period, Viard's result is not surprising. We get much larger benefits of indexation because we model indexation as elimination of the inflation risk in long-term assets, and study the benefits to long-term investors.[9]

An apparently paradoxical result is that for some investors welfare is higher when only indexed bonds are available (in the top panel) than when both nominal and indexed bonds are available (in the third panel). The explanation is that in the top panel the short-term bond is indexed, whereas in the bottom panel it is nominal. The small benefits of short-term indexation in the top panel are enough to outweigh the small benefits of the additional long-term nominal asset that is available in the third panel.

The case where two long-term nominal bonds and one short-term nominal bond are available is illustrated in the bottom panel of Table 7. This asset menu delivers the highest welfare for investors with low or moderate risk aversion, but is much less satisfactory for investors with high risk aversion (above 10 or so). Such investors have a strong demand for the consumption insurance provided by long-term indexed bonds.

Advocates of bond indexation have sometimes argued that the availability of indexed assets will stimulate saving. However this effect depends on the elasticity of intertemporal substitution, ψ. If $\psi = 1$, then the consumption-wealth ratio is constant regardless of the available asset menu. If ψ is close to zero, as many empirical estimates suggest, then the consumption-wealth ratio approximately equals the value function. Thus the welfare benefit of indexation is accompanied by an increase in consumption and a decline in saving. This point can be appreciated by comparing the average consumption-wealth ratios in Table 5 with the welfare measures in Table 7.

Finally, we note that welfare calculations for the 1983-96 sample period, not reported here, deliver qualitatively similar results but considerably smaller welfare benefits of bond indexation. The Volcker-Greenspan monetary regime has greatly reduced long-run uncertainty about inflation, and has correspondingly reduced the benefits of eliminating inflation risk entirely.

[9] Campbell and Shiller (1996) also emphasize the benefits of indexation to long-run investors, but they do not present a formal welfare analysis of the type attempted here.

5 Bond Demand in the Presence of Equities

The realism of the preceding analysis is limited by the fact that we have not allowed investors to hold equities. We now consider a scenario in which both bonds and equities are available to the investor. Table 8 reports optimal demands for equities and for 10-year indexed or nominal bonds by investors who are unconstrained (in panel A) or subject to borrowing and short-sales constraints (in panel B). For simplicity we do not allow investors to hold equities and both types of long-term bonds simultaneously.[10]

In a world with full indexation, the unconstrained demand for both long-term indexed bonds and equities is positive and often above 100%, implying that the investor optimally borrows to finance purchases of equities and indexed bonds. The portfolio share of indexed bonds exceeds that of equities, despite the higher Sharpe ratio of equities, because indexed bonds are much less risky than equities.[11] As the coefficient of relative risk aversion increases, the demands for both long-term indexed bonds and equities fall, but the share of equities falls faster. In the limit the infinitely risk-averse investor holds a portfolio equivalent to an indexed perpetuity as we have already discussed. When there are borrowing and short-sale constraints, investors with low risk aversion invest fully in equities as a way to maximize their risk and expected return without using leverage, while more risk-averse investors hold both indexed bonds and equities. Cash plays only a minor role and only in the portfolios of the most risk-averse investors, who are almost fully invested in indexed bonds.

In a world with no indexation, bonds play a much smaller role in optimal portfolios. Unconstrained investors with low risk aversion hold modest bond positions, but constrained investors hold only equities. As risk aversion increases, investors move into cash rather than long-term nominal bonds.

These findings are related to the "asset allocation puzzle" of Canner, Mankiw, and Weil (1997). Popular investment advisers often suggest that more conservative investors should have a higher ratio of long-term bonds to stocks in their portfolios. Canner, Mankiw, and Weil point out that this is inconsistent with the mutual fund theorem of static portfolio analysis, according to which risk aversion should affect

[10] Our solution method fails to converge in a few cases with very low risk aversion and high elasticity of intertemporal substitution. This corresponds to a violation of the transversality condition; when equities are available, investors with these preferences are able to achieve a growth rate of utility that exceeds the time discount rate.

[11] A myopic investor facing independent risks allocates a share to each risk that is proportional to its mean divided by its variance, or equivalently its Sharpe ratio divided by its standard deviation. Although equities have a higher Sharpe ratio than indexed bonds, their standard deviation is much higher so the optimal equity share is lower.

only the ratio of cash to risky assets and not the relative weights on different risky assets.

Our analysis shows that static portfolio analysis can be seriously misleading when investment opportunities are time-varying and investors have long time horizons. The portfolio allocations to equities and indexed bonds in Table 8 are strikingly consistent with popular investment advice. Aggressive long-term investors should hold stocks, while conservative ones should hold long-term bonds and small amounts of cash. The explanation is that long-term bonds, and not cash, are the riskless asset for long-term investors.[12]

A weakness in this resolution of the asset allocation puzzle is that it assumes that long-term bonds are indexed, or equivalently, that there is no inflation uncertainty. The portfolio allocations to nominal bonds in Table 8 do not correspond well with popular investment advice. In order to rationalize the popular investment advice for long-term nominal bonds, one must assume that future interest rates will be generated by a different process than the one estimated in 1952–96, a process with less uncertainty about future inflation.[13] Interestingly, we have estimated just such a process over the Volcker-Greenspan sample period 1983-96. Table 9 repeats Table 8 using our 1983-96 estimates and finds that even when only nominal bonds are available, aggressive long-term investors should hold stocks, while conservative ones should hold primarily long-term nominal bonds along with small quantities of stocks.[14] These results support the conventional wisdom about optimal portfolio choice for long-term investors.

