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INSIDER INFORMATION, ARBITRAGE AND OPTIMAL PORTFOLIO AND CONSUMPTION POLICIES

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Abstract

This article extends the continuous time financial market model pioneered by Samuelson (1969) and Merton (1971) to allow for insider information.

We prove that if the investment horizon of an insider ends after his initial information advantage has disappeared, an insider has arbitrage opportunities if and only if the anticipative information is so informative that it contains zero-probability events given initial public information. When it ends before or when anticipative information does not contain such events we derive expressions for optimal consumption and portfolio policies and examine the effects of anticipative information on the optimal policies of an insider. Optimal insider policies are shown not to be fully revealing. Anticipative information is of no value and therefore does not affect the optimal behavior of insiders if and only if it is independent from public information. We show that arbitrage opportunities allow to replicate arbitrary consumption streams such that the insider's budget constraint is not binding. Consequently, Merton's consumption-investment problem has no solution whenever investment horizons are longer than resolution times of signals and insider information contains events whose occurrence is not believed. If the true signal is perturbed by independent noise this problem can be avoided. But since in this case investors never learn the true anticipative information we argue that this does not capture an important feature of insider information. We also show that the valuation of contingent claims measurable with respect to public information at maturity is invariant to insider information if the latter does not allow for arbitrage opportunities. In contrast contingent claims have no value for insiders with anticipative information generated by signals with continuous distributions.

Key Words: insider information, free lunch, arbitrage, contingent claim, utility maximization, portfolio policies, enlargements of filtrations, Malliavin calculus

1 INTRODUCTION AND SUMMARY

THIS ARTICLE EXTENDS the standard continuous time financial market model pioneered by Samuelson (1969) and Merton (1971) to allow for investors with anticipative information. Such investors are called insiders, since they can already tell today whether or not an event unknown to the public will occur at some date in the future. The effects of insiders on the social efficiency of mechanisms for intertemporal risk sharing has been of great concern in the financial market literature. The traditional models used to address this issue are based on strong assumptions about the nature of the insider signal and the preferences of investors. In the classical equilibrium models following Grossman (see the articles in Grossman (1989)) and the market microstructure literature based on the market game model of Kyle (1985) it is assumed that insiders have a constant absolute risk aversion and preferences for terminal wealth only. Insider signals are restricted to the liquidation value of the risky asset. Since prices of risky assets are determined endogenously and therefore depends on information, preferences and beliefs of other market participants such anticipative information might be rare. Furthermore these models do not allow for investment horizons for which the uncertainty about the insider signal is resolved given public information.

In this paper we generalize results of Karatzas and Pikovsky (1996a) and show how in a standard continuous time financial market model, techniques of the theory of enlargements of filtrations can be used to analyze the effects of arbitrary insider signals on dynamic portfolio and consumption policies. We illustrate our approach with an example in which an investor knows already today whether or not the risky asset stops to pay dividends after a certain time. Such signals could not been considered previously. For an insider with constant relative risk aversion we derive an explicit expression for the demand of risky assets which is due to his/her anticipative information. More generally the techniques introduced in this paper enable us to analyze how the valuation of contingent claims and optimal consumption and portfolio policies are affected by the precision of the information about the true state of nature contained in insider signals and can therefore also be used for purposes of dynamic risk management.

Since the seminal paper of Harrison and Kreps (1979), it is well-known that the existence of a probability measure for which prices are martingales is equivalent to the existence of a viable market model, that is a model where there exists an optimal trading strategy for some agent with strictly monotonic, continuous and convex preferences. This is often referred to as the fundamental theorem of asset pricing [1]. As shown by Kreps (1981) viability is a stronger requirement than the absence of free lunches. If the market model is viable, there cannot exist strategies which provide positive gains from trade with positive probability but no initial investment.

We show that the existence of a viable market model for an investor who already knows today whether or not an event unknown to the public will occur, depends crucially on the information about states of nature contained in his/her anticipative information. We prove that for investment horizons which do not end before the first moment in time the informational advantage of an insider has disappeared, there exist always free lunches with vanishing risk but no arbitrage opportunities [2] whenever the insider's anticipative information does not contain zero probability events (atomic insider information). On the other hand if the insider information is so informative about the state of nature that it contains events which are not believed to occur given public information (non-atomic insider information), a viable market model exists only if the investment horizon ends before the uncertainty about the insiders information is resolved. This implies that no viable market model exists for insider whose anticipative information is generated by signals with continuous distributions when the investment horizon is unrestricted. We show that this problem can be avoided if we add independent noise to the insider signal. In this case we can always derive optimal portfolio and consumption strategies. But since in this case uncertainty about the true signal is never resolved it does not capture an important feature of insider information in financial markets and resembles more market models with differential information [3]. Clearly insiders with imperfect anticipative information exist in

[1] The result of Harrison and Kreps (1979) has been generalized in various directions. See Dalang, Morton and Willinger (1990) , Back and Pliska (1991), Schachermayer (1994) for models with discrete trading, Delbaen and Schachermayer (1994) respectively (1998) for general semi-martingale models and continuous trading and Jouini and Kallal (1995a) and (1995b), Pham and Touzi (1996) and Wang (1998) for models with constraints on trading strategies. Dybvig and Huang (1988) and Delbaen, Monat, Schachermayer, Stricker and Schweizer (1997) have shown that that the absence of arbitrage can be enforced if gains from trade are bounded in the $\mathbf{L}^p$ norm.

[2] As shown by Delbaen and Schachermayer (1995) there is a difference between the absence of free lunches with vanishing risk and the absence of arbitrage. An investor has free lunches with vanishing risk whenever there exists a sequence of portfolio policies with associated discounted gains from trade bounded from below, such that discounted terminal wealth associated with the sequence of portfolio policies does not converge in probability to the initial wealth. In contrast a portfolio policy is an arbitrage if associated gains form trade a bounded from below with probability one and discounted wealth is bigger than initial wealth with positive probability. See section 1 for precise definitions.

[3] This result also explains how Elliott, Geman and Korkie (1997) prevent the existence of arbitrage opportunities for insiders by introducing insider information on incomplete

financial markets, but it seems less realistic that uncertainty about their private signal is never resolved given public information. If we add noise to the true signal we cannot analyze the effects of anticipative information which will be also known by non-insiders at some future point in time. The introduction of independent noise allows just do model noisy insider information such that the information advantage never disappears.

An extension of the fundamental theorem of asset pricing to differential information has previously been considered by Duffie and Huang (1986). They show that in a fully revealing rational expectation equilibrium where better informed agents do not have free lunches, insiders must fully reveal their information to guarantee the existence of a viable market model. Our result shows that their assumption about the existence of absolutely continuous local martingale measures for better informed agents holds true only if the difference of agents' flows of information is atomic when the information advantage can disappear.

We show that the absolutely continuous local martingale measure of an insider and of a non-insider are identical if restricted to public information. Consequently contingent claims whose payoff is measurable with respect to public information must also be valued identically. In contrast, since we show that contingent claims for insiders with non-atomic anticipative information have no value, option pricing by arbitrage is invariant to atomic but not to non-atomic anticipative information.

To prove the existence of an optimal trading strategy and therefore the existence of a viable market model for an investor with insider information poses additional difficulties. If we want to solve Merton's consumption-investment problem for such an investor, we face the problem that portfolio policies may depend on anticipative information. Consequently, if we allow for general continuous time investment strategies these may not be adapted to the filtration generated by returns of risky assets and therefore gains from trade can no longer be defined as Itô integrals [4]. Therefore, if we want to allow for general trading strategies we first have to find the representation of the processes relevant for the investment and consumption decision with respect to the insider's enlarged flow of information. If such a representation exists, we can allow for general portfolio policies adapted to the insider's filtration

information.

[4] One way to deal with this problem would be to use results from anticipative stochastic calculus and to define gains from trade as Skorohod integrals. The drawback of such an approach is that the semi-martingale property of wealth which is linked to the absence of free lunches is completely lost. In discrete time models the definition of gains from trade with anticipative strategies is still possible but the resulting process will not be a martingale.

and answer questions about the existence of free lunches simply by checking whether or not local martingale measures exist on the enlarged flow of information.

Arbitrage opportunities for non-atomic information are such that associated gains from trade can replicate any desired wealth process with zero cost. Therefore, an insider who has non-atomic anticipative information does not face any budget constraint and consequently will attain infinite expected utility[5]. Consequently no viable model can exist when the insiders investment horizon ends after his/her private information is completely revealed and insider information is non-atomic.

The solutions of Merton's consumption-investment problem for insider information previously presented in Karatzas and Pikovsky (1996a), Eliott, Geman and Korkie (1997) as well as Amendinger, Imkeller and Schweizer (1998) are based on von Neumann-Morgenstern preferences for terminal wealth with logarithmic utility. Since such preferences lead to myopic portfolio policies which can be obtained by a sequence of one-period optimization problems, they avoid questions about the existence of hedging portfolio policies which finance optimal cumulative consumption adapted to insider information. For logarithmic preferences we do not have to be concerned about the perfect replicability of the cumulative consumption. Since perfect hedging for insider information given as an initial enlargement of a Brownian filtration is still possible, we are able to solve Merton's consumption-investment problem for more general utility functions, independently of whether preferences are defined over terminal wealth, consumption or both.

All the results in this paper are based on partial equilibrium considerations. We focus on the consumer's problem who takes a certain flow of information as given. In this article we are not asking if this information is actually implementable in equilibrium. Clearly the insider's optimal strategies for atomic anticipative information can always be supported in a fully revealing rational expectation equilibrium. But since we show that optimal consumption and portfolio policies do not reveal all anticipative information, a Walrasian auctioneer will not learn all the insider information from the individual demands. Consequently, such an equilibrium will not exist without other channels of information transmission. Such issues will be discussed in more detail in Rindisbacher (1998).

Similarly our results have important consequences for many market microstructure models. Such models are generally principal-agent models where

[5]This result for non-atomic insider information therefore explains why Karatzas and Pikovsky (1996a) find infinite additional logarithmic utility from final wealth in a Gaussian model for signals corresponding to final states or final prices without noise.

the principal is the market maker and the agent is the insider. For the regulation of insider trading these models imply that the allocative efficiency is improved whenever a mechanism can be found by which insiders reveal more information as in any model with adverse selection. Our results about the non-existence of a viable market model for non-atomic insider information imply that this is not necessarily the case. For such signals full revelation of insider information will lead to a breakdown of markets for intertemporal risk sharing. In a companion paper (Rindisbacher (1998)) we show that this effect more generally known as the Hirshleifer (1971) effect in the insurance literature may play an important role if we want to analyze the social costs of insider trading [6].

The paper is organized as follows. In section 2 we introduce the models for public information and derive the representation of the price, endowment and wealth processes for the insider information. In section 3 we analyze whether or not anticipative information allows for arbitrage opportunities. Then in section 4 we consider the pricing of contingent claims for insiders. The results in section 4 are necessary for the existence of a solution of Merton's consumption-investment problem for an insider. Explicit expression for optimal consumption and portfolio policies for insiders are presented in section 5. In section 6 the results are illustrated with two examples. Section 7 concludes.

In Appendix A we show in detail how the model relevant for insiders can be obtained from the "Girsanov approach" to initial enlargements of filtrations. In Appendix B we show how random variables measurable with respect to an enlarged filtration can be represented as product-measurable functions of signals and states. The two appendices contain all results necessary to solve Merton's consumption-investment problem for an insider. Appendix C contains some definitions from Malliavin calculus. Finally we present the proofs of our results in Appendix D.

2 A MODEL FOR PUBLIC AND INSIDER INFORMATION

A financial market model can be characterized by $(\Omega, \mathcal{F}_1, \mathbb{F}, \mathbf{P}, \mathcal{C}, \succeq)$ where $(\Omega, \mathcal{F}_1, \mathbb{F}, \mathbf{P})$ corresponds to a stochastic basis consisting of a state space Ω, possible information $\mathcal{F}$, flow of information $\mathbb{F}$ and beliefs $\mathbf{P}$. The space $\mathcal{C}$

[6] We argue that this effect is the source for the findings of Back (1993), that in the presence of asymmetric information it may be impossible to price options by arbitrage and is not captured in principal-agent models for insider trading.

denotes the consumption space on which a preference ordering $\succeq$ is defined. In this section we first introduce the model for public information and then show how from this we obtain the model for the insider by initial enlargements of filtrations.

2.1 The Model for Public Information

We consider a frictionless market where each investor has the choice between d dividend paying risky assets and one asset without risk. Possible states of nature are given as points in a d-dimensional Wiener space $\Omega = C^0([0,1];\mathbb{R}^d)$. Possible information is given by the Borel σ- field $\mathcal{F}_1$ on Ω. Investors' beliefs are homogeneous and given by the standard Wiener measure $\mathbf{P}$, the measure for which observed states given as trajectories of the coordinate process $W = (\omega(t), t \in [0,1])$ on Ω correspond to a d-dimensional Wiener process. The flow of information $\mathbb{F} := (\mathcal{F}_t)_{t\in[0,1]}$ available to all investors is defined as the $\mathbf{P}$-completion of the Wiener filtration, the natural filtration of the coordinate process W.

Given the topology of the state space Ω, all processes on the stochastic basis $(\Omega, \mathcal{F}_1, \mathbb{F}, \mathbf{P})$ relevant for investors' decisions are given as Brownian functionals adapted to the flow of information which is given by the Wiener filtration $\mathbb{F}$.

The only risk free asset B (i.e. the predictable asset of bounded total variation) is defined as an exponential of a bounded adapted interest rate process $r = (r_t, t \in [0,1])$ by

$$B_t := \exp(\int_0^t r_s ds) \tag{2.1}$$

whereas the $(\mathbf{P}, \mathbb{F})$- Doob-Meyer decomposition of the d-dimensional vector of risky assets $P = ((P_t^j)_{j=1,\dots,d}; t \in [0,1])$ is given by

$$P_t + \int_0^t D_s ds = P_0 + \int_0^t diag[P_s^j][b_s ds + \sigma_s d\omega(s)] \tag{2.2}$$

where $D = ((D^j)_{j=1,\dots,d}; t \in [0,1])$ denotes the dividend process. This process is assumed to be exogenous and given by

$$D_t = D_0 + \int_0^t diag[D_s^j]\mu^D(s, D_s)ds + \int_0^t diag[D_s^j]\gamma^D(s, D_s)d\omega(s). \tag{2.3}$$

The other exogenously given process in the economy is the endowment rate process which satisfies the following stochastic differential equation

$$e_t = e_0 + \int_0^t \mu^e(s, e_s)ds + \int_0^t (\gamma^e(s, e_s))^* d\omega(s). \tag{2.4}$$

It follows that if the initial dividend D_0 is positive dividends will be always positive whereas endowments are allowed to be negative. For both equations we assume that coefficients satisfy global Lipschitz conditions that guarantee existence and linear growth conditions that are sufficient to get unique solutions [7]. Furthermore we assume that coefficients are differentiable to an appropriate degree.

The coefficients (r, b, σ) of the model are assumed to be bounded and $\mathcal{F}_t$-adapted. The volatility coefficient σ is assumed to be positive definite $\mathbf{P} \otimes \lambda$ a.e. on the product space $L^2([0,1] \times \Omega)$ of random functions, where λ corresponds to the Lesbegue measure. Furthermore we assume that $b_t^i \in \mathbb{L}^{1,2}(\mathbb{R}^q)$ and $\sigma_t^{i,j} \in \mathbb{L}^{1,2}(\mathbb{R}^q)$ for all $i, j \in \{1, \dots, d\}$ where $\mathbb{L}^{1,2}$ corresponds to the domain of the Skorohod integral which is defined in Appendix C[8].

An admissible trading strategy for risky assets is a d-dimensional adapted vector process $\pi = ((\pi^j)_{j=1,\dots,d}; t \in [0,1])$ such that $\pi \in \mathbb{L}_a^{1,2}(\mathbb{R}^q)$ where the subscript denotes the restriction to adapted processes in $\mathbb{L}^{1,2}$. To exclude free lunches from doubling strategies for an investment horizon $T \in [0,1]$ we assume that optimal strategies are tame[9], meaning that the corresponding wealth $X^{\pi,c}$ at any moment in time $t \in [0,T]$ must be bounded from below

$$\frac{X_t^{\pi,c}}{B_t e_0} \geq -K \tag{2.5}$$

$\mathbf{P}$- a.s. for some $K > 0$.

The consumption space is given by $\mathcal{C} = \mathbb{L}_a^{1,2}(\mathbb{R}^q)$. We assume that consumption is absolutely continuous with respect to the Lesbegue measure and can therefore be written as $C_t = \int_0^T c_s ds$. The wealth process $X^{\pi,c} = (X_t^{\pi,c}; t \in [0,1])$ satisfies the following stochastic differential equation.

$$X_t^{\pi,c} = e_0 + \int_0^t X_s^{\pi,c} r_s ds + \int_0^t (\pi_s)^* \sigma_s [\theta_s ds + d\omega(s)] - \int_0^t (c_s - e_s) ds, \tag{2.6}$$

where $\theta = (\theta_t; t \in [0,1])$ denotes the market price of risk or conditional Sharpe ratio, defined by $\sigma_t \theta_t := b_t - 1_d r_t$.

[7] see Nualart (1995) p.99 for example

[8] The assumptions made about price coefficients are stronger than required to solve the consumption-investment problem for Brownian information $\mathbb{F}$, but enable us to find explicit expressions for optimal portfolio policies from Malliavin's calculus. Furthermore, the techniques presented in this paper to get explicit hedging strategies for contingent claims measurable with respect to insider information apply to claims H for which Malliavin derivatives exist, that is $H \in \mathbb{D}^{1,1}(\mathbb{R}^q)$, where $\mathbb{D}^{1,1}$ denotes the domain of the Malliavin derivative defined in Appendix C.

[9] Tameness of portfolio policies is necessary to exclude doubling strategies (see Dybvig and Huang (1988))

Final wealth must be non-negative at the end of the investment horizon for strategies to be admissible

$$\frac{X_T^{\pi,c}}{B_T e_0} \geq 0. \tag{2.7}$$

Preferences $\succeq$ are of the von Neumann-Morgenstern type with an additive state independent utility function

$$U(T,c,\mathbf{P},\mathbb{F}) := \mathbf{E}^{\mathbf{P}}[\int_0^T u(s,c_s)ds|\mathcal{F}_0], \tag{2.8}$$

where the utility function $u \in C^{1,2}([0,T] \times \mathbb{R}_+;\mathbb{R})$ is strictly increasing and strictly concave. It also satisfies the Inada conditions $\lim_{c\to 0} \partial_2 u(t,c) = +\infty$ and $\lim_{c\to +\infty} \partial_2 u(t,c) = 0$. The corresponding absolute risk aversion is given by $A(t,c) := -\partial_2 \log \partial_2 u(t,c)$. The inverse of marginal utility I is defined by $\partial_2 u(t, I(t,y)) = y$. We require that investment-consumption strategies (π, c) satisfy the condition

$$\mathbf{E}^{\mathbf{P}}[\int_0^T u^-(t,c_t)dt|\mathcal{F}_0] < \infty \tag{2.9}$$

where $u^- := -min(0,u)$. Without this technical condition the consumption-investment problem is not well posed [10]. If (π, c) satisfies the budget constraints (2.6) as well as conditions (2.5), (2.7) and (2.9) we call $\mathcal{F}_t$-adapted (π, c) admissible and write $(\pi, c) \in \mathcal{A}(\mathbf{P}, \mathbb{F}, e)$ meaning that admissibility holds for beliefs $\mathbf{P}$, flow of information $\mathbb{F}$ and endowment process e.

Finally we introduce the following deflator process $S = (S_t; t \in [0,1])$ where

$$S_t := -\int_0^t r_s ds - \int_0^t (\theta_s)^* d\omega(s). \tag{2.10}$$

The corresponding state price density process $\mathcal{E}(S) = (\mathcal{E}(S)_t; t \in [0,1])$ can then be written as a stochastic exponential of the deflator process, that is as the unique solution of the following stochastic differential equation

$$\mathcal{E}(S)_t = 1 + \int_0^t \mathcal{E}(S)_v dS_v.$$

[10] see Karatzas, Lehocky, Sethi and Shreve (1986) for a discussion.

2.2 The Model for Insider Information

To analyze the effects of anticipative information on consumption and portfolio policies, we propose a model for the insider given by $(\Omega, \mathcal{F}_1, \mathbb{G}, \mathbf{P}, \mathcal{C}, \succeq)$, where his/her flow of information $\mathbb{G} := (\mathcal{G}_t)_{t\in[0,1]}$ is obtained by an initial enlargement of filtration

$$\mathcal{G}_t := \bigcap_{\epsilon>0} [\mathcal{F}_{t+\epsilon} \bigvee \sigma(G)], \quad t \in [0,1], \tag{2.11}$$

with G a q-dimensional random vector with law $\mathbf{P}_G$. The random vector G corresponds to the signal which, at the beginning of the investment horizon, is only known by the insider. Signals previously considered in the literature correspond to terminal states of nature (Karatzas and Pikovsky (1996a) and Elliott, Geman and Korkie (1997)), terminal values of risky assets (see the articles in Grossman (1990), Kyle (1989) and Karatzas and Pikovsky (1996a)) or indicators of these variables (Karatzas and Pikovsky (1996a)). In what follows we allow for any kind of $\mathcal{F}_1$- measurable signals.

To the signal G we can associate the resolution time $T_G := inf\{t \in [0,1] : \mathbf{E}[\mathbf{1}_E|\mathcal{F}_t] = \mathbf{1}_E \forall E \in \mathcal{G}_t\}$, meaning that the stopping time T_G designates the first moment in time at which the information advantage about the signal G that was known at the beginning ($t = 0$) has disappeared. Clearly, the signal G is $\mathcal{F}_{T_G}$ measurable.

In Appendix A we show in detail how we obtain the representation of processes on the stochastic basis $(\Omega, \mathcal{F}_1, \mathbb{G}, \mathbf{P})$ relevant for the insider's investment decision. The assumptions used to derive these decompositions and the results in the following sections are as follows

Assumption 1 ***("Condition A" Jacod (1980))*** *There exists a common measure ν on the Borel field $\mathcal{B}_{R^q}$ such that $\mathbf{P}_t^\omega \ll \nu$ for all $t \in [\![0, T_G[\![$ where $\mathbf{P}_t^\omega$ corresponds to the conditional law of G given the initial filtration $\mathcal{F}_t$.*

This assumption basically guarantees the stability of semi-martingales under initial enlargements of filtrations ("Hypothèse H" in Jacod (1980)) on the stochastic interval $[\![0, T_G[\![$. It is necessary since if the insider has a private signal such that prices with respect to the enlarged flow of information are not any longer semi-martingales no viable market model exists. As shown by Delbaen and Schachermayer (1994) the semi-martingale property is necessary for the absence of free lunches. Our assumptions on the coefficients (r, b, σ) guarantee that after resolution time all processes relevant for the insider's investment decisions are semi-martingale since on $[\![T_G, 1]\!]$ as we will see below the insider's model is identical to the model for public information.

If "condition A" is satisfied it follows from the Radon-Nikodym theorem that $\mathbf{P}_t^\omega(dz) = p(\omega,t,z)\nu(dz)$ for all $t \in [\![0,T_G[\![$. For $t = 0$ we get the unconditional density of the signal as Radon-Nikodym derivative $\mathbf{P}_0^\omega(dz) = q(z)\nu(dz)$, that is $q(z) := p(\omega,0,z)$. If such a measure ν exists we can with out loss of generality assume that it corresponds to the unconditional law of the signal $\nu = \mathbf{P}_G$. The following assumption from Imkeller (1996) guarantees that the contemporaneous Malliavin derivative used to represent the conditional density process as a non-negative martingale is well defined.

Assumption 2 *The conditional density of the signal $p(\omega,t.z)$ is such that (i) $p(\omega,s,z) \in \mathbb{L}^{1,2}(\mathbb{R}^d)$ as well as (ii) the mapping $r \mapsto \mathcal{D}_r^j p(\omega,s,z)$ is left-continuous in $L^1(\Omega)$ at $s \in [0,t]$ and $z \in \mathbb{R}^q$, for all $j \in \{1,\cdots,d\}$, where $\mathcal{D}_t^j p(\omega,t,z)$ denotes the Malliavin derivative of the conditional density.*

This assumption will be used to derive the drift of the processes relevant for the insider's investment decision. Given the conditional density of the signal it allows to define the process $\alpha_t^z(\omega) := \lim_{s\uparrow t} \frac{\mathcal{D}_s p(\omega,t,z)}{p(\omega,t,z)}$. As we explain in Appendix A this process is used to derive the $(\mathbf{P},\mathbb{G})$- Brownian motion W^G given in (A.34). The Brownian motion W^G on the insider information is such that $dW_t^G = d\omega(t) - \alpha_t^G dt$ for $t \in [\![0,T_G[\![$ and $dW_t^G = d\omega(t)$ for $t \in [\![T_G,1]\!]$.

Finally the next assumption is necessary for $(\mathbf{P},\mathbb{G})$- semi-martingales to exist up to the moment at which the information advantage of the insider has disappeared.

Assumption 3 *Signals G are such that*

$$\int_0^{T_G} \left| \left[\frac{\mathcal{D}_t^j p(\omega,t,z)}{p(\omega,t,z)} \right]_{|z=G(\omega)} \right| ds < \infty \tag{2.12}$$

$\mathbf{P}$-a.s. for all $j \in \{1,\cdots,d\}$, where $\mathcal{D}_t^j p(\omega,t,z)$ denotes the Malliavin derivative of the conditional density.

Without this assumption price processes for insider information are not semi-martingales on $[\![0,T_G[\![$. Consequently the insider will have free lunches and no viable model for such investors exists.

The following theorem uses the results derived in Appendix A and provides the market model from the point of view of the insider.

Theorem 1 *For insider signals G such that assumption 1, 2 and 3 are satisfied, the representation of price, dividend rate and endowment processes on* *the stochastic basis relevant for the insider* $(\Omega, \mathcal{F}_1, \mathbb{G}, \mathbf{P})$ *is for* $t \in [0,1]$ *as follows.*

Risk free assets are given by:

$$B_t = \exp(\int_0^t r_s ds). \tag{2.13}$$

The prices of risky assets P *satisfy:*

$$P_t + \int_0^t D_s ds = P_0 + \int_0^t diag[P_s^j][b_s + \sigma_s dW_s^G] + \int_0^{t\wedge T_G} \sigma_s \alpha_s^G ds, \tag{2.14}$$

where the dividend process D *is given by*

$$D_t = D_0 + \int_0^t diag[D_s^j][\mu^D(s, D_s)ds + \gamma^D(s, D_s)dW_s^G] + \int_0^{t\wedge T_G} \gamma^D(s, D_s)\alpha_s^G ds. \tag{2.15}$$

The endowment rate satisfies the following stochastic differential equation

$$e_t = e_0 + \int_0^t [\mu^e(s, D_s)ds + (\gamma^e(s, e_s))^* dW_s^G] + \int_0^{t\wedge T_G} \gamma^e(s, e_s)^* \alpha_s^G ds. \tag{2.16}$$

For strategies $(\pi, c) \in \mathcal{A}(\mathbb{G}, \mathbf{P}, e)$ *such that* $\int_0^{T_G} |\pi_s^* \sigma_s \alpha_s^G| ds < +\infty$ $\mathbf{P}$- *a.s. we get the following discounted wealth process*

$$\frac{X_t^{\pi,c}}{B_t} = e_0 + \int_0^t \frac{(\pi_s)^* \sigma_s}{B_s}[(\theta_s + dW_s^G] - \int_0^t \frac{(c_s - e_s)}{B_s} ds + \int_0^{t\wedge T_G} \frac{(\pi_s)^* \sigma_s \alpha_s^G}{B_s} ds, \tag{2.17}$$

We see that anticipative information affects the insider's view about the individual market price of risk, the appreciation rate of the dividend and price processes of the risky assets as well as the growth rate of the endowments in a way which depends on the "contemporaneous elasticity with respect to changes of the state of nature" (i.e. logarithmic Malliavin derivative) of the signal's conditional density.

The anticipative information does not change the quadratic variation of the processes relevant for the portfolio choice. If we compare the price processes of insiders and non-insiders we see that they agree on the volatility of the asset prices but not on their appreciation rates. A priori this seems to be surprising but it simply stems from the fact that for price processes given as semi-martingales the quadratic variation is locally already known by non-insiders (i.e. $\mathbb{F}$- predictable). As a consequence additional information cannot locally reduce the conditional volatility of the process. Exactly for the same reason the difference in information lets the price process of a risk free (i.e. predictable) asset unchanged.

Since the initial information of an insider $\mathcal{G}_0 = \sigma(G)$ is, in contrast to the public information $\mathcal{F}_0 = \{\Omega, \emptyset\}$ non trivial, his/her beliefs are not any longer given by $\mathbf{P}$ but by the conditional probabilities $\mathbf{P}(\cdot|\mathcal{G}_0)$. In Appendix A we show that $\mathbf{P}(\cdot|\mathcal{G}_0) = \mathbf{P}^z_{|z=G}$ where $\mathbf{P}^z$ denotes the conditional Wiener measure which concentrates its probability mass on $\{G = z\}$. The conditional Wiener measure is obtained from the joint law of the signal and states of nature, defined on the product space $\Omega \times \mathbb{R}^d$. Therefore, states relevant for an insider can be described in form of pairs (ω, G) and beliefs take all its probability mass along the diagonal of the product state space. In what follows this fact will play an important role. Because of a decoupling property of the insider's local martingale measure, we can derive optimal hedging policies first by conditioning on the realization of the signal. If evaluated at the true signal, these policies will be shown to be optimal for an insider.