[12]Canner, Mankiw, and Weil are aware of the potential importance of intertemporal hedging demand for the asset allocation puzzle. They write "In principle, intertemporal hedging of the sort discussed by Merton could point in the right direction.... Unfortunately, the magnitude of this effect is not evident a priori, and the empirical literature on intertemporal hedging lags far behind the theoretical literature" (p. 187). This paper attempts to bridge the gap they identify between empirical and theoretical work on intertemporal hedging.

[13]Canner, Mankiw, and Weil argue in the NBER Working Paper version of their paper (1994) that money illusion might help to resolve the asset allocation puzzle. However they consider money illusion in the context of short-term mean-variance analysis and do not relate it to intertemporal hedging as we do here.

[14]During the 1983-96 period the interest-rate sensitivity of a 10-year indexed zero-coupon bond is considerably less than that of an indexed perpetuity. Therefore an infinitely risk-averse investor would like to hold a leveraged position in 10-year indexed zeros, which was not the case in our 1952-96 model. For greater comparability with that model, in Table 9 we replace the 10-year zero-coupon bond with a 20-year zero-coupon bond. This ensures that the optimal portfolio for an infinitely risk-averse investor is available even when borrowing and short-sales constraints are imposed.

6 Conclusion

In this paper we have shown that investors may hold long-term bonds for two reasons. First, if long-term bonds offer a term premium then investors may hold them for speculative purposes, to increase their expected portfolio return even at the cost of some extra short-term risk. This "myopic demand" for long-term bonds can be large when risk aversion is small, because long-term bonds have attractive Sharpe ratios. Second, long-term investors may hold long-term bonds for hedging purposes. Long-term bonds can finance a stable long-run consumption stream even in the face of time-varying short-term interest rates, and this is attractive to risk-averse long-term investors. In the extreme cases where there is no term premium, or where investors are infinitely risk-averse, the myopic demand for long-term bonds is zero and all bond demand is accounted for by the hedging demand.

We have shown that indexed bonds are particularly suitable for hedging purposes, because they do not impose extraneous inflation risk on long-term investors seeking a stable real consumption path. When long-term indexed bonds are available, an infinitely risk-averse long-term investor with zero intertemporal elasticity of substitution holds a bond portfolio that is equivalent to an indexed perpetuity. The indexed perpetuity is the riskless asset for a long-term investor, since it finances a constant consumption stream forever. When only nominal bonds are available, highly risk-averse investors shorten their bond portfolios in order to reduce their exposure to inflation risk. Less risk-averse investors hold long-term nominal bonds for speculative purposes if there is a positive inflation risk premium.

We have extended our approach to solve the intertemporal portfolio choice problem imposing short-sale and borrowing constraints. This is possible because our solution takes the same form as the solution to a static portfolio choice problem for which standard mean-variance analysis is appropriate. Therefore we can solve our constrained problem using methods that have been developed to solve static problems with portfolio constraints.

Our constrained solution enables us to study the welfare effects of bond indexation in a realistic framework. When portfolio constraints are in place, and both nominal and indexed bonds are available to investors, more conservative investors hold in their portfolios relatively more indexed bonds than nominal bonds. These investors benefit substantially from the consumption insurance provided by long-term indexed bonds.

We have also studied the demand for bonds when equities are available as an alternative investment. We find that the ratio of bonds to stocks in the optimal portfolio increases with risk aversion, very much in line with popular investment advice but contrary to the mutual fund theorem of static portfolio analysis. However the demand for long-term bonds is only large when these bonds are indexed, or when inflation uncertainty is low as it has been in the Volcker-Greenspan monetary policy regime since 1983.

Our approach can be extended in several ways. We can explore alternative term-structure models, adding factors or allowing for changing interest-rate volatility. A particularly tractable possibility is a discrete-time version of the Cox, Ingersoll, and Ross (1985) model, in which interest-rate volatility rises with the level of the interest rate. Since term premia and bond return variances move in proportion to one another, this model delivers constant portfolio allocations.

We can consider the effect of investors' horizons more explicitly by solving a finite-horizon version of our model, or by varying the time discount factor δ. In a model with a constant probability of death each period, an investor with a high death probability has a low δ. Our model predicts that this investor has a high optimal consumption-wealth ratio, a low value for the log-linearization parameter ρ, and an optimal portfolio that is close to the optimal portfolio for a myopic single-period investor.

We can allow for multiple consumption goods, and consider assets that are indexed to the price of one of these goods. A house, for example, can be regarded as an asset that delivers a constant flow of housing services, in the same way that an indexed bond delivers a constant flow of consumption. This perspective might explain why conservative investors are willing to own houses despite the short-run variability of house prices (Flavin and Yamashita 1998).

Our analysis also has interesting implications for the design of pension plans and annuities. Our results suggest that conservative investors should favor indexed defined-benefit plans, while more risk-tolerant investors may be willing to accept some inflation or equity risk in their retirement income in exchange for higher average payments.

Our ultimate goal is to build a more fully realistic model of portfolio choice by combining the results in several of our recent papers. This paper explores the effects of interest-rate risk on long-term portfolio choice, while Campbell and Viceira (1999) studies time-variation in the equity premium, and Viceira (1997) considers uninsurable risk in labor income. A complete model accounting for all these effects offers the exciting prospect that financial economists will at last be able to offer realistic but scientifically grounded investment advice.