We have already seen that anticipative information does not affect the volatility coefficient of risky assets. The following corollary gives necessary and sufficient conditions which guarantee that the drift coefficients of the processes relevant for the portfolio choice remain unchanged.

Corollary 1 *Under the conditions of theorem 1 an insider's decompositions of price, dividend and endowment processes on* $(\Omega, \mathcal{F}_1, \mathbb{F}, \mathbf{P})$ *and* $(\Omega, \mathcal{F}_1, \mathbb{G}, \mathbf{P})$ *are the same if and only if for all* $t \in [\![0, T_G[\![$ *the information generated by the signal is independent from the common available information.*

$$\mathcal{F}_t \perp \sigma(G) \tag{2.18}$$

It follows that information which is independent of the flow of common information for given beliefs is irrelevant for the insider's investment decision since it does not change his/her wealth process. Then if the events revealed by his/her side information do not increase knowledge about initial pay-off relevant events, the observation of such a signal does not reduce uncertainty concerning states of nature relevant for his/her consumption and investment decision. Such information is therefore simply not taken into account. Insider information only affects optimal strategies if it helps to get a finer flow of information concerning events the he cares about. We will call independent signals redundant, meaning that they are irrelevant for investors' decisions. The redundancy of independent signals implies that consumption and investment strategies are unaffected by idiosyncratic shocks which are independent from the common flow of information. Consequently, such shocks will have no impact on equilibrium prices [11].

3 INSIDER INFORMATION AND ARBITRAGE

Based on the insider's model obtained in theorem 1 we now investigate whether or not an insider has necessarily free lunches. We first define two notions of free lunches previously considered in the literature: the concept of free lunches with vanishing risk and the concept of arbitrage opportunities. We also introduce the concept of conditional arbitrage which seems more appropriate when initial information is non-trivial as for an insider. Then we explain how we distinguish between atomic and non-atomic insider information. The next section shows that dependent on the investment horizon these distinctions matter for an insider. Our results illustrate that it is the precision of the anticipative information about the state of nature which provides an insider with arbitrage opportunities and not the fact that he has more information than a non-insider alone.

3.1 Definitions

If assumption 3 is not satisfied, prices of risky assets are not any longer semi-martingales and as a consequence it follows from results in Delbaen and Schachermayer (1994) that the insider has free lunches with vanishing risk, that is their NFLVR condition is not satisfied. In our context we can state the NFLVR condition for an investment horizon T as follows.

Definition 1 *The price process of risky assets satisfies the NFLVR condition if for all sequences of portfolio strategies $(\pi^n)_{n\in N}$ with associated wealth process such that*

$$\frac{X_T^{\pi^n,e}}{e_0 B_T} < +\infty \tag{3.1}$$

$\mathbf{P}$*-a.s. and positive sequences $(\delta_n)_{n\in N}$ converging to 1 such that*

$$\frac{X_t^{\pi^n,e}}{e_0 B_t} > \delta_n, \tag{3.2}$$

[11] See Rindisbacher (1998) for more on this point.

$\mathbf{P}$- *a.s. for all* $t \in [0, T]$, *we have that*

$$\mathbf{P} - \lim_{n \to +\infty} \frac{X_T^{\pi^n, e}}{e_0 B_T} = 1. \tag{3.3}$$

Condition (3.1) excludes portfolio policies for which wealth is not defined and the sequence δ^n in (3.2) guarantees that gains from trade are bounded from below and therefore excludes doubling strategies. As shown in Delbaen and Schachermayer (1994) and also in Back and Pliska (1991) this condition is more restrictive than the no arbitrage condition. Insiders do not necessarily have arbitrage opportunities if they have free lunches with vanishing risk. As we will show below the distinction between free lunches with vanishing risk and arbitrage opportunities is important for insider information.

In our model we define an arbitrage opportunity as follows

Definition 2 *A tame portfolio policy* π *with associated wealth process* $X^{\pi,e}$ *such that*

$$\mathbf{P}(\frac{X_T^{\pi,e}}{e_0 B_T} \geq 1) = 1 \tag{3.4}$$

and

$$\mathbf{P}(\frac{X_T^{\pi,e}}{e_0 B_T} > 1) > 0 \tag{3.5}$$

for some $T \in [0, 1]$ *is called an arbitrage.*

This basically states that if gains from trade are almost surely believed to be non-negative but not strictly positive there is no arbitrage. Since convergence in probability does not imply almost sure convergence a free lunch is not necessarily an arbitrage. Behind both definitions is the idea that a reasonable model of the financial market should not allow investors to make positive gains from trade out of nothing.

The definitions above previously considered in the literature are based on the assumption that the information of the investor at the beginning of his/her investment horizon corresponds to the trivial information set $(\Omega, \emptyset)$. Since we want to analyze the flow of information of an insider who has anticipative information given by $\sigma(G)$ already at the initial date, we need a notion of conditional arbitrage.

Definition 3 *A conditionally tame portfolio policy (*π *such that for all* $t \in [0, 1]$ *we* $\mathbf{P}$- *a.s. have that* $\mathbf{P}(\frac{X_t^{\pi,c}}{B_t e_0} \geq -K | \mathcal{G}_0) = 1$*) with associated wealth process* $X^{\pi,e}$ *such that* $\mathbf{P}$- *a.s.*

$$\mathbf{P}(\frac{X_T^{\pi,e}}{e_0 B_T} \geq 1 | \mathcal{G}_0) = 1 \tag{3.6}$$

and for some $E \in \mathcal{G}_0$ *where* $\mathbf{P}(E) > 0$

$$\mathbf{P}(\frac{X_T^{\pi,e}}{e_0 B_T} > 1 | \mathcal{G}_0) > 0 \tag{3.7}$$

for some $T \in [0, 1]$ *is called a conditional arbitrage.*

Therefore, for an absence of arbitrage it is necessary that there is no arbitrage given any event known at the beginning of the investment horizon, whereas for a conditional arbitrage to exist it is sufficient that gains from trade are believed to be positive given the initial information. This shows that the absence of arbitrage is a stronger assumption than the absence of conditional arbitrage.

Proposition 1 *For a given flow of information an investor who has a conditional arbitrage has necessarily an arbitrage, whereas an investor who has an arbitrage may not have a conditional arbitrage.*

This proposition does not compare arbitrage opportunities for insiders and non-insiders since it assumes that portfolio policies are adapted to the same flow of information. It implies that it is sufficient to prove the absence of an arbitrage for the enlarged flow of information to guarantee the existence of a viable model for the insider.

As shown by Harrison and Kreps (1979) it is necessary for the existence of a pricing kernel that prices any asset that there is no arbitrage. In what follows we analyze how this result depends on the flow of information available to investors. We will show how the informational content of the insider signal determines whether or not an insider has arbitrage opportunities. It will be important to distinguish between atomic and non-atomic anticipative information

Definition 4 *An investor has atomic insider information if there exists an event* $E \in \mathcal{G}_0$ *such that* $\mathbf{P}(E) > 0$ *and for all* $F \in \sigma(G)$ *we have that* $F \subset E$ *implies either* $F = E$ *or* $F = \emptyset$*. If there does not exist such an event* E *we say that the insider information is non-atomic.*

Clearly if there exists a countable partition of the state space which generates the anticipative information $\sigma(G)$ then there exists a countable number of atoms. In contrast an investor who has non-atomic insider information knows whether or not an event has occurred out of an uncountable number of events. In this sense atomic insider information is less informative than non-atomic insider information. Signals with discrete distributions generate atomic insider information, whereas signals with continuous distributions generate non-atomic insider information. Non-atomic insider information in contrast to atomic insider information contains events whose occurrence is not believed given public information. This fact will be crucial for whether or not an insider has an arbitrage opportunity.

3.2 Main Result

We now show that the distinction between different concepts of free lunches for insiders matters. The main result of this section shows how the existence of arbitrage opportunities for insiders does depend on the complexity of their anticipative information.

Theorem 2 *Under the assumptions of theorem 1 we have for insider information which is not independent of the common available flow of information and non-trivial ($\sigma(G) \neq \{\Omega, \emptyset\}$) that*

1. *Any investor who knows the information revealed by G and who has an investment horizon such that $T \in [\![T_G, 1]\!]$ has necessarily free lunches with vanishing risk.*
2. *Any investor who knows the information revealed by G has no arbitrage if and only if either $T \in [\![0, T_G[\![$ or $T \in [\![T_G, 1]\!]$ and insider information $\mathbb{G}$ is atomic.*

As it follows from the proof, the results of the theorem are basically a consequence of the fact that the unconditional law of the signal cannot any longer be absolutely continuous with respect to the conditional law of the signal when the uncertainty about the insider information is resolved. This implies that the state price density process which depends on the conditional and unconditional law of the signal is not anymore strictly positive after the realization of the signal is known to the public. It follows that there can exist at most an absolutely continuous martingale measure. On the other hand if there is no free lunch with vanishing risk the density process of the local martingale measure must be strictly positive [12] and therefore the unconditional law of the signal would be absolutely continuous with respect to the conditional law after resolution time. Since this is only possible if the signal is constant there must exist free lunches with vanishing risk for such investment horizons. Furthermore since if there does not exist any atoms, the conditional and unconditional laws of the insider signal are even mutually singular, not even an absolutely continuous local martingale measure will

[12] See also Delbaen and Schachermayer (1995).

exist when the investment horizon does not end before his/her information advantage has disappeared. In contrast, we show that the equivalence of the conditional and unconditional law of the signal before the information advantage is lost, also excludes arbitrage opportunities if the investment horizon ends before the insider information is fully known by the other investors.

If an insider is only allowed to trade before the resolution time, he cannot attain arbitrage opportunities. Similarly, if he is not allowed to trade before the uncertainty about the information contained in his signal is resolved he has no anticipative information and cannot obtain an arbitrage. We show in the proof that a mean-variance demand for risky assets at the resolution time and zero elsewhere is an arbitrage if insider information is non-atomic [13]. Therefore, if an insider is able to realize arbitrage opportunities he can do this with strategies which trade just immediately before his/her information advantage has disappeared. This implies that any effective regulation of insider trading must prevent insiders from trade contingent upon anticipative information on a time span that includes resolution time.

The fact that for insiders without restrictions on the investment horizon only absolutely continuous local martingale measures exist, seems problematic from the point of view of financial innovation. In this case there exist binary options paying zero dollars or one dollar contingent upon events of the form $\{G = z\}$ some $z \in \mathbb{R}^d$ which are known not to occur by the insider only. Such a claim has no value for insiders but a positive value for non-insiders. There will be an infinite offer for such claims and consequently infinite profits for insiders issuing them. But to develop such an argument we must take into account that the marketing of those claims will necessarily change the distribution of information in the economy since the claim is written on an event unknown to non-insiders. This will affect the flow of information and therefore the set of local martingale measures, for non-insiders in such a way that these claims may have no value for initial non-insiders as well.

In many market models following the work of Grossman (see the articles in Grossman (1990)) it is assumed that the insider's signal is given by the

[13] The existence of an absolutely continuous martingale measure depends critically on the behavior of $\lim_{T \mapsto T_G} \int_{]t,T]} \|\theta_s + \alpha_s^z\|^2 ds$ some $t \in [\![0, T_G[\![$ under the conditional Wiener measure $\mathbf{P}^z$ as it follows from the arguments in Delbaen and Schachermayer (1995) and Levental and Skorohod (1995) . If the signal is discrete this expression reaches smoothly $+\infty$ whereas for non-atomic signals it jumps to $+\infty$. In the proof of their main result they use that the existence of an absolutely but non-equivalent local martingale measure is equivalent to the NFLVR condition under a probability which restricts its positive mass on events on which we have absolute continuity. In our model these events are given by $\{G = z\}$ which are of positive probability only if the signal is discrete. As a consequence a local martingale measure will only exist in this case.

"true signal" G^0 perturbed by some independent noise Z. The following corollary shows that this prevents arbitrage opportunities for the insider.

Corollary 2 *Under the assumptions of theorem 1 we have for signals G^0 which are perturbed by independent noise Z*

$$G = G^0 + Z \tag{3.8}$$

that there is no arbitrage opportunity.

As we show in the proof, in the presence of independent noise, the insider's information advantage never disappears. From this point of view the result of the corollary seems surprising. But since the information advantage always exists, the information contained in the true signal G^0 is never fully revealed and as a consequence perturbed insider information is less informative than true anticipative information and does not allow for arbitrage opportunities. This illustrates that it is not the fact that an insider has more information than the non-insider which provides him arbitrage opportunities but the precision of the information about the states of nature contained in his/her signal.

Before we show how theorem 2 explains why Merton's consumption-investment problem for insiders' having non-atomic anticipative information has no solution we prove that claims adapted to insider information can be replicated without tracking error. This result is crucial for the existence of a solution to the portfolio choice problem with "non-myopic" preferences. It guarantees that insiders can finance any consumption policy adapted to their enlarged flow of information.

4 INSIDER INFORMATION AND HEDGING OF CLAIMS

In this section we consider the effects of anticipative information on the valuation of contingent claims. Since an insider has an enlarged set of admissible portfolio and consumption strategies he is able to replicate every claim a non-insider can synthetize and therefore his/her willingness to pay for a contingent claim is necessarily bounded from above by the highest price at which a non-insider is willing to buy. We first prove that an investor can replicate any claim measurable with respect to his/her enlarged information. This is the key result to solve Merton's consumption-investment problem in the next section. Then we show that the valuation of contingent claims by arbitrage is invariant with respect to atomic insider information. In contrast the implicit price of contingent claims is zero if anticipative information contains zero probability events given public information.

4.1 Claims measurable with respect to Insider Information

Since we want to use the results presented in this section to solve Merton's consumption-investment problem in a non-Markovian market for an insider we have to allow for claims whose pay-off at maturity may be uncertain for non-insiders. From a technical point of view the basic problem is that in contrast to a Wiener filtration it is not clear whether any $(\mathbf{P}, \mathbb{G})$ local martingale can be written as a stochastic integral with respect to the $(\mathbf{P}, \mathbb{G})$-Brownian motion W^G. If $\mathbb{G}$ does not have the predictable representation property, hedging without tracking error will not be possible for contingent claims which are measurable with respect to the enlarged filtration only. As a consequence the martingale techniques used to solve the consumption-investment problem in a non-Markovian market will not work for general convex preferences other than logarithmic utility for final wealth.

Karatzas and Pikovsky (1996b) have shown that for enlargements with random vectors G such that $G = \eta + \int_0^1 g(t)d\omega(t)$ where η is a random vector independent of $\mathcal{F}_1^\omega$ and $g(\cdot)$ is a deterministic matrix function locally bounded in the operator norm and such that $\int_0^1 \|g(t)\|^2 dt < \infty$, any local $(\mathbb{P}, \mathbb{G})$ martingale starting at zero can be represented as a stochastic integral with respect to W^G. It follows that $\mathcal{G}_T$- measurable random variables ("contingent claims") that can be written as the sum of a random variable which lies in the first Wiener chaos plus a random variable which is independent of the flow of publicly available information can be perfectly replicated if markets are complete. We show that this remains true for initial enlargements and contingent claims that are sufficiently smooth.

The basic idea exploited to hedge a $\mathcal{G}_T$- measurable contingent claim H uses theorem B.1, where we show that claims adapted to insider information can be written as $\mathcal{B}_{R^q} \otimes \mathcal{F}_T$- measurable random function $C^z(\omega)$ given as shown by

$$C^z(\omega) = \mathbf{E}[H|\mathcal{F}_T](\omega) + \mathbf{E}^{\mathbf{P}^z}[\int_T^{T_G} \mathbf{E}[\mathcal{D}_t H|\mathcal{F}_t]\alpha_t^z dt|\mathcal{F}_T](\omega) \tag{4.1}$$

$\mathbf{P}$- a.s.. As a consequence we are able to replicate contingent claims in two steps. First conditional on the event $\{G = z\}$ for some $z \in \mathbb{R}^d$ we get hedging strategies to hedge C^z. The hedging strategies obtained this way are measurable functions of the realization of the signal. If we evaluate these functions at the true signal $z = G$ we get hedging strategies for H since the absolutely continuous local martingale measure of the insider decouples the signal and the states of nature. The associated minimal cost of the replicating strategies provides the implicit price of the claim.

Theorem 3 *If "condition A" and assumption 2 are satisfied then*

1. *For any investment horizon $T \in [\![0, T_G[\![$ a $\mathcal{G}_T$- measurable contingent claim H such that $\mathcal{E}(S)_T C^z \in \mathbb{D}^{1,1}(\mathbb{R}^q)$ for all $z \in \mathbb{R}^q$ where C^z is given by (4.1) can be perfectly hedged*

$$X_t^{\hat{\pi},\hat{c}} = Y_t \quad \mathbf{P} \otimes \lambda \ a.e., \tag{4.2}$$

where $Y_t = (Y_t : t \in [0, T])$ denotes the value process of the contingent claim given by

$$Y_t = \mathbf{E}^{\tilde{\mathbf{Q}}}[\frac{B_t}{B_T} H | \mathcal{G}_t], \tag{4.3}$$

and where $\tilde{\mathbf{Q}}$ corresponds to the insider's local martingale measure given by

$$\frac{d\tilde{\mathbf{Q}}}{d\mathbf{P}^z}_{|\mathcal{G}_t, z=G} = \frac{q(G)}{p(\omega, t, G)} \frac{d\mathbf{Q}}{d\mathbf{P}}_{|\mathcal{F}_t}.$$

Corresponding replicating strategies $(\hat{\pi}, \hat{c})$ are given by

$$\hat{c}_t = e_t \tag{4.4}$$

for consumption and

$$(\hat{\pi}_t)^* = \mathcal{D}_t \mathbf{E}^{\mathbf{P}^z}[\frac{\mathcal{E}(S)_T}{\mathcal{E}(S)_t} C^z | \mathcal{F}_t]_{|\{z=G\}} \sigma_t^{-1} \tag{4.5}$$

for portfolio investments.

2. *For an investment horizon $T \in [\![T_G, 1]\!]$ the contingent claim has a positive implicit price if and only if insider information $\mathbb{G}$ is atomic.*

It follows from the expression for the hedging strategy (4.5) that the fraction of wealth that has to be invested in each risky asset is given by the contemporaneous elasticity of the value process (4.2) with respect to changes in the state of nature multiplied by the inverse of the elasticity of prices with respect to changes in the state of nature

$$\frac{\hat{\pi}_t}{X_t^{\hat{\pi},\hat{c}}} = ((\mathcal{D}_t[[\log P_t^j]]_{j=1,\dots,d})^*)^{-1}(\mathcal{D}_t \log Y_t)^*. \tag{4.6}$$

From lemma B.1 and the proof of the theorem we see that the initial price an investor with an enlarged flow of information is willing to pay for the contingent claim given by $\mathbf{E}[\mathcal{E}(-\int_0^{\cdot}(\alpha_s^z)^* dW_s^G)_T \mathcal{E}(S)_T H|\mathcal{G}_0]$ corresponds to the price an investor who is "gambling" upon the event $\{G = z\}$ is willing to pay $\mathbf{E}[\mathcal{E}(S)C^z]$ evaluated at $z = G$. Furthermore as we show in the proof a non-insider faces a tracking error given by $\int_T^{T_G} \mathbf{E}[\mathcal{D}_u[\mathcal{E}(S)_T H|\mathcal{F}_u]d\omega(u)$ even if there are no constraints and markets are dynamically complete. This stems from the fact that the claim considered in the theorem is supposed to be $\mathcal{G}_T$- and not $\mathcal{F}_T$- measurable and therefore at maturity non-insiders do not know its actual pay-off with certainty.

4.2 Claims measurable with respect to Public Information

Claims such that even at maturity their pay-off is unknown for common available information exist whenever there are some non-traded goods in the economy. But the existence of a market for options with uncertain pay-offs even at maturity for some investors may be unrealistic. Independently, The results of the previous theorem have their own interest since they will be used for the solution of the consumption-investment problem for insiders in the next section.

If insiders' information is revealed through market prices in equilibrium, contingent claims written on prices will in fact be $\mathcal{G}_T$- and not $\mathcal{F}_T$- measurable. In this case theorem 3 provides the relevant pricing formulas and hedging strategies.

For contingent claims whose cash flow at maturity is known also for non-insiders we have the following result:

Theorem 4 *In the absence of an arbitrage opportunity an insider and a non-insiders have the same valuation for a claim* $H \in \mathbf{L}^1(\Omega, \mathcal{F}_T, \mathbf{P})$. *Its implicit price is given by*

$$Y_0 = \mathbf{E}[\mathcal{E}(S)_T H]. \tag{4.7}$$

It follows that if the contingent claim is $\mathcal{F}_T$- adapted its pay-off structure does not depend on the random element the flow of information is sharpened with. Its value is then the same for investors knowing the random vector G, and for investors who just have common available information to replicate the contingent claim. Obviously, replication costs of insiders are not higher than those of non-insiders since they can replicate the contingent claim by strategies that depend on the coarser common available flow of information.

The reason why prices of $\mathcal{F}_T$- measurable contingent claims are unaffected by insider information follows from the fact that absolutely continuous local martingale measure $\tilde{\mathbf{Q}}$ coincides on $\mathcal{F}_T$ for $T \in [\![0, T_G[\![$ with the equivalent local martingale measure $\mathbf{Q}$ of investors having just public information.

As we have seen in theorem 1 enlargements of filtrations do not affect the quadratic variation of the prices of risky assets. Consequently the invariance of the implicit price with respect to insider information is basically just a consequence of the well-known fact first discovered for the Black and Scholes formula that option prices do not depend on the drift coefficient of the risky assets. Since as we have seen initial enlargements of filtrations can be derived from a Girsanov transformation with respect to a conditional measure this result is not surprising since Girsanov transformations do not affect the quadratic variation which implies invariance of the option prices with respect to changes of measures and initial enlargements of filtrations.The invariance with respect to changes of filtrations and heterogeneity of equivalent beliefs will disappear if perfect replication of the claim is impossible or as we have seen if insider information is non-atomic such that no local martingale measure for the insider will exist. In this case the investor with insider information may be capable of constructing an investment strategy with smaller tracking error or with the same tracking error but higher a probability to replicate the pay-off structure of the claim than the non-insiders. To analyze such effects we need to relax the assumption that initial available information is given by a Wiener filtration.

Duffie and Huang (1986) have shown that it is necessary for a fully revealing rational expectation equilibrium to exist that the hedging costs of agents having ordered flows of information be the same. Otherwise better informed investors would have an arbitrage opportunity. Our result shows that if the difference of flows of information is atomic an insider and a non-insider agree on the implicit price of the contingent claim, independently of whether or not the insider takes into account that his/her information will be fully revealed through equilibrium prices. It follows that their conclusion, that the advantage of better information is not such that the costs of hedging are lower for the better informed, but such that the set of claims that can be replicated is enlarged, does not depend on the fact that the information about the insider signal is contained in equilibrium prices. Our analysis shows that it is already a consequence of the fact that the absolutely continuous local martingale measure of an insider does correspond to the risk neutral probability of a non-insider on public information.

On the other hand theorem 2 shows that the conclusions of Duffie and Huang are only valid if the differences in the information flows are atomic or never disappear. Then whenever an insider has non-atomic anticipative information and an investment horizon which includes the first moment in time his/her insider information will be known publicly, any contingent claim can be replicated with zero initial capital, and as a consequence asset prices in a fully revealing rational expectations equilibrium will necessarily be zero.

5 INSIDER INFORMATION, PORTFOLIO AND CONSUMPTION POLICIES

In this section we solve Merton's consumption-investment problem for an investor with information given by a Wiener filtration and additional information generated by a $\mathcal{F}_{T_G}$- measurable random vector G. Then we derive explicit expressions for portfolio policies, which show that the insider's demand for risky assets can be decomposed in a part which is dependent on the state price density, another part which depends on the endowment process and a last part which purely depends on his/her anticipative information. Finally, we analyze the information about insider signals contained in optimal strategies.

5.1 Merton's Problem for an Insider

Since we have shown in the previous section that smooth claims measurable with respect to enlarged information can be perfectly hedged we can generalize previous results of of Karatzas and Pikovsky (1996a), Elliott, Geman and Korkie (1997) and Amendinger, Imkeller and Schweizer (1998) and allow for more general preferences and for consumption before the final period.

The approach presented here differs from those previously considered by the choice of a conditional criterion function, that is we maximize expected utility of consumption conditional on initial information. Furthermore we assume that insiders do not care about final wealth. Given that initial Wiener filtrations are trivial this is equivalent to unconditional optimization if there is no side information. As it can easily be seen optimal policies for the conditional criterion must be optimal for the unconditional one. As a consequence the marginal value of wealth for insiders in our model will be state dependent.

In our model Merton's consumption investment problem with admissible strategies for an insider $(\pi, c) \in \mathcal{A}(\mathbf{P}, \mathbb{G}, e)$ can be formulated as follows.

$$\sup_{(\pi,c)\in\mathcal{A}(\mathbf{P},G,e)} \mathbf{E}[\int_0^T u(v, c_v)dv|\mathcal{G}_0]. \tag{5.1}$$

The following assumption is necessary for the existence of a solution.

Assumption 4

$$\mathcal{X}(y, z) < \infty \tag{5.2}$$

for all $z \in \mathbb{R}^q$ *where*

$$\mathcal{X}(y,z) := \mathbf{E}^{\mathbf{P}^z}[\int_0^T \frac{q(z)}{p(\cdot,t,z)}\mathcal{E}(S)_v I(t, y\mathcal{E}(S)_v \frac{q(z)}{p(\cdot,v,z)})dv] \tag{5.3}$$

Merton's consumption-investment problem without liquidity constraints in an economy with future endowments can be interpreted as a problem in an economy where initial liquid wealth is given by the implicit price of cumulative endowments at the initial date and no future endowments. If there are no arbitrage opportunities marketed wealth e^G at time zero for an insider is as follows

$$e^G = e_0 + \mathbf{E}^{\tilde{\mathbf{Q}}}[\int_0^T \frac{e_s}{B_s}ds|\mathcal{G}_0]. \tag{5.4}$$

Similarly, for conditional beliefs $\mathbf{P}^z$ we define the process $Y^z = (Y_t^z; t \in [0,T])$ by

$$Y_t^z := \mathbf{E}^{\mathbf{P}^z}[\int_t^T \frac{\mathcal{E}(S^z)_v}{\mathcal{E}(S^z)_t}(\hat{c}_v^z - e_v)dv|\mathcal{F}_t], \tag{5.5}$$

and where the conditional state price density $\mathcal{E}(S^z)$ is given by

$$\mathcal{E}(S^z)_t = \frac{q(z)}{p(\omega,t,z)}\mathcal{E}(S)_t. \tag{5.6}$$

The process Y^z corresponds to the implicit price of optimal net consumption $\hat{c}^z$ for an investor with beliefs given by the conditional Wiener measure.

The following theorem provides the solution of Merton's consumption-investment problem for an insider. It shows that for the existence of a viable market model without any restrictions on the investment horizon it is necessary that insider information be atomic. The anticipative information affects optimal strategies through the likelihood ratio between the conditional and unconditional density of the signal. It follows that an insider will never consume more or less in all states of nature than a non-insider.

Theorem 5 *Under the assumptions of theorem 1 we have the following:*

1. *An investor with investment horizon $T \in [\![T_G, 1]\!]$ who has non-atomic insider information attains ex ante infinite expected utility.*

2. *If the insider has only atomic anticipative information or the investment horizon terminates before the uncertainty about the signal is resolved $T \in [\![0, T_G[\![$, the solution to Merton's consumption-investment problem is if assumption 4 is satisfied as follows:*

   The optimal consumption policy is given by:

$$\hat{c}_t = I(t, \hat{y}^G \frac{q(G)}{p(\omega, t, G)} \mathcal{E}(S)_t), \tag{5.7}$$

   where $\hat{y}^G$ corresponds to the marginal value of initial wealth and satisfies:

$$\mathcal{X}(\hat{y}^G, G) = e^G. \tag{5.8}$$

   The optimal portfolio policy is given by:

$$\hat{\pi}_t = \hat{\pi}_t^G \tag{5.9}$$

   where for all $z \in \mathbb{R}^q$

$$(\hat{\pi}_t^z)^* = \mathcal{D}_t Y_t^z (\sigma_t)^{-1}. \tag{5.10}$$

   The insider's optimal wealth $X_t^{\hat{\pi}, \hat{c}}$ satisfies

$$X^{\hat{\pi}, \hat{c}_t} = Y_t^G, \mathbf{P} \otimes \lambda \quad a.e.. \tag{5.11}$$

For logarithmic utility for final wealth only Amendinger, Imkeller and Schweizer (1998) show that the existence of a solution to Merton's problem depends on the relative entropy between the conditional and unconditional law of the signal. They show for these particular preferences that a solution exists whenever the relative entropy is finite. It follows from theorem 2 that if there is an arbitrage the conditional and unconditional laws of the signal at resolution time are mutually singular, and consequently the corresponding relative entropy is infinite.