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8 Appendix

A. Pricing Nominal Bonds

The pricing of default-free nominal bonds follows the same steps as the pricing of indexed bonds. The relevant stochastic discount factor to price nominal bonds is the nominal SDF $M^{\$}_{t+1}$, whose log is given by (10): $m^{\$}_{t+1} = m_{t+1} - \pi_{t+1}$. Since both M_{t+1} and Π_{t+1} are jointly lognormal and homoskedastic, $M^{\$}_{t+1}$ is also lognormal. The log nominal return on a one-period nominal bond is $r^{\$}_{1,t+1} = -\log \mathrm{E}_t[M_{t+1}]$, or

$$\begin{aligned} r^{\$}_{1,t+1} &= -\mathrm{E}_t\left[m^{\$}_{t+1}\right] - \frac{1}{2}\mathrm{Var}_t\left[m^{\$}_{t+1}\right] \\ &= x_t + z_t - \frac{1}{2}\left[(\beta_{mx} + \beta_{\pi x})\, 2\sigma_x^2 + \beta_{\pi z}^2\sigma_z^2 + (1+\beta_{\pi m})^2\sigma_m^2 + \sigma_\pi^2\right], \qquad (47) \end{aligned}$$

a linear combination of the expected log real SDF and expected inflation.

The risk premium on a 1-period nominal bond over a 1-period real bond can be written as

$$\mathrm{E}_t\left[r^{\$}_{1,t+1} - \pi_{t+1} - r_{1,t+1}\right] + \frac{1}{2}\mathrm{Var}_t\left[\pi_{t+1}\right] = -\beta_{mx}\beta_{\pi x}\sigma_x^2 - \beta_{\pi m}\sigma_m^2, \qquad (48)$$

which has the same form as equation (4) for equities.

The log price of an n-period nominal bond, $p^{\$}_{n,t}$, also has an affine structure. It is a linear combination of x_t and z_t whose coefficients are time-invariant, though they vary with the maturity of the bond. As shown in equation (11), $-p^{\$}_{n,t} = A^{\$}_n + B^{\$}_{1,n}x_t + B^{\$}_{2,n}z_t$,where

$$\begin{aligned} B^{\$}_{1,n} &= 1 + \phi_x B^{\$}_{1,n-1} = \frac{1-\phi_x^n}{1-\phi_x} \\ B^{\$}_{2,n} &= 1 + \phi_z B^{\$}_{2,n-1} = \frac{1-\phi_z^n}{1-\phi_z} \\ A^{\$}_n - A^{\$}_{n-1} &= (1-\phi_x)\,\mu_x B^{\$}_{1,n-1} + (1-\phi_z)\,\mu_z B^{\$}_{2,n-1} \\ &\quad -\frac{1}{2}\left(\beta_{mx} + \beta_{\pi_x} + B^{\$}_{1,n-1} + \beta_{zx}B^{\$}_{2,n-1}\right)^2\sigma_x^2 \\ &\quad -\frac{1}{2}\left(\beta_{\pi z} + B^{\$}_{2,n-1}\right)^2\sigma_z^2 - \frac{1}{2}\left(1+\beta_{\pi m}+\beta_{zm}\right)^2\sigma_m^2 \qquad (49) \\ &\quad -\frac{1}{2}\sigma_\pi^2, \end{aligned}$$

and $A_0^\$ = B_{1,0}^\$ = B_{2,0}^\$ = 0$.

The excess return on a n-period bond over the one-period log nominal interest rate is

$$
\begin{aligned}
r_{n,t+1}^\$ - r_{1,t+1}^\$ &= p_{n-1,t+1}^\$ - p_{n,t}^\$ + p_{1,t}^\$ \\
&= -\left(B_{1,n-1}^\$ + B_{2,n-1}^\$\beta_{zx}\right)(\beta_{mx} + \beta_{\pi x})\,\sigma_x^2 - B_{2,n-1}^\$\beta_{\pi z}\sigma_z^2 - (1+\beta_{\pi m})\,\beta_{zm}\sigma_m^2 \\
&\quad -\frac{1}{2}\left(B_{1,n-1}^\$ + B_{2,n-1}^\$\beta_{zx}\right)2\sigma_x^2 - \frac{1}{2}\left(B_{2,n-1}^\$\right)^2\sigma_z^2 - \frac{1}{2}\beta_{zm}^2\sigma_m^2 \\
&\quad - \left(B_{1,n-1}^\$ + B_{2,n-1}^\$\beta_{zx}\right)\varepsilon_{x,t+1} - B_{2,n-1}^\$\beta_{zm}\varepsilon_{m,t+1} - B_{2,n-1}^\$\varepsilon_{z,t+1}. \qquad (50)
\end{aligned}
$$

The terms in $B_{2,n-1}^\$\beta_{zx}$ and $B_{2,n-1}^\$\beta_{zm}$ arise because shocks to expected inflation are correlated with shocks to the expected and unexpected log real SDF. Thus risk premia in the nominal term structure are different from risk premia in the real term structure because they include compensation for inflation risk. Like real risk premia, however, nominal risk premia are constant over time.