Since as we have seen in theorem 2 arbitrage opportunities occur just immediately before a resolution time it follows that for investment horizons which end before there is no arbitrage.

If the investment horizon covers the resolution time it is necessary that all possible events revealed by the insider information are non-zero probability events. In this case additional information is not sufficiently fine to realize an arbitrage and therefore the demand for risky assets is not infinite.

The results of theorem 5 illustrate that even if there exists only an absolutely continuous but not equivalent local martingale measure, Merton's consumption-investment problem still has a solution. The static budget constraint associated with the dynamic problem with conditional beliefs and investment horizon T can be written

$$\mathbf{E}^{\mathbf{P}^z}[\mathbf{1}_{\{\frac{dP}{dP^z}|_{\mathcal{F}_T}\in]0,+\infty[\}}\frac{d\mathbf{P}}{d\mathbf{P}^z}_{|\mathcal{F}_T}\int_0^T \mathcal{E}(S)_t c_t dt] \leq e^z, \tag{5.12}$$

or equivalently

$$\mathbf{E}[\mathbf{1}_{\{\frac{dP}{dP^z}|_{\mathcal{F}_T}\in]0,+\infty[\}}\int_0^T \mathcal{E}(S)_t c_t dt] \leq e^z \tag{5.13}$$

since on $\{\frac{d\mathbf{P}}{d\mathbf{P}^z}_{|\mathcal{F}_T}\in]0,+\infty[\}$ we have that $\mathbf{P}^z \sim \mathbf{P}$. This proves that though $\mathbf{P}(\{\frac{d\mathbf{P}}{d\mathbf{P}^z}_{|\mathcal{F}_T} = +\infty\}) > 0$ and $\mathbf{P}^z(\{\frac{d\mathbf{P}}{d\mathbf{P}^z}_{|\mathcal{F}_T} = +\infty\}) = 0$ for $T \in [\![T_G, 1]\!]$ binary options $\mathbf{1}_{\{\frac{dP}{dP^z}|_{\mathcal{F}_T}=+\infty\}}$ will not add current consumption without violating the budget constraint [14].

[14]This seems to contradict results in Dybvig and Willard (1996) . They claim that constraints, which prevent "empty promises" in states which are not believed to occur given the public beliefs, are necessary for the existence of a solution to the static consumption-investment problem associated with the dynamic problem. They assume that there are no future endowments. The difference with our results arises since they consider the budget constraint under the common beliefs $\mathbf{P}$, given by $\mathbf{E}[\int_0^T \mathcal{E}(S)_t c_t dt] = e_0$, which would correspond to the static budget constraint associated with the dynamic consumption-investment problem only if $\mathbf{P} \sim \mathbf{P}^z$ on $\mathcal{F}_T$ for all $T \in [0,1]$. Now suppose there exists a $\mathcal{F}_t$- adapted consumption policy $\tilde{c}_t$ which satisfies the budget constraint for public beliefs and is such that

$$\mathbf{E}^{\mathbf{P}^z}[\int_0^T u(t,\tilde{c}_t)dt] > \mathbf{E}^{\mathbf{P}^z}[\int_0^T u(t,\hat{c}_t^z)dt].$$

It follows from Lebesgue's decomposition

$$\mathbf{E}[\int_0^{T_G} \mathcal{E}(S)_t \tilde{c}_t dt] = \mathbf{E}^{\mathbf{P}^z}[\frac{d\mathbf{P}}{d\mathbf{P}^z}_{|\mathcal{F}_{T_G}}\int_0^{T_G} \mathcal{E}(S)_t \tilde{c}_t dt] + \mathbf{E}[\mathbf{1}_{\{\frac{dP}{dP^z}|_{\mathcal{F}_{T_G}}=+\infty\}}\int_0^{T_G} \mathcal{E}(S)_t \tilde{c}_t dt]$$

that the consumption policies $\tilde{c}$ will also satisfy the budget constraint for the conditional beliefs. The existence of such a consumption policy contradicts therefore the optimality of $\hat{c}_t^z$. As a consequence the value of the static optimization problem with feasible consumption policies for the public beliefs must be bounded from above by the value of the static problem given conditional beliefs. This illustrates that in our model non-empty promises constraints will not be binding.

5.2 Optimal Portfolio Policies for Insiders

Optimal investment policies of insiders (5.10) in theorem 5 are given as Malliavin derivative of the respective implicit price of net-consumption of an investor who "gambles" upon the event $\{G = z\}$ evaluated at the true realization of the signal. It follows that if we can find an explicit expression for optimal wealth for an investor with beliefs given by the conditional Wiener measure $\mathbf{P}^z$, we are also capable to get explicit expressions for the optimal investment policies of an insider since these are given as instantaneous Malliavin derivatives of "gambler's" optimal wealth evaluated at the true signal. This result is similar to the way optimal investment policies without insider information are obtained in Markov markets as described by Cowell, Elliott and Kopp (1991) and Benoussan and Elliott (1995).

Under a Markovian assumption, optimal wealth is given as a function of state variables and the corresponding optimal portfolio policy is derived with Itô's rule from the derivatives of this function. Our setup is non-Markovian and we obtain optimal portfolio policies from Malliavin derivatives of the optimal wealth with respect to states of nature. The Wiener processes play in a non-Markovian market the role of prices in the determination of optimal portfolio policies and the "delta hedging term" is given as an instantaneous Malliavin derivative of the claim's implicit price with respect to the Wiener processes. Malliavin derivatives measure the "contemporaneous sensitivity" of wealth with respect to changes in states of natures.

To get explicit expressions for the conditional expectation in the expression for the optimal wealth process in a non-Markovian setup may be difficult. In this case, to get more explicit expressions for optimal portfolio policies, we can interchange the conditional expectation and derivative operator to get representations similar to those obtained by Karatzas and Ocone (1991) for Brownian filtrations. This allows us to analyze how investment strategies depend on anticipative information.

The following expression shows that portfolio policies depend on the "sensitivity" of optimal consumption policies with respect to changes in states of nature (i.e. Malliavin derivatives).

$$\begin{aligned}(\hat{\pi}_t^G)^*\sigma_t = \mathbf{E}[\int_t^T \frac{\partial_2 u(v,\hat{c}_v^G)}{\partial_2 u(t,\hat{c}_t^G)}(\hat{c}_v^G - e_v)dv|\mathcal{G}_t]A(t,\hat{c}_t^G)(\mathcal{D}_t\hat{c}_t^z)_{|z=G} \\ + \mathbf{E}[\int_t^T \frac{\partial_2 u(v,\hat{c}_v^G)}{\partial_2 u(t,\hat{c}_t^G)}[1-(\hat{c}_v^G - e_v)A(v,\hat{c}_v^G)](\mathcal{D}_t\hat{c}_v)_{|z=G}dv|\mathcal{G}_t] \\ - \mathbf{E}[\int_t^T \frac{\partial_2 u(v,\hat{c}_v^G)}{\partial_2 u(t,\hat{c}_t^G)}\mathcal{D}_t e_v)dv|\mathcal{G}_t] \quad (5.14)\end{aligned}$$

The instantaneous "sensitivity" $(\mathcal{D}_t\hat{c}_t^z)_{|z=G}$ is related to an expression similar to the CCAPM equation, then:

$$\mathcal{D}_t\hat{c}_t^z = \partial_2 I(t, \hat{y}^z\mathcal{E}(S^z)_t)\hat{y}^z\mathcal{E}(S_t^z)\mathcal{D}_t S_t^z, \tag{5.15}$$

where we have used that $\mathcal{D}_t[S^z, S^z]_t = 0$. Since

$$\partial_2 I(t, \hat{y}^z\mathcal{E}(S^z)_t)\hat{y}^z\mathcal{E}(S_t^z) = -\frac{1}{A(t, \hat{c}_t^z)}, \tag{5.16}$$

where $A(t,c)$ denotes absolute risk aversion and

$$\mathcal{D}_t S_t^z = -(\alpha_t^z + \theta_t^z)^*, \tag{5.17}$$

we see that the conditional Sharpe ratio $\theta + \alpha^G$ for an insider is proportional to the "sensitivity" of optimal consumption of an investor who conditions on the event $\{G = z\}$ with respect to changes in states of nature, evaluated at the true signal. The proportionality factor is determined by the insider's absolute risk aversion.

$$[A(t, \hat{c}_t^z)(\mathcal{D}_t\hat{c}_t^z)^*]_{|z=G} = \alpha_t^G + \theta_t. \tag{5.18}$$

If we calculate all the Malliavin derivatives of consumption and endowment rate processes for insiders, we can break down the individual demand for risky assets into three components: one depending on state price density, another depending on the individual endowment process and a component depending on the anticipative information.

Proposition 2 *Under the assumptions of theorem 5 we have that the insider's demand for risky assets $\hat{\pi} = \hat{\pi}^G$ can be written as follows*

$$\hat{\pi}_t^G = \hat{\pi}_t^{G,S} + \hat{\pi}_t^{G,E} + \hat{\pi}_t^{G,I}, \tag{5.19}$$

as long as $\mathcal{E}(S)_T X_T^{\hat{\pi},\hat{c}} \in \mathbb{D}^{1,1}(\mathbb{R}^d)$. The demand $\hat{\pi}^{G,S}$ denotes the demand for risky assets that depends on the sensitivity of the state price density with respect to changes in the state of nature, $\hat{\pi}^{G,E}$ corresponds to the demand that depends on the sensitivity of the endowment process with respect to states of nature and $\pi^{G,I}$ describes the demand which depends on the insider signal G.

In what follows we give explicit expressions for the different parts of the demand for risky assets. As we mentioned before optimal policies of an insider can be derived as those of an investor who conditions on $\{G = z\}$ for some $z \in$

$\mathbb{R}^d$. Such an investor has a financial market model given by $(\Omega, \mathcal{F}_1, \mathbb{F}, \mathbf{P}^z, \mathcal{C}, \succeq)$. We see that his/her beliefs correspond to the conditional Wiener measure derived in Appendix A. The consumption-investment problem of such an investor can be derived for atomic insider signals on the stochastic basis for non-insiders if we introduce the state dependent utility function

$$v(t, c, \omega, z) := \frac{p(\omega, t, z)}{q(z)} u(t, c). \tag{5.20}$$

The "discount factor" $\frac{p(\omega,t,z)}{q(z)}$ in this representation corresponds to the density process $\frac{d\mathbf{P}^z}{d\mathbf{P}}_{|\mathcal{F}_t}$. With this utility function it is possible to represent agents with heterogeneous beliefs on the same stochastic basis.

Corollary 3 *The demand $\pi^{G,S}$ in proposition 2 that depends on the state price density has itself a "myopic" and a "dynamic" component*

$$\hat{\pi}_t^{G,S} = \hat{\pi}_t^{G,S,m} + \hat{\pi}_t^{G,S,d}. \tag{5.21}$$

The "myopic" component given by

$$\hat{\pi}_t^{G,S,m} := X_t^{\hat{\pi}^G, \hat{c}^G} (\sigma_t \sigma_t^*)^{-1} (b_t - 1_d r_t), \tag{5.22}$$

and the "dynamic" component $\hat{\pi}^{G,S,d}$ is such that on $\{z = G\}$

$$\hat{\pi}_t^{z,S,d} := (\sigma_t^*)^{-1} \mathbf{E}[\int_t^T \frac{\partial_2 v(u, \hat{c}_u^z, \cdot, z)}{\partial_2 v(t, \hat{c}_t^z, \cdot, z)} [e_u - \hat{c}_u^z + \frac{1}{A(u, \hat{c}_u^z)}] (\mathcal{D}_t \log \mathcal{E}(S)_u)^* du | \mathcal{F}_t], \tag{5.23}$$

where the logarithmic Malliavin derivative of the state price density is given by

$$(\mathcal{D}_t \log \mathcal{E}(S)_u)^* = -[\theta_t + \int_t^u (\mathcal{D}_t r_s)^* ds + \int_t^u (\mathcal{D}_t \theta_s)^* (\theta_s ds + d\omega(s))]. \tag{5.24}$$

The myopic component describes the demand of an insider with logarithmic utility function. This part depends on the deflator process S. We see that whenever the state price density is Gaussian the hedging demand is zero.

A similar decomposition exists for the demand that depends on the endowment rate process. The demand depends on the sensitivity of endowment with respect to states of nature. Since the endowment rate process is given exogenously and therefore given as a stochastic differential equation we obtain more explicit expressions for Malliavin derivatives.

Corollary 4 *The demand $\pi^{G,E}$ in proposition 2 that depends on the sensitivity of the endowment rate process with respect to changes in the state of nature is such that on $\{G = z\}$*

$$\hat{\pi}_t^{z,E} := -(\sigma_t^*)^{-1}\mathbf{E}[\int_t^T \frac{\partial_2 v(u, \hat{c}_u^z, \cdot, z)}{\partial_2 v(t, \hat{c}_t^z, \cdot, z)}(\mathcal{D}_t e_u)^* du|\mathcal{F}_t], \tag{5.25}$$

where the Malliavin derivative of the endowment rate process satisfies the diffusion for the endowment rate process with linearized coefficients.

$$(\mathcal{D}_t e_u)^* = \gamma^e(t, e_t) + \int_t^u (\mathcal{D}_t e_s)^*[(\partial_2 \mu^e(s, e_s))ds + (\partial_2 \gamma^e(s, e_s))^* d\omega(s)]. \tag{5.26}$$

Finally we present the part of the demand in the decomposition that determines how the optimal demand depends on the anticipative information. The demand which depends on the market price of risk $\pi^{G,S}$ depends on the change of measure $\frac{d\mathbf{Q}}{d\mathbf{P}}$ which determines the martingale measure of a non-insider. Similarly we see below that the demand which depends exclusively on insider information depends on the density process of conditional and unconditional law of the signal.

Corollary 5 *The demand $\pi^{G,I}$ in proposition 2 that depends on the insider's signal has a "myopic" and a "dynamic" component*

$$\hat{\pi}_t^{G,I} = \hat{\pi}_t^{G,I,m} + \hat{\pi}_t^{G,I,d}, \tag{5.27}$$

where the "myopic" component of the demand is given by

$$\hat{\pi}_t^{G,I,m} = X_t^{\hat{\pi}^G, \hat{c}^G}(\sigma_t^*)^{-1}\alpha_t^G, \tag{5.28}$$

whereas the "dynamic" component of the demand $\hat{\pi}^{G,I,d}$ is such

$$\hat{\pi}_t^{z,I,d} := (\sigma_t^*)^{-1}\mathbf{E}[\int_0^T \frac{\partial_2 v(u, \hat{c}_u^z, \cdot, z)}{\partial_2 v(t, \hat{c}_t^z, \cdot, z)}[\hat{c}_u^z - e_u - \frac{1}{A(u, \hat{c}_u)}]\frac{\mathcal{D}_t p(\omega, u, z)}{p(\omega, u, z)}du|\mathcal{F}_t], \tag{5.29}$$

We see that only the myopic demand for risky assets [15] of the optimal portfolio policies can be unambiguously signed. The myopic demand for risky assets is increasing in the conditional spread between risky and risk free assets with respect to the conditional Wiener measure. The insider increases his/her

[15] By "myopic" we mean the demand of an investor with logarithmic Bernoulli indicator.

demand for risky assets whenever risky assets and the conditional density of the signal are locally positively correlated.

Since $\alpha_t^z = 0$ for all $t \in [\![T_G, 1]\!]$ we see that after the resolution the insider's strategies only differ from $\mathcal{F}_t$- adapted optimal policies trough different optimal consumption processes.

From the expression for the optimal portfolio policies we can see how different assumptions can considerably simplify terms.

1. We obtain the demand for risky assets with non-anticipative information by putting $\alpha_t^z = 0$, which implies $\hat{\pi}_t^{\cdot,I,\cdot} = 0$ and $v = u$.

2. For deterministic coefficients ("Gaussian model") the Malliavin derivatives $\mathcal{D}_t\theta_s = 0$ and $\mathcal{D}_t r_s = 0$ and therefore

$$\hat{\pi}_t^{z,S} = \mathbf{E}[\int_t^T \frac{\partial_2 v(u, \hat{c}_u^z, \cdot, z)}{\partial_2 v(t, \hat{c}_t^z, \cdot, z)} \frac{1}{A(u, \hat{c}_u^z)} du|\mathcal{F}_t](\sigma_t\sigma_t^*)^{-1}(b_t - 1_d r_t), \quad (5.30)$$

   and the corresponding part of the demand for insiders in Gaussian models is found by $\hat{\pi}_t^{G,S} = (\hat{\pi}_t^{z,S})_{|z=G}$. Since the Malliavin derivatives of Gaussian random variables are deterministic, Gaussianity is a strong simplifying assumption.

3. Utility functions of the HARA type simplify expressions since these satisfy by definition $A(t,c)^{-1} = (1-\nu(t))c + \eta(t)$ and as a consequence:

$$\hat{\pi}_t^{z,S,d} = (\sigma_t^*)^{-1}\mathbf{E}[\int_t^T \frac{\partial_2 v(u, \hat{c}_u^z, \cdot, z)}{\partial_2 v(t, \hat{c}_t^z, \cdot, z)} \times [e_u + \nu(u)\hat{c}_u^z + \eta(u)](\mathcal{D}_t \log \mathcal{E}(S)_u)^* du|\mathcal{F}_t],$$

   whereas in this case,

$$\hat{\pi}_t^{z,I,d} = (\sigma_t^*)^{-1}\mathbf{E}[\int_t^T \frac{\partial_2 v(u, \hat{c}_u^z, \cdot, z)}{\partial_2 v(t, \hat{c}_t^z, \cdot, z)}[e_u - (1-\nu(u))\hat{c}_u^z + \eta(u)] \times (\mathcal{D}_t \log \mathcal{E}(-\int_0^{\cdot} (\alpha_s^z)^* dW_s^z)_u)^* du|\mathcal{F}_t].$$

   Corresponding demands for insiders are obtained if these expressions are evaluated at $z = G$. The most simple case in this class is logarithmic utility $u(t,c) = h(t)\log c$ since then $(A(t,c))^{-1} = c$ and in the above expressions $\nu = 1$ and $\eta = 0$.

4. If endowments are deterministic $e_t = e(t)$ for all $t \in [0,1]$ we have that $\mathcal{D}_s e_t = 0$ and as a consequence $\hat{\pi}_t^{\cdot,E} = 0$.

5. The simplest expression is obtained if we assume logarithmic utility and deterministic endowments then in this case we have just myopic demand for risky assets $\hat{\pi}_t^{\cdot} = \hat{\pi}_t^{\cdot,S,m} + \hat{\pi}_t^{\cdot,I,m}$. This case with the additional assumption of preferences for final wealth was the only one considered in Karatzas and Pikovsky (1996a), Elliott, Geman and Korkie (1997) as well as Amendinger, Imkeller and Schweizer (1998) .

5.3 The Value of Insider Information

To rationalize ex-ante the use of additional side information is easy. A finer flow of information reduces at each moment in time uncertainty about the true state of nature. Since investors are risk averse they will like this. That they indeed do can be seen from a comparison of the value functions for a normal investor with common beliefs and information

$$V(\mathbf{P}, \mathbb{F}, e) := \sup_{(\pi,c)\in\mathcal{A}(\mathbf{P},F,e)} \mathbf{E}[\int_0^T u(t, c_t)dt], \tag{5.31}$$

with that of an insider

$$V(\mathbf{P}, \mathbb{G}, e) := \sup_{(\pi,c)\in\mathcal{A}(\mathbf{P},G,e)} \mathbf{E}[\int_0^T u(t, c_t)dt]. \tag{5.32}$$

Since an insider can always choose his optimal strategies to be just $\mathcal{F}_t$- adapted we must by optimality of his strategies have that

$$V(\mathbf{P}, \mathbb{G}, e) - V(\mathbf{P}, \mathbb{F}, e) \geq 0. \tag{5.33}$$

It follows that the left hand side of this inequality can be interpreted as the individual value of better information.

In section 3 we have seen that insiders do not change the representation of market data on the stochastic basis $(\Omega, \mathcal{F}_1, \mathbb{G}, \mathbf{P})$ which is relevant for their decisions if and only if their anticipative information is independent of public information $G \perp \mathcal{F}_t$ for all $t \in [\![0, T_G[\![$. An equivalent result holds for the individual value of side information

Proposition 3 *Information generated by a random element G that satisfies assumption 1, 2 and 3 does not have any individual value:*

$$V(\mathbf{P}, \mathbb{G}, e) = V(\mathbf{P}, \mathbb{F}, e), \tag{5.34}$$

if it is independent of the public flow of information

$$G \perp \mathcal{F}_t, \quad \textit{for all} \;\; t \in [\![0, T_G[\![. \tag{5.35}$$

Then it is never possible to ex-ante Pareto improve an allocation by allowing the optimal strategies to depend upon information that cannot be learned from the common flow of information. For this to happen we would need for at least one investor a strict inequality between the value functions with and without side information.

5.4 The Informational Content of Insider Portfolio and Consumption Policies

For many equilibrium results it is crucial to determine whether optimal insider strategies reveal all private information or not. To show that optimal strategies are fully revealing it is necessary that the filtration generated by the optimal strategies correspond to the insider's flow of information. The following proposition shows that optimal portfolio and consumption policies are never fully revealing if insider information is given by initial enlargements of Wiener filtration.

Proposition 4 *If the insider information in non-redundant, optimal insider strategies (5.7) and (5.10) in theorem 5 are never fully revealing, since*

$$\mathcal{G}_t \not\subset \mathcal{F}_t^{\hat{c}^G} \bigvee \mathcal{F}_t^{\hat{\pi}^G} \tag{5.36}$$

for some $t \in [0, T]$.

This result shows that if the insider can only reveal his superior information through his state contingent investment and consumption demand, rational expectation equilibria in our model will never be fully revealing. A Walrasian auctioneer or market maker who just obtains the contingent investment demand in the form of any kind of market orders will never be able to fully infer the information about the state of nature contained in the investor's anticipative signals. For a fully revealing equilibrium to exist it is necessary that insiders communicate their information through other channels.

6 TWO EXAMPLES OF INSIDER INFORMATION

In this section we consider first an investor who already knows at the beginning of the investment horizon the time at which a stock stops to pay a dividend. Then we will derive optimal portfolio strategies for an investor who knows already at the beginning whether or not a dividend will be paid after a certain time. This kind of insider information has previously not been considered in the literature.

For simplicity we consider in both examples the following dividend process

$$dD_t = D_t[-2\gamma D_t^{-\frac{1}{2\gamma}} dW_t - 2\gamma(1-2\gamma) D_t^{-\frac{1}{\gamma}} dt] \text{ with } D_0 = a^{2\gamma}, \tag{6.1}$$

some $\gamma \geq 1$ and $a > 0$.

6.1 Non-Atomic Insider Information

Since the solution of this stochastic differential equation is $D_t = (a - W_t)^{2\gamma}$ an investor who knows at the beginning whether or not the stock does pay a dividend up to a certain moment in time has anticipative information generated by the signal $G = \inf\{t : D_t = 0\}$. Since $\{D_t = 0\} = \{W_t = a\}$ the signal G corresponds to the passage time $T_a = \inf\{t : W_t = a\}$ of the Brownian motion W at a. It is well known (see Karatzas and Shreve (1987) proposition 8.2 p.96) that the density of the passage time of a Brownian motion starting at zero T_a has the following density

$$\mathbf{P}^0(T_a \in dz) = \frac{a}{\sqrt{2\pi z^3}} \exp\{-\frac{a^2}{2z}\} dz, \tag{6.2}$$

and consequently it follows from the Markovian property of Brownian motion and the expression for the solution D_t that in our context $p(\omega, t, z) =: \tilde{p}(D_t(\omega), t, z)$ with

$$\tilde{p}(d, t, z) = \frac{d^{\frac{1}{2\gamma}}}{\sqrt{2\pi(z-t)^3}} \exp\{-\frac{d^{\frac{1}{\gamma}}}{2(z-t)}\} \mathbf{1}_{z>t}, \tag{6.3}$$

and therefore the logarithmic Malliavin derivative is given as follows $\alpha_t^z(\omega) =: \tilde{\alpha}(D_t, t, z)$ where

$$\tilde{\alpha}(d, t, z) = (-\frac{1}{d^{\frac{1}{2\gamma}}} + \frac{d^{\frac{1}{2\gamma}}}{z-t}) \mathbf{1}_{z>t}, \tag{6.4}$$

and consequently

$$\alpha_t^G = (\frac{1}{D_t^{\frac{1}{2\gamma}}} + \frac{D_t^{\frac{1}{2\gamma}}}{G-t}) \mathbf{1}_{G>t}. \tag{6.5}$$

If we use this expression in theorem 1 we get the model relevant for an insider knowing exactly the time of the last dividend payment. By left continuity of

α_t^G we $\mathbf{P}$-a.s. have that $(\alpha_{G-}^G)^2 = +\infty$ and consequently that the wealth process corresponding to the myopic strategy $\frac{\tilde{\pi}_t^G}{X^{\tilde{\pi}^G}} = \frac{\alpha_t^G}{\sigma_t}\mathbf{1}_{t<G}$; the corresponding wealth

$$X_t^{\tilde{\pi}^G,e} = \frac{\exp\{-\int_0^t \alpha_s^G \theta_s ds\}}{\mathcal{E}(-\int_0^{\cdot} \alpha_s^G dW_s^G)_t}, \tag{6.6}$$

converges $\mathbf{P}$-a.s. to $X_G^{\tilde{\pi}^G,e} = +\infty$. Clearly such an investor has an arbitrage opportunity given by $\tilde{\pi}$. The arbitrage is due to the fact that the information which is generated by the signal consists of events $\{T_a = z\}$ for some $z \in \mathbb{R}^d$ and is so precise that it contains events which are not believed to be a possible occurrence given initial public information $\mathbf{E}[\mathbf{1}_{\{T_a=z\}}] = 0$.

6.2 Atomic Insider Information

To illustrate the difference between atomic and non-atomic insider information we consider an insider who knows already at the beginning whether or not the last dividend payment will be after a certain time $T^\star$. The signal of such an insider is therefore $G(\omega) = \mathbf{1}_{]T^\star,+\infty[}(T_a(\omega))$. The information revealed by this signal is equivalent to the information whether or not the minimal dividend rate during the period $[0, T^\star]$ is positive. The important difference relative to the previous example is that the event $\{G = z\}$ is of positive probability when $z \in \{0, 1\}$. The interpretation of this is that the insider signal contains information which was considered a possible occurrence given initial public information and in this sense contains less information about the true state of nature. The conditional density in this case is obtained from

$$p(\omega, t, z) = \int_{T^\star}^{+\infty} \tilde{p}(D_t(\omega), t, x) dx \mathbf{1}_{t<T^\star}, \tag{6.7}$$

which gives $p(\omega, t, z) =: \hat{p}(D_t(\omega), t, z)$ where for $t < T^\star$

$$\hat{p}(d, t, z) = ((2\Phi(\frac{d^{\frac{1}{2\gamma}}}{\sqrt{T^\star - t}}) - 1)\mathbf{1}_{\{1\}}(z) + 2(1 - \Phi(\frac{d^{\frac{1}{2\gamma}}}{\sqrt{T^\star - t}})\mathbf{1}_{\{0\}}(z)), \tag{6.8}$$

where Φ denotes the cumulative Gaussian distribution with corresponding kernel ϕ. Consequently taking Malliavin derivatives we obtain in this case $\alpha_t^G(\omega) =: \hat{\alpha}(D_t(\omega), t, G)$ where

$$\hat{\alpha}(D_t, t, G) = \frac{\frac{1}{\sqrt{T^\star - t}}\phi(\frac{D_t^{\frac{1}{2\gamma}}}{\sqrt{T^\star - t}})}{\Phi(0)^G - \Phi(\frac{D_t^{\frac{1}{2\gamma}}}{\sqrt{T^\star - t}})}\mathbf{1}_{t<T^\star}. \tag{6.9}$$

If we put these expressions into those of theorem 1 we obtain again the model relevant for an insider who knows whether or not the dividend of a stock is always positive up to a given date. We see that for states such that the dividend is strictly positive up to $T^\star$ the conditional return of the stock is reduced. For states such that dividends fall to zero for the first time before $T^\star$ the conditional appreciation rate of stocks increases.