B. Solution of the Model

Our guess (26) and the expression for the log excess return on the long-term bond given in equation (8) imply that $\text{Cov}_t\left(r_{n,t+1}, c_{t+1} - w_{t+1}\right) = -B_{n-1}b_1\sigma_x^2$. Our term structure model implies that $\text{E}_t[r_{n,t+1} - r_{1,t+1}] = -B_{n-1}^2\sigma_x^2/2 - \beta_{mx}B_{n-1}\sigma_x^2$ and $\text{Var}_t(r_{n,t+1}) = B_{n-1}^2\sigma_x^2$. Substituting these expressions into (25) we obtain:

$$
\alpha_{n,t} = \alpha_n = \frac{-1}{\gamma B_{n-1}}\left(\beta_{mx} + \frac{1-\gamma}{1-\psi}b_1\right), \qquad (51)
$$

which does not depend on the future portfolio and consumption choices of the investor.

Given the optimal portfolio rule (51) we can now solve for the parameters b_0 and b_1 of the consumption rule. The expected return on the wealth portfolio is a linear function of the state variable:

$$
\text{E}_t\left[r_{p,t+1}\right] = p_0 + x_t, \qquad (52)
$$

where the intercept

$$
p_0 \equiv -(\alpha_n B_{n-1})(\beta_{mx}\sigma_x^2) - (\alpha_n B_{n-1})^2\sigma_x^2/2 - (\beta_{mx}^2\sigma_x^2 + \sigma_m^2)/2 \qquad (53)
$$

does not vary with t or n. The consumption intercept term given in (21) becomes

$$
v_{p,t} = v_p \equiv -((1-\gamma)/(1-\psi))[b_1 - (1-\psi)(\alpha_n B_{n-1})]^2\sigma_x^2/2, \qquad (54)
$$

which also does not vary with t or n.

Substituting (52), (53), and (54) into (20), we get a linear expectational difference equation for $(c_t - w_t)$,

$$c_t - w_t = \rho \, \mathrm{E}_t \left[c_{t+1} - w_{t+1} \right] + \rho \left(1 - \psi\right) x_t - \rho \left[\left(1 - \psi\right) p_0 - \psi \log \delta - v_p + k \right], \quad (55)$$

from which we can identify the coefficients of the consumption rule.

C. The Cox-Huang method with recursive utility

With recursive utility the optimal invested wealth-consumption ratio satisfies

$$\left(\frac{W_t^*}{C_t} \right)^{\theta} = \mathrm{E}_t \left[\left(1 + \frac{W_{t+1}^*}{C_{t+1}} \right)^{\theta} \delta^{\theta} \left(\frac{C_{t+1}}{C_t} \right)^{\theta\left(1 - \frac{1}{\psi}\right)} \right]. \quad (56)$$

The consumer's IMRS is now $\delta^{\theta} (C_{t+1}/C_t)^{-\theta/\psi} R_{p,t+1}^{\theta}$, which under complete markets must equal the SDF M_{t+1}, so that

$$\delta^{\theta} \left(\frac{C_{t+1}}{C_t} \right)^{\theta\left(1 - \frac{1}{\psi}\right)} = \delta^{\theta\psi} M_{t+1}^{1-\psi} R_{p,t+1}^{(1-\theta)(1-\psi)}, \quad (57)$$

and we can rewrite (56) as

$$\left(\frac{W_t^*}{C_t} \right)^{\theta} = \mathrm{E}_t \left[\left(1 + \frac{W_{t+1}^*}{C_{t+1}} \right)^{\theta} \delta^{\theta\psi} M_{t+1}^{1-\psi} R_{p,t+1}^{(1-\theta)(1-\psi)} \right]. \quad (58)$$

We can loglinearize this equation in a similar fashion to (34) and obtain a linear expectational difference for the log consumption-wealth ratio whose solution is exactly the same as the one we obtain using our Euler-equation methodology.

TABLE 1

Term Structure Model Estimation

Parameter	1952.I - 1996.III est.	1952.I - 1996.III s.e.	1983.I - 1996.III est.	1983.I - 1996.III s.e.
μ_x	0.0620	0.0125	0.0196	0.0301
μ_z	0.0100		0.0087	
ϕ_x	0.8702	0.0070	0.9860	0.0070
ϕ_z	0.9985	0.0010	0.8660	0.0342
β_{mx}	-100.5374	19.7436	-31.0690	140.3320
β_{zx}	0.0545	0.0454	-0.3994	0.4428
β_{zm}	-0.0012	0.0004	0.0006	0.0017
$\beta_{\pi x}$	0.9947	0.2728	-0.0175	0.1884
$\beta_{\pi m}$	-0.0103	0.0027	0.0016	0.0078
$\beta_{\pi z}$	1.7932	0.4600	-1.5596	1.7914
β_{ex}	-4.1449	2.6678	-11.1761	5.2311
β_{em}	0.3154	0.0477	0.4780	0.2697
σ_x	0.0023	0.0001	0.0023	0.0008
σ_m	0.2578	0.0363	0.1424	0.0791
σ_z	0.0012	0.0001	0.0016	0.0002
σ_π	0.0075	0.0004	0.0071	0.0047
log-lik.	26.6128		27.1526	
no. obs.	179		55	
$E[r_{1,t+1}]$	0.85%		2.72%	
$E[r^{\$}_{1,t+1}]$	5.31%		6.19%	
$\sigma(r_{1,t+1})$	0.93%		2.82%	
$\sigma(r^{\$}_{1,t+1})$	4.80%		2.69%	
$E[\pi_{t+1}]$	3.99%		3.49%	
$\sigma_t(\pi_{t+1})$	1.72%		1.51%	

TABLE 2

Sample and Implied Moments of the Term Structure

	Moment		1952.I - 1996.III			1983.I - 1996.III		
			1 yr.	3 yr.	10 yr.	1 yr.	3 yr.	10 yr.
		A: Nominal Term Structure						
(1)	$E[r^{\$}_{n,t+1} - r^{\$}_{1,t+1}] + \sigma^2(r^{\$}_{n,t+1} - r^{\$}_{1,t+1})/2$	sample	0.372	0.602	0.784	0.659	1.957	5.184
		implied	0.613	1.411	2.058	0.130	0.546	1.876
(2)	$\sigma(r^{\$}_{n,t+1} - r^{\$}_{1,t+1})$	sample	1.474	4.227	10.383	1.052	3.898	11.556
		implied	1.486	4.099	10.535	1.213	4.143	13.008
(3)	$SR^{\$} = (1)/(2)$	sample	0.252	0.142	0.076	0.626	0.502	0.449
		implied	0.412	0.344	0.195	0.107	0.132	0.144
(4)	$E[y^{\$}_{n,t+1} - y^{\$}_{1,t+1}]$	sample	0.422	0.759	1.121	0.493	1.183	1.927
		implied	0.322	0.825	1.324	0.060	0.230	0.654
(5)	$\sigma(y^{\$}_{n,t+1} - y^{\$}_{1,t+1})$	sample	0.204	0.377	0.569	0.163	0.316	0.511
		implied	0.167	0.449	0.767	0.129	0.355	0.738
(6)	$\sigma(y^{\$}_{n-1,t+1} - y^{\$}_{n,t+1})$	sample	0.474	0.377	0.263	0.344	0.350	0.295
		implied	0.498	0.375	0.271	0.407	0.378	0.334
		B: Real Term Structure						
(7)	$E[r_{n,t+1} - r_{1,t+1}] + \sigma^2(r_{n,t+1} - r_{1,t+1})/2$	implied	0.556	1.278	1.624	0.203	0.704	2.073
(8)	$\sigma(r_{n,t+1} - r_{1,t+1})$	implied	1.206	2.770	3.520	1.390	4.822	14.196
(9)	$SR = (7)/(8)$	implied	0.461	0.461	0.461	0.146	0.146	0.146
(10)	$E[y_{n,t+1} - y_{1,t+1}]$	implied	0.288	0.763	1.273	0.098	0.318	0.742
(11)	$\sigma(y_{n,t+1} - y_{1,t+1})$	implied	0.166	0.446	0.753	0.059	0.207	0.649
(12)	$\sigma(y_{n-1,t+1} - y_{n,t+1})$	implied	0.406	0.255	0.092	0.464	0.439	0.364
		C: Equities						
(13)	$E[r_{e,t+1} - (r^{\$}_{1,t+1} - \pi_{t+1})]$ $+\sigma^2(r_{e,t+1} - (r^{\$}_{1,t+1} - \pi_{t+1}))/2$	sample		7.093			8.951	
		implied		8.714			4.688	
(14)	$\sigma(r_{e,t+1} - (r^{\$}_{1,t+1} - \pi_{t+1}))$	sample		15.904			14.643	
		implied		15.876			14.715	
(15)	$SR = (13)/(14)$	sample		0.446			0.611	
		implied		0.549			0.319	

TABLE 3

Optimal Percentage Allocation to n-Period Bond

$\boldsymbol{\alpha_n \times 100}$

Model	R.R.A.	E.I.S.						E.I.S.					
		1/.75	1.00	1/2	1/5	1/10	1/5000	1/.75	1.00	1/2	1/5	1/10	1/5000
	0.75	1715	1717	1719	1721	1721	1722						
	1	1311	1311	1311	1311	1311	1311						
Indexed	2	703	702	701	700	700	700						
Only	5	337	337	337	337	337	337						
	10	215	215	216	216	216	216						
	5000	93	94	95	96	96	96						
	0.75							245	245	245	245	245	245
	1							185	185	185	185	185	185
Nominal	2							96	96	96	96	96	96
Only	5							42	42	43	43	43	43
	10							24	25	25	25	25	25
	5000							+	7	-	-	-	-
	0.75	1124	1125	1126	1126	1127	1127	160	160	160	160	160	160
Indexed	1	867	867	867	867	867	867	120	120	120	120	120	120
and	2	481	481	480	480	480	480	60	60	59	59	59	59
Nominal	5	249	249	249	249	249	249	23	24	24	24	24	24
	10	171	172	172	173	173	173	11	11	12	12	12	12
	5000	94	95	96	97	97	97	-1	0	0	0	0	0
	0.75	3486	3490	3496	3499	3500	3501	-1005	-1006	-1008	-1009	-1009	-1010
Both	1	2662	2662	2662	2662	2662	2662	-769	-769	-769	-769	-769	-769
Nominal	2	1421	1419	1416	1414	1414	1413	-413	-413	-412	-411	-411	-411
(3,10y)	5	673	673	672	672	672	672	-199	-199	-199	-199	-199	-198
	10	424	424	425	426	426	426	-128	-128	-128	-128	-128	-128
	5000	+	176	-	-	-	-	+	-57	-	-	-	-

Note: " - " indicates that the recursion for ρ converged to $\rho = 1$ and "+" that it converged to a negative value.