The events revealed by such a signal are countable and it follows from theorem 2 that there is no arbitrage for insiders whenever there is no arbitrage for non-insiders in this case. To focus on the portfolio strategies which do just depend on the information we assume for simplicity in what follows that the market price of risk $\theta_t = 0$, the spot rate $r_t = 0$ and the endowment rate process $e_t = 0$ all the time. In such a model non-insiders have in contrast to insiders no demand for risky assets. Therefore, the only demand for risky assets stems from the insider's anticipative information. Furthermore we assume that the utility function is from the CRRA family $u(t,c) = e^{-\rho t}\log c$ if the relative risk aversion is one $R = 1$ and $u(t,c) = e^{-\rho t}\frac{c^{1-R}}{1-R}$ else. Using the results of corollary 5 we obtain the myopic part of the demand for the stock as follows

$$\hat{\pi}_t^{G,I,m} = X_t^{\hat{\pi}^G,\hat{c}^G}\frac{\hat{\alpha}(\omega(t),t,G)}{\sigma_t}\mathbf{1}_{t<T^\star}, \tag{6.10}$$

The myopic part is such that on states for which dividends remain positive up to $T^\star$ he wants to short the stock. In contrast he wants to hold a long position in the stock if the dividend falls to zero before $T^\star$. Such a behavior is only characteristic for an investor with logarithmic utility. For other non-myopic preferences we have for $t < T^\star$ the following additional demand for risky assets

$$\hat{\pi}_t^{G,I,d} = \frac{-(1-1/R)}{\sigma_t}\int_t^{T^\star} dv \int_0^{+\infty} \hat{c}(y,v,G)\hat{\alpha}(y,v,G)d\Phi\left(\frac{D_t^{\frac{1}{2\gamma}} - y^{\frac{1}{2\gamma}}}{\sqrt{v-t}}\right). \tag{6.11}$$

where the $\hat{c}(D_t,t,G)$ corresponds to the optimal consumption policy of the insider given by $\hat{c}(d,t,z) = I(t,\hat{y}^z\frac{q(z)}{\hat{p}(d,t,z)})$. An insider who is less risk averse than an insider with logarithmic preferences will have higher short or long positions than a myopic insider. In contrast an insider who is more risk averse than a myopic insider want to hedge their short and long positions. Their net demand in the two cases cannot be signed. For more general Markovian setups, where explicit expressions cannot be found, the Monte Carlo techniques presented in Detemple, Garcia and Rindisbacher (1998) can be extended to study the different components of the portfolio strategies in proposition 2. Our example illustrates that the effects of anticipative information on optimal portfolio policies can depend on the assumptions about preferences. The decomposition of portfolio policies in this paper helps to clarify this dependence on preferences. Similarly, they show whether or not a conclusion is robust with respect to the probabilistic structure of the insider signal.

7 CONCLUSION

The results of this article can be seen as a necessary step for the construction of general dynamic models with anticipative information. The expressions for optimal portfolio and consumption policies enable us to investigate exactly how pieces of anticipative information affect the optimal behavior of an investor. They can therefore be used to analyze whether or not conclusions about the efficiency of trading mechanisms in the presence of insiders are robust to the probabilistic specification of the model. Furthermore, since the whole setup is non-Markovian, that is prices are not state variables, our results seem more appropriate to address questions about the informational efficiency of prices in financial markets. Such issues are considered in Rindisbacher (1998) where we use the analysis at the individual level of this paper to construct equilibrium models.

The result about the existence of arbitrage opportunities for insiders illustrates that without restrictions on the investment horizon careful modeling of such investors is required. It shows that viability is an important issue for any model of intertemporal risk sharing.

The techniques introduced in this paper have interesting applications in other fields. In many problems of dynamic risk management it is required to quantify the effects of a worst case scenario on the value of contingent claims and optimal wealth. In this case we can interpret the vector of signals as realizations of such a scenario and the value of insider information as a measure of the value at risk. Such a measure will incorporate the investor's attitude towards risk. This issue will be of interest for future research where we plan to consider enlargements of filtrations not only by random variables but by continuous stochastic processes. Such a generalization will enable us to study exactly the structure of a given flow of information.

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A Appendix A:The "Girsanov Approach" to Initial Enlargements of Wiener filtrations

The aim of this appendix is to derive in detail all necessary results to get the semi-martingale representation of the wealth process on the stochastic basis $(\Omega, \mathcal{F}_1, \mathbb{G}, \mathbf{P})$ relevant for the choice of consumption and portfolio policies of an insider. In contrast to the filtering problem where Stricker's theorem guarantees that any $\mathbb{G}$ semi-martingale is also a $\mathbb{F}$- semi-martingale, stability of semi-martingales with respect to enlargements of filtration does not necessarily hold (see Jacod (1979) for a discussion). The presentation is based on Jacod (1980) , Foellmer and Imkeller (1993) and Imkeller (1996), but allows for enlargements with respect to random variables which are measurable with respect to Brownian filtration before the terminal date. We first derive a conditional Wiener measure and find semi-martingale decompositions for this measure. Corresponding compensators depend on a parameter which represents the realization of the signal. Using results in Stricker and Yor (1978) (proposition 2 and théorème 1) we can always pick a version which depends measurably on this parameter. We will always consider this version of such processes in what follows. The general idea behind the "Girsanov approach" to the enlargements of filtrations is due to Song (1987) . He also shows how the same idea can be exploited for progressive enlargements of filtrations.

A sufficient condition for the existence of a semi-martingale representation for initial enlargements ("hypothèse H") is Jacod's criterion ("condition A") (see Jacod (1980) page 15), which in our model can be stated as follows

Assumption A.1 *(**"Condition A"**) There exists a common measure ν on $\mathcal{B}_{R^q}$ such that $\mathbf{P}_t^\omega \ll \nu$ for all $t \in [\![0, T_G[\![$ where $\mathbf{P}_t^\omega$ corresponds to the conditional law of G given the initial filtration $\mathcal{F}_t$.*

Remarks

1. If "condition A" is satisfied we can without loss of generality choose $\nu = \mathbf{P}_G$.
2. Since for $t \in [\![T_G, 1]\!]$ the conditional law of the signal corresponds to $\mathbf{1}_{\{z\}}(G(\omega))$ "condition A" cannot be satisfied after resolution time unless G is constant. Furthermore it follows that if $\sigma(G)$ does not contain any atom then not only $\mathbf{P}_G \not\ll \mathbf{P}_t^\omega$ but even $\mathbf{P}_G \perp \mathbf{P}_t^\omega$ for $t \in [\![T_G, 1]\!]$. This follows since the conditional law takes all its positive mass on $\{G = z\}$, a null set of the unconditional law.

3. Imkeller (1996) has recently given sufficient conditions for which random variables $G \in \mathbb{D}^{1,2}(\mathbb{R}^q)$ on Wiener space have conditional laws that are absolutely continuous with respect to the Lebesgue measure λ.

We now derive the compensator of a $\mathbb{F}$- martingale on $(\Omega, \mathcal{F}_1, \mathbb{G}, \mathbf{P})$. As it was sketched by Jacod (1980) and shown by Foellmer and Imkeller (1993) this can be done by means of a Girsanov transformation with a conditional (Wiener) measure. Since this approach provides the key to derive optimal strategies for an insider we present it in detail.

To understand the link between the conditional and unconditional law with the Wiener measure and conditional Wiener measure we introduce the joint law of insider signal G and states of nature W

$$B \times E \mapsto \mathbf{R}(B, E); \quad B \times E \in \mathcal{F}_1 \otimes \sigma(G), \tag{A.1}$$

where

$$\mathbf{R}(B, E) := \mathbf{E}[\mathbf{1}_{G^{-1}(B)} \mathbf{1}_E]. \tag{A.2}$$

Lemma A.1 *1. For all $t \in [0,1]$ the measure*

$$\mathbf{P}_t^{\omega}(dz) := \frac{d\mathbf{R}}{d\mathbf{P}}_{|\mathcal{F}_t}(\omega, dz) \tag{A.3}$$

on $\sigma(G)$ corresponds to the conditional law of G given $\mathcal{F}_t$

2. For all $t \in [0,1]$ the probability measure

$$\mathbf{P}^z(d\omega) := \frac{d\mathbf{R}}{d\mathbf{P}_G}(d\omega, z) \tag{A.4}$$

corresponds to the Wiener measure restricted to $\{G = z\}$.

3. For all $t \in [0,1]$ we have

$$\frac{d\mathbf{P}^z}{d\mathbf{P}}_{|\mathcal{F}_t}(\omega) = \frac{d\mathbf{P}_t^{\omega}}{d\mathbf{P}_G}(z) \tag{A.5}$$

$\mathbf{P} \otimes \mathbf{P}_G$*- a.e..*

Proof

Since $\mathbf{P}(E) = \mathbf{R}(\mathbb{R}^q, E)$, respectively $\mathbf{P}_G(B) = \mathbf{R}(B, \Omega)$ it follows that $\mathbf{R}(\cdot, E) \ll \mathbf{P}_G$ for all $E \in \mathcal{F}_t$, respectively $\mathbf{R}(B, \cdot) \ll \mathbf{P}$ for all $B \in \sigma(G)$, we have that $\mathbf{P}_t^{\omega}$ respectively $\mathbf{P}^z$ are both absolutely continuous with respect to

$\mathbf{P}$ and therefore by the Radon-Nikodym theorem that $\mathbf{P}_t^\omega$ and $\mathbf{P}^z$ exist and are given as Radon-Nikodym derivatives of $\mathbf{R}$ with respect to $\mathbf{P}$.

Furthermore since

$$\mathbf{P}^z(d\omega)\mathbf{P}_G(dz) = d\mathbf{R}(dz, d\omega) = \mathbf{P}_t^\omega(dz)\mathbf{P}(d\omega), \tag{A.6}$$

it follows that (A.5) must hold $\mathbf{P} \otimes \mathbf{P}_G$- a.e..

Q.E.D.

Remarks

1. The conditional Wiener measure $\mathbf{P}^z$ can be interpreted as beliefs of an investor "gambling" upon the event $\{G = z\}$.

2. The equality in (A.5) allows to represent processes relevant for decisions conditional on the event $\{G = z\}$ in terms of properties of the signal's conditional law only.

3. If "condition A" holds we have for $t \in [\![0, T_G[\![$ that

$$\mathbf{P}_t^\omega(dz) = p(\omega, t, z)\nu(dz), \tag{A.7}$$

   and consequently that

$$\frac{d\mathbf{P}^z}{d\mathbf{P}}_{|\mathcal{F}_t}(\omega) = \frac{p(\omega, t, z)}{q(z)}, \tag{A.8}$$

   where $q(z) := p(\omega, 0, z)$ corresponds to the density of $\mathbf{P}_G$ with respect to ν.

4. Since for $t \in [\![T_G, 1]\!]$ we have that $\mathbf{P}_G \not\ll \mathbf{P}_t^\omega$ and $\mathbf{P}_G \perp \mathbf{P}_t^\omega$ if $\sigma(G)$ is non-atomic it follows from (A.5) that equivalently $\mathbf{P} \not\ll \mathbf{P}^z$ and $\mathbf{P}^z \perp \mathbf{P}$ on $\mathcal{F}_t$ for $t \in [\![T_G, 1]\!]$ if $\sigma(G)$ is non-atomic.

On Wiener space Imkeller (1996) has recently shown how to get explicit expressions for the compensator using the Clark-Ocone representation formula. The following lemma presents his result and summarizes complementary results from Jacod (1980) .

Lemma A.2 *If "condition A" and (i) $p(\omega, s, z) \in \mathbb{L}^{1,2}(\mathbb{R}^d)$ as well as (ii) the mapping $r \mapsto \mathcal{D}_r p(\omega, s, z)$ is left-continuous in $L^1(\Omega)$ at $s \in [0, t]$ and $z \in \mathbb{R}^q$ are satisfied then we have for $t < \tau^z$ that*

$$p(\omega, t, z) = q(z)\mathcal{E}\left(\int_0^\cdot (\alpha_s^z)^* d\omega(s)\right)_t \tag{A.9}$$

where the stopping time τ^z *is given by*

$$\tau^z := \inf\{u \in [0,1] : p(\omega,u,z) = 0\}, \tag{A.10}$$

whereas

$$(\alpha_t^z(\omega))^* := \frac{\mathcal{D}_t p(\omega,t,z)}{p(\omega,t,z)}. \tag{A.11}$$

Furthermore for all $t \in [\![0, T_G[\![$ *we have that* $p(\omega,t,G) > 0$ $\mathbf{P}$*- a.s. and therefore*

$$(\alpha_t^G(\omega))^* = \left[\frac{\mathcal{D}_t p(\omega,t,z)}{p(\omega,t,z)}\right]_{z=G} \tag{A.12}$$

is well defined for all $t \in [\![0, T_G[\![$.

Proof

Since under "condition A" $\mathbf{P}_t^\omega(dz) = p(\omega,t,z)\nu(dz)$ for $t \in [\![0, T_G[\![$ and therefore

$$\mathbf{E}[\mathbf{P}_t^\omega(B)|\mathcal{F}_s] = \mathbf{E}[\int_B p(\omega,t,z)\nu(dz)|\mathcal{F}_s] \tag{A.13}$$

and by Fubini's theorem

$$\mathbf{E}[\int_B p(\omega,t,z)\nu(dz)|\mathcal{F}_s] = \int_B \mathbf{E}[p(\omega,t,z)|\mathcal{F}_s]\nu(dz) \tag{A.14}$$

But since at the same time

$$\mathbf{E}[\mathbf{P}_t^\omega(B)|\mathcal{F}_s] = \mathbf{P}_s^\omega(B) = \int_B p(\omega,s,z)\nu(dz) \tag{A.15}$$

we have $\mathbf{P}(d\omega) \otimes \nu(dz)$ a.e. that $\mathbf{E}[p(\omega,t,z)|\mathcal{F}_s] = p(\omega,s,z)$ and we have established that the conditional density process $p(\omega,t,z)$ is a non-negative martingale.

Since $p(\omega,s,z) \in \mathbb{L}^{1,2}(\mathbb{R}^d)$ we can use the Clark-Ocone formula (see Nualart (1995) proposition 1.3.5. page 42) to represent the conditional density as

$$p(\omega,t,z) = q(z) + \int_0^t \mathbf{E}[\mathcal{D}_v p(\cdot,t,z)|\mathcal{F}_v]d\omega(v) \tag{A.16}$$

and since the conditional expectation operator and the Malliavin derivative commute (see Nualart (1995) proposition 1.2.4 page 32) we have for all $r \in [0,v]$ that

$$\mathbf{E}[\mathcal{D}_r p(\cdot,t,z)|\mathcal{F}_v] = \mathcal{D}_r \mathbf{E}[p(\cdot,t,z)|\mathcal{F}_v] = \mathcal{D}_r p(\omega,v,z) \tag{A.17}$$

Then since $\mathcal{D}_r p(\omega, v, z)$ is assumed to be left-continuous $\lim_{r \uparrow v} \mathcal{D}_r p(\omega, v, z) = \mathcal{D}_v p(\omega, v, z)$ exists in $L^1(\Omega)$ and we can write

$$p(\omega, t, z) = q(z) + \int_0^t \mathcal{D}_v p(\cdot, v, z) d\omega(v) \tag{A.18}$$

We clearly have that $p(\omega, t, z) > 0$ for $t < \tau^z$. Suppose that for some $t \geq \tau^z$ we also have that $p(\omega, t, z) > 0$. Since the conditional density process is a martingale and τ^z is a $\mathcal{F}_t$- stopping time, we would have that $p(\omega, \tau^z, z) > 0$ since by the optional sampling theorem $p(\omega, \tau^z, z) = \mathbf{E}[p(\cdot, t, z)|\mathcal{F}_{\tau^z}]$, a contradiction since obviously $p(\omega, \tau^z, z) = 0$.We therefore have shown that $p(\omega, t, z) > 0$ for $t < \tau^z$ and $p(\omega, t, z) = 0$ for $t \geq \tau^z$.