TABLE 4

Percentage Hedging Demand Over Total Demand

$$\alpha_{n,hedging}(\gamma,\psi)/\alpha_n(\gamma,\psi) = [1 - \alpha_n(1,\psi)/(\gamma\alpha_n(\gamma,\psi))] \times 100$$

Model	R.R.A.	E.I.S.						E.I.S.					
		1/.75	1.00	1/2	1/5	1/10	1/5000	1/.75	1.00	1/2	1/5	1/10	1/5000
Indexed Only	0.75	-1.9	-1.8	-1.6	-1.6	-1.5	-1.5						
	1	0.0	0.0	0.0	0.0	0.0	0.0						
	2	6.8	6.6	6.5	6.4	6.4	6.3						
	5	22.2	22.2	22.1	22.1	22.1	22.1						
	10	39.0	39.1	39.2	39.3	39.3	39.4						
	5000	99.7	99.7	99.7	99.7	99.7	99.7						
Nominal Only	0.75							-0.9	-0.9	-0.9	-0.9	-0.9	-0.9
	1							0.0	0.0	0.0	0.0	0.0	0.0
	2							3.4	3.5	3.6	3.6	3.6	3.7
	5							12.2	12.5	13.0	13.3	13.4	13.5
	10							23.8	24.4	25.3	25.9	26.1	26.3
	5000							+	99.4	-	-	-	-
Indexed and Nominal	0.75	-2.9	-2.8	-2.7	-2.6	-2.6	-2.6	0.1	0.1	0.2	0.2	0.2	0.2
	1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	2	9.9	9.8	9.8	9.7	9.7	9.7	-0.4	-0.4	-0.5	-0.5	-0.5	-0.5
	5	30.3	30.4	30.4	30.5	30.5	30.5	-1.9	-1.8	-1.7	-1.6	-1.6	-1.5
	10	49.4	49.5	49.7	49.8	49.9	49.9	-4.5	-4.1	-3.5	-3.2	-3.1	-3.0
	5000	99.8	99.8	99.8	99.8	99.8	99.8	104.1	104.8	106.3	107.5	108.1	108.7
Both Nominal (3,10y)	0.75	-1.8	-1.7	-1.5	-1.4	-1.4	-1.4	-2.0	-1.9	-1.7	-1.6	-1.6	-1.6
	1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	2	6.3	6.2	6.0	5.9	5.9	5.8	7.0	6.8	6.6	6.5	6.5	6.5
	5	21.0	20.9	20.8	20.8	20.7	20.7	22.7	22.7	22.6	22.6	22.5	22.5
	10	37.2	37.3	37.4	37.5	37.5	37.5	39.7	39.8	39.9	39.9	39.9	40.0
	5000	+	99.7	-	-	-	-	+	99.7	-	-	-	-

Note: " - " indicates that the recursion for ρ converged to $\rho = 1$ and "+" that it converged to a negative value.

TABLE 5

Percentage Mean Optimal Consumption-Wealth Ratio and Percentage Volatility of Consumption Growth

Model	R.R.A.	$E[C_t/W_t] \times 100$ E.I.S. 1/.75	1.00	1/2	1/5	1/10	1/5000	$\sigma(\Delta c_{t+1} - E_t[\Delta c_{t+1}]) \times 100$ E.I.S. 1/.75	1.00	1/2	1/5	1/10	1/5000
Indexed Only	0.75	0.10	0.98	2.29	3.07	3.33	3.59	30.76	30.21	29.51	29.15	29.04	28.93
	1	0.35	0.98	1.91	2.47	2.65	2.83	23.64	23.07	22.30	21.89	21.76	21.63
	2	0.73	0.98	1.34	1.56	1.63	1.70	12.93	12.36	11.54	11.06	10.91	10.76
	5	0.96	0.98	1.00	1.02	1.03	1.03	6.48	5.93	5.11	4.61	4.45	4.29
	10	1.03	0.98	0.89	0.84	0.83	0.81	4.33	3.79	2.97	2.47	2.31	2.14
	5000	1.11	0.98	0.78	0.66	0.63	0.59	2.17	1.65	0.84	0.34	0.17	0.00
Nominal Only	0.75	0.98	0.98	0.97	0.96	0.96	0.95	13.52	13.27	12.94	12.76	12.71	12.65
	1	1.03	0.98	0.89	0.84	0.82	0.81	10.39	10.14	9.81	9.64	9.59	9.54
	2	1.11	0.98	0.77	0.65	0.61	0.57	5.71	5.46	5.16	5.03	5.00	4.98
	5	1.17	0.98	0.69	0.52	0.46	0.40	2.97	2.70	2.46	2.45	2.47	2.50
	10	1.20	0.98	0.63	0.43	0.36	0.29	2.10	1.82	1.66	1.76	1.83	1.92
	5000	+	0.98	-	-	-	-	+	1.05	-	-	-	-
Indexed and Nominal	0.75	0.53	0.98	1.64	2.04	2.17	2.30	23.49	22.99	22.30	21.92	21.80	21.68
	1	0.68	0.98	1.42	1.69	1.78	1.87	18.14	17.62	16.89	16.48	16.35	16.22
	2	0.89	0.98	1.10	1.18	1.20	1.23	10.09	9.57	8.81	8.37	8.22	8.08
	5	1.02	0.98	0.91	0.87	0.86	0.85	5.29	4.77	3.99	3.53	3.37	3.22
	10	1.06	0.98	0.85	0.77	0.74	0.72	3.71	3.18	2.39	1.92	1.76	1.61
	5000	1.13	0.98	0.76	0.65	0.61	0.57	2.19	1.67	0.85	0.35	0.19	0.03
Both Nominal (3,10y)	0.75	0.03	0.98	2.39	3.24	3.52	3.80	31.40	30.85	30.16	29.80	29.69	29.59
	1	0.29	0.98	1.99	2.60	2.80	3.00	24.13	23.55	22.79	22.38	22.25	22.13
	2	0.69	0.98	1.40	1.65	1.74	1.82	13.19	12.62	11.80	11.33	11.17	11.02
	5	0.93	0.98	1.04	1.08	1.10	1.11	6.62	6.07	5.26	4.77	4.61	4.45
	10	1.01	0.98	0.92	0.88	0.87	0.86	4.44	3.91	3.11	2.64	2.48	2.32
	5000	+	0.98	-	-	-	-	+	1.82	-	-	-	-

Note: " - " indicates that the recursion for ρ converged to $\rho = 1$ and "+" that it converged to a negative value.

TABLE 6

Optimal Percentage Allocation to n-Period Bond

Under Borrowing and Short-Sale Constraints

$\alpha_n \times 100$

Model	R.R.A.	E.I.S.						E.I.S.					
		1/.75	1.00	1/2	1/5	1/10	1/5000	1/.75	1.00	1/2	1/5	1/10	1/5000
Indexed Only	0.75	100	100	100	100	100	100						
	1	100	100	100	100	100	100						
	2	100	100	100	100	100	100						
	5	100	100	100	100	100	100						
	10	100	100	100	100	100	100						
	5000	93	94	95	96	96	96						
Nominal Only	0.75							100	100	100	100	100	100
	1							100	100	100	100	100	100
	2							96	96	96	96	96	96
	5							42	42	43	43	43	43
	10							24	25	25	25	25	25
	5000							+	7	-	-	-	-
Indexed and Nominal	0.75	0	0	0	0	0	0	100	100	100	100	100	100
	1	11	11	11	11	11	11	89	89	89	89	89	89
	2	56	56	56	56	56	56	44	44	44	44	44	44
	5	82	82	82	82	82	82	17	17	18	18	18	18
	10	91	91	91	91	91	91	9	9	9	9	9	9
	5000	93	95	96	97	97	97	0	0	0	0	0	0
Both Nominal (3,10y)	0.75	0	0	0	0	0	0	100	100	100	100	100	100
	1	14	14	14	14	14	14	86	86	86	86	86	86
	2	80	80	80	80	80	80	20	20	20	20	20	20
	5	100	100	100	100	100	100	0	0	0	0	0	0
	10	100	100	100	100	100	100	0	0	0	0	0	0
	5000	+	42	-	-	-	-	+	0	-	-	-	-

Note: " - " indicates that the recursion for ρ converged to $\rho = 1$ and "+" that it converged to a negative value.

TABLE 7

Percentage Mean Value Function

$\boldsymbol{E[V_t] \times 100}$

Model	R.R.A.	**Unconstrained** E.I.S. 1/.75	1.00	1/2	1/5	1/10	1/5000	**Constrained** E.I.S. 1/.75	1.00	1/2	1/5	1/10	1/5000
Indexed Only	0.75	*	14.51	5.37	4.09	3.82	3.60	0.68	0.67	0.64	0.62	0.61	0.60
	1	21.43	6.68	3.74	3.11	2.96	2.84	0.68	0.67	0.64	0.62	0.61	0.60
	2	2.35	2.08	1.85	1.75	1.73	1.70	0.68	0.67	0.64	0.62	0.61	0.60
	5	1.04	1.04	1.03	1.03	1.03	1.03	0.68	0.67	0.64	0.62	0.61	0.60
	10	0.82	0.82	0.82	0.81	0.81	0.81	0.68	0.67	0.64	0.62	0.61	0.60
	5000	0.66	0.65	0.62	0.60	0.60	0.59	0.66	0.65	0.62	0.60	0.60	0.59
Nominal Only	0.75	0.96	0.97	0.95	0.95	0.95	0.95	0.77	0.77	0.75	0.74	0.74	0.74
	1	0.82	0.82	0.81	0.81	0.81	0.81	0.75	0.74	0.72	0.71	0.71	0.70
	2	0.66	0.63	0.61	0.59	0.58	0.57	0.66	0.63	0.61	0.59	0.58	0.57
	5	0.57	0.53	0.49	0.44	0.42	0.40	0.57	0.53	0.49	0.44	0.42	0.40
	10	0.52	0.49	0.41	0.35	0.32	0.29	0.52	0.49	0.41	0.35	0.32	0.29
	5000	+	0.00	-	-	-	-	+	0.00	-	-	-	-
Indexed and Nominal	0.75	5.98	3.80	2.75	2.45	2.37	2.30	0.96	0.97	0.95	0.95	0.95	0.95
	1	2.94	2.45	2.08	1.94	1.90	1.87	0.75	0.74	0.72	0.71	0.71	0.71
	2	1.28	1.27	1.25	1.23	1.23	1.23	0.71	0.70	0.68	0.67	0.66	0.65
	5	0.85	0.85	0.85	0.85	0.85	0.85	0.69	0.68	0.65	0.64	0.63	0.62
	10	0.75	0.75	0.73	0.73	0.72	0.72	0.69	0.67	0.65	0.63	0.62	0.61
	5000	0.62	0.62	0.60	0.59	0.58	0.57	0.61	0.62	0.60	0.59	0.58	0.57
Both Nominal (3,10y)	0.75	*	17.93	5.86	4.37	4.06	3.80	0.96	0.97	0.95	0.95	0.95	0.95
	1	36.86	7.96	4.08	3.32	3.15	3.00	0.75	0.74	0.72	0.71	0.71	0.71
	2	2.78	2.35	2.01	1.89	1.85	1.82	0.71	0.70	0.68	0.66	0.65	0.65
	5	1.13	1.12	1.12	1.11	1.11	1.11	0.68	0.62	0.63	0.61	0.60	0.59
	10	0.87	0.87	0.86	0.86	0.86	0.86	0.63	0.53	0.57	0.54	0.52	0.51
	5000	+	0.00	-	-	-	-	+	0.00	-	-	-	-

Note: " - " indicates that the recursion for ρ converged to $\rho = 1$ and "+" that it converged to a negative value.

TABLE 8

Optimal Percentage Allocation to Equities

and to n-Period Bond

$\alpha \times 100$

Model	R.R.A.	**Equities** E.I.S.						**n-Period Bond** E.I.S.					
		(A) Unconstrained											
		1/.75	1.00	1/2	1/5	1/10	1/5000	1/.75	1.00	1/2	1/5	1/10	1/5000
	0.75	-	423	423	423	423	423	-	1488	1495	1498	1498	1499
	1	-	317	317	317	317	317	-	1140	1140	1140	1140	1140
Indexed	2	159	159	159	159	159	159	619	616	613	611	610	610
Only	5	63	63	63	63	63	63	304	303	301	300	299	299
	10	32	32	32	32	32	32	198	198	197	197	197	197
	5000	0	0	0	0	0	0	93	94	95	96	96	96
	0.75	-	450	450	450	450	450	-	61	61	62	62	62
	1	-	337	337	337	337	337	-	47	47	47	47	47
Nominal	2	168	168	168	168	168	168	28	27	27	27	27	27
Only	5	66	66	66	66	66	66	15	15	15	15	15	15
	10	32	32	32	32	32	32	11	11	12	12	12	12
	5000	+	-2	-	-	-	-	+	8	-	-	-	-
		(B) Constrained											
		1/.75	1.00	1/2	1/5	1/10	1/5000	1/.75	1.00	1/2	1/5	1/10	1/5000
	0.75	-	100	100	100	100	100	-	0	0	0	0	0
	1	100	100	100	100	100	100	0	0	0	0	0	0
Indexed	2	100	100	100	100	100	100	0	0	0	0	0	0
Only	5	58	58	58	58	58	58	42	42	42	42	42	42
	10	29	29	29	29	29	29	71	71	71	71	71	71
	5000	0	0	0	0	0	0	93	94	95	96	96	96
	0.75	100	100	100	100	100	100	0	0	0	0	0	0
	1	100	100	100	100	100	100	0	0	0	0	0	0
Nominal	2	100	100	100	100	100	100	0	0	0	0	0	0
Only	5	66	66	66	66	66	66	15	15	15	15	15	15
	10	32	32	32	32	32	32	11	11	12	12	12	12
	5000	+	0	-	-	-	-	+	7	-	-	-	-

Note: " - " indicates that the recursion for ρ converged to $\rho = 1$ and "+" that it converged to a negative value.

TABLE 9

Optimal Percentage Allocation to Equities and to n-Period Bond Sample Period: 1983-1996

$\alpha \times 100$

Model	R.R.A.	Equities E.I.S.						n-Period Bond E.I.S.					
(A) Unconstrained													
		1/.75	1.00	1/2	1/5	1/10	1/5000	1/.75	1.00	1/2	1/5	1/10	1/5000
Indexed Only	0.75	279	279	279	279	279	279	-15	-8	-1	1	2	3
	1	209	209	209	209	209	209	16	16	16	16	16	16
	2	105	105	105	105	105	105	55	52	48	46	45	44
	5	42	42	42	42	42	42	75	73	71	70	69	69
	10	21	21	21	21	21	21	81	80	80	79	79	79
	5000	0	0	0	0	0	0	87	87	89	90	90	90
Nominal Only	0.75	275	275	275	275	275	275	-13	-6	1	3	4	5
	1	207	207	207	207	207	207	18	18	18	18	18	18
	2	105	105	105	105	105	105	58	55	51	49	48	47
	5	44	44	44	44	44	44	78	77	75	74	73	73
	10	23	23	23	23	23	23	84	84	84	84	84	84
	5000	+	3	-	-	-	-	+	92	-	-	-	-
(B) Constrained													
		1/.75	1.00	1/2	1/5	1/10	1/5000	1/.75	1.00	1/2	1/5	1/10	1/5000
Indexed Only	0.75	100	100	100	100	100	100	0	0	0	0	0	0
	1	100	100	100	100	100	100	0	0	0	0	0	0
	2	58	59	62	63	63	64	42	41	38	37	37	36
	5	29	30	31	32	33	33	71	70	69	68	67	67
	10	20	20	21	21	21	21	80	80	79	79	79	79
	5000	0	0	0	0	0	0	87	87	89	90	90	90
Nominal Only	0.75	100	100	100	100	100	100	0	0	0	0	0	0
	1	100	100	100	100	100	100	0	0	0	0	0	0
	2	58	59	61	63	63	64	42	41	39	37	37	36
	5	27	28	29	30	30	30	73	72	71	70	70	70
	10	18	18	17	17	17	17	82	82	83	83	83	83
	5000	+	3	-	-	-	-	+	92	-	-	-	-

Note: " - " indicates that the recursion for ρ converged to $\rho = 1$ and "+" that it converged to a negative value.