It follows for $t < \tau^z$ that

$$p(\omega, t, z) = q(z) + \int_0^t p(\cdot, v, z) \frac{\mathcal{D}_v p(\cdot, v, z)}{p(\cdot, v, z)} d\omega(v) \tag{A.19}$$

or equivalently that

$$p(\omega, t, z) = q(z) \mathcal{E}\left(\int_0^\cdot \frac{\mathcal{D}_v p(\cdot, v, z)}{p(\cdot, v, z)} d\omega(v)\right)_t \tag{A.20}$$

This establishes (A.11).

To show that for all $t \in [\![0, T_G[\![$ we $\mathbf{P}$- a.s. have that $p(\omega, t, G) > 0$ is equivalent to show that $\mathbf{P}(\tau^G = 1) = 1$. Clearly for all $t \in [0, 1[$

$$\mathbf{E}\left[\int_{R^q} p(\cdot, t, z) \nu(dz)\right] = 1 \tag{A.21}$$

But since $\mathbf{P}$- a.s.

$$\int_{R^q} p(\omega, t, z) \nu(dz) = \int_{R^q} \mathbf{1}_{\{\tau^z > t\}} p(\omega, t, z) \nu(dz) \tag{A.22}$$

and

$$\mathbf{E}\left[\int_{R^q} \mathbf{1}_{\{\tau^z > t\}} p(\cdot, t, z) \nu(dz)\right] = \mathbf{E}[\mathbf{1}_{\{\tau^G > t\}}] \tag{A.23}$$

we have established for all $t \in [\![0, T_G[\![$ that $\mathbf{P}(\tau^G > t) = 1$ and therefore $\tau^G = T_G$ $\mathbf{P}$- a.s..

Q.E.D.

Remarks

1. Since for non-atomic signals we have for $t \in [\![T_G, 1]\!]$ that $\mathbf{P}^z \perp \mathbf{P}$ on $\mathcal{F}_t$ it follows for all $z \in \mathbb{R}^q$ that

$$\lim_{t \uparrow T_G} \mathcal{E}(\int_0^{\cdot} (\alpha_s^z)^* d\omega(s))_t = \begin{cases} +\infty & \mathbf{P}^z\text{- a.s.} \\ 0 & \mathbf{P}\text{- a.s.} \end{cases} \tag{A.24}$$

whereas for atomic signals

$$\lim_{t \uparrow T_G} \mathcal{E}(\int_0^{\cdot} (\alpha_s^z)^* d\omega(s))_t = \frac{1}{\mathbf{P}_G(\{z\})} \tag{A.25}$$

$\mathbf{P}^z$- a.s. and

$$\mathbf{P}(\{\lim_{t \uparrow T_G} \mathcal{E}(\int_0^{\cdot} (\alpha_s^z)^* d\omega(s))_t = +\infty\}) > 0 \tag{A.26}$$

and therefore $\mathbf{P}^z \ll \mathbf{P}$ on $\mathcal{F}_t$ for all $t \in [0, 1]$ in this case.

2. For signals which have laws that are absolutely continuous with respect to the Lebesgue measure Imkeller's (1996) theorem 5 gives for signals absolutely continuous with respect to the Lebesgue measure sufficient conditions for which (i) and (ii) are satisfied.

From the law of W given $G = z$ we can find the $(\mathbf{P}^z, \mathbb{F})$- compensator of the Wiener process W by a Girsanov transformation. Since

$$\frac{\mathbf{E}[\mathcal{E}(\int_0^{\cdot} (\alpha_s^z)^* d\omega(s))_{t \wedge \tau^z}]}{q(z)} = 1 \tag{A.27}$$

it follows that for $t < \tau^z$

$$W(\omega)_t^z := \omega(t) - \int_0^t \alpha_s^z(\omega) ds \tag{A.28}$$

is a $(\mathbf{P}^z, \mathbb{F})$- Wiener process for all $z \in \mathbb{R}^q$. Furthermore if $\mathbf{P}^z$- a.s. we have that

$$\int_0^{\tau^z} \alpha_s^z ds < +\infty \tag{A.29}$$

then W^z can be extended for $t \geq \tau^z$ as follows

$$W(\omega)_t^z := \omega(t) - \int_0^{t \wedge \tau^z} \alpha_s^z(\omega)^* ds \tag{A.30}$$

An investor "gambling" upon the event $\{G = z\}$ has states of nature given by W^z. Since an insider can be seen as a "gambler" who knows with certainty the realization of the event he is "gambling" upon we get states for insiders by evaluating "gamblers" states at $z = G$. That W^G on $[\![0, T_G[\![$ are indeed the state of natures for an insider follows from theorem 1 of Foellmer and Imkeller (1993) . They show that W^G corresponds to the $(\mathbf{P}, \mathbb{G})$- decomposition of the Wiener process W. Since we want to analyze hedging and investment policies for investment horizons longer than the resolution time of the signal without loosing the semi-martingale property of processes relevant for insiders' decisions we have to assume the following

Assumption A.2 *Signals G are such that*

$$\int_0^{T_G} \alpha_s^G ds < \infty \tag{A.31}$$

$\mathbf{P}$-*a.s.*

Remarks

1. An example where assumption A.2 is satisfied is $G = \omega(T)$ some $T \in [0, 1]$. In this case the resolution time is $T_G = T$ and

$$\int_0^{T_G} \alpha_s^G ds = \int_0^{T_G} \log(T_G - s) d\omega(s) \tag{A.32}$$

   $\mathbf{P}$-a.s. such that

$$W_t^G = \mathbf{E}[\int_0^{T_G} (T_G + \log(T_G - s)) d\omega(s) | \mathcal{G}_t] \tag{A.33}$$

   which proves that W^G is an uniformly integrable $(\mathbf{P}, \mathbb{G})$- martingale. It follows by arguments similar to those in Jeulin and Yor (1977) that W_T^G is independent of $\omega(T)$ and that $\mathbb{G}$ corresponds for $t \in [0, T]$ to the filtration of the Brownian bridge $\beta_t(\omega) := \omega(t) - \frac{t}{T} G(\omega)$.

2. In the paper of Elliott,Geman and Korkie (1997) (A.31) is satisfied, since they introduce insider information on incomplete information such that non-insiders never get to know the signal. Consequently $T_G = 1$ and since for their signal "condition A" holds for all $t \in [0, 1]$ assumption A.2 is satisfied for any investment horizon in their model.

Theorem A.1 *If "condition A" and (i) and (ii) of lemma A.2 and assumption A.2 are satisfied then the process* $W^G = (W_t^G, t \in [0,1])$ *given by*

$$W^{G(\omega)}(\omega)_t = \omega(t) - \int_0^{t\wedge T_G} (\alpha_s^{G(\omega)}(\omega)))^* ds \tag{A.34}$$

is a $(\mathbf{P}, \mathbb{G})$*- Wiener process.*

Proof

Obviously W^G is $\mathcal{G}_t$ adapted. Since for all $A = A_1 \times A_2 \in \mathcal{F}_s \otimes \sigma(G)$ where $s \in [0,t]$ and $t < T_G$

$$\mathbf{E}[\mathbf{1}_A \mathbf{E}[W_t^G - W_s^G | \mathcal{G}_s]] = \int_{\Omega\times R^q} \mathbf{1}_{A_1\times A_2}(w,z)(W_t^z - W_s^z)\mathbf{P}^z(d\omega)\mathbf{P}_G(dz) \tag{A.35}$$

it follows that

$$\mathbf{E}[\mathbf{1}_A \mathbf{E}[W_t^G - W_s^G | \mathcal{G}_s]] = \int_{R^q} \mathbf{1}_{A_2}(z)\mathbf{E}^{\mathbf{P}^z}[\mathbf{1}_{A_2}\mathbf{E}^{\mathbf{P}^z}[W_t^z - W_s^z|\mathcal{F}_s]]\mathbf{P}_G(dz) \tag{A.36}$$

From Girsanov's theorem we know that W^z is a $\mathbb{F}$- Wiener process. It follows that $\mathbf{E}^{\mathbf{P}^z}[W_t^z - W_s^z|\mathcal{F}_s] = 0$ for $s < t$ and therefore that

$$\mathbf{E}[\mathbf{1}_A \mathbf{E}[W_t^G - W_s^G | \mathcal{G}_s] = 0 \tag{A.37}$$

The statement that W^G is a $(\mathbf{P}, \mathbb{G})$- Wiener process on $[\![0, T_G[\![$ follows then from Lévy's characterization of Brownian motion since the quadratic variation of W^G is

$$[(W^G)^i, (W^G)^j]_t = \delta_{i,j} t \tag{A.38}$$

for all $i, j \in \{1, \dots, d\}$.

For $t \geq T_G$ we have $\mathcal{G}_t = \mathcal{F}_t$ and $dW_t^G = d\omega(t)$. It follows that W^G corresponds to a Wiener process starting at $-\int_0^{T_G} \alpha_s^G ds$ on $[\![T_G, 1]\!]$.

Q.E.D.

Remarks

1. Given the expression for the process W^G on $[\![0, T_G[\![$ we might from Girsanov's theorem conclude that the measure determined by the density process $\mathcal{E}(-\int_0^{\cdot}(\alpha_s^G)^* dW_s^G)$. should correspond to the Wiener measure $\mathbf{P}$ and therefore be constant. Imkeller and Foellmer (1993) have shown

that in fact this density process with respect to the Wiener measure defines a measure which is not even equivalent to the Wiener measure, a paradox, which is explained by the fact that $\frac{d\mathbf{P}}{d\mathbf{P}^G}$ corresponds on $\mathcal{G}_t$ for $t \in [0,1]$ to the Radon-Nikodym derivative of the product measure $\mathbf{P} \otimes \mathbf{P}_G$ with respect to the joint law $\mathbf{R}$, that is defines a product measure on $\sigma(G) \otimes \mathcal{F}_t$ which corresponds to the Wiener measure only if projected on its second coordinate. It is this fact which is crucial for whether or not on insider information local martingale measures exist.

2. If we compare the process W^G and W on $[\![T_G, 1]\!]$ we see that they have the same increments but not the same starting point $W_{T_G} - W^G_{T_G} = \int_0^{T_G} \alpha^G_s ds$.

3. Since any $\mathcal{F}_t$- local martingale is given as $\int_0^{\cdot} \phi^*_s d\omega(s)$ its $\mathcal{G}_t$- compensator is given by $\int_0^{\cdot \wedge T_G} \phi^*_s \alpha^G_s ds$ as long this integral is finite $\mathbf{P}$- a.s..

B Appendix B: The Representation of Contingent Claims as Two-Parameter Random Variables

The results of appendix A show how using Girsanov transformations the Doob-Meyer decompositions of $(\mathbf{P}, \mathbb{F})$- local martingales can be obtained on $(\Omega, \mathcal{F}_1, \mathbb{G}, \mathbf{P})$ in two steps. First conditional on an arbitrary realization of the signal we get $(\mathbf{P}^z, \mathbb{F})$- semi-martingales by a change of measure. Secondly we get desired representations for enlarged filtration simply be evaluating this decomposition at the realization of the signal. This shows that processes on insiders' stochastic basis can be interpreted as two-parameter processes, one parameter for the state of nature and the other for the realization of the signal. We will show how this can be used to get optimal hedging strategies for insiders. To do this we need to establish that $\mathcal{G}_T$- measurable (some $T < T_G$) contingent claim can be regarded as $\mathcal{F}_T \otimes \sigma(G)$ measurable mappings from $\Omega \times \mathbb{R}^q$ to $\mathbb{R}^l$ some $l \in \mathbb{N}$. The next theorem will show that this is possible for sufficiently smooth signals.

Theorem B.1 *Under "condition A", (i) and (ii) of lemma A.2 and assumption A.2 we have for any $\mathcal{G}_T$- measurable random variable H such that $H \in \mathbb{D}^{1,1}(\mathbb{R}^q)$ if*

$$\int_0^{T_G} |\mathbf{E}[\mathcal{D}_t H | \mathcal{F}_t] \alpha_t^G| dt < +\infty \tag{B.1}$$

$\mathbf{P}$- *a.s. that*

$$H(\omega) = C^{G(\omega)}(\omega) \tag{B.2}$$

where for all $z \in \mathbb{R}^q$

$$C^z(\omega) = \mathbf{E}[H|\mathcal{F}_T](\omega) + \mathbf{E}^{\mathbf{P}^z}[\int_T^{T_G} \mathbf{E}[\mathcal{D}_t H|\mathcal{F}_t] \alpha_t^z dt | \mathcal{F}_T](\omega) \tag{B.3}$$

$\mathbf{P}$- *a.s..*

In the proof and for other results we need the following lemma.

Lemma B.1 *For $\mathcal{G}_{T_G -}$- measurable random variables $H(\omega)$ that can be written as $H(\omega) = C^{G(\omega)}(\omega)$ where $C^z(\omega)$ is $\mathbf{F}_{T_G -} \otimes \sigma(G)$- measurable we have under "condition A" for all $t \in [0, T_G[$ that*

$$\left(\mathbf{E}^{\mathbf{P}^z}[C^z|\mathcal{F}_t]\right)_{|z=G} = \mathbf{E}[C^G|\mathcal{G}_t] \tag{B.4}$$

whereas

$$(\mathbf{E}[C^z|\mathcal{F}_t])_{|z=G} = \mathbf{E}[\frac{p(\cdot,t,G)}{p(\cdot,T,G)}H|\mathcal{G}_t] \tag{B.5}$$

Proof

Since for $E \in \mathcal{G}_t$ there exist $F \in \mathcal{F}_t$ and $B \in \mathcal{B}(\mathbb{R}^d)$ such that $\mathbf{1}_E(\omega) = \mathbf{1}_{F\times B}(\omega, G(\omega))$ we have for all $E \in \mathcal{G}_t$ that

$$\mathbf{E}[\mathbf{1}_E\mathbf{E}[C^G|\mathcal{G}_t]] = \int_{B\times F} C^z(\omega)\mathbf{R}(dz, d\omega) \tag{B.6}$$

Then since at the same time

$$\mathbf{E}[\mathbf{1}_E\left(\mathbf{E}^{\mathbf{P}^z}[C^z|\mathcal{F}_t]\right)_{|z=G}] = \int_B \mathbf{P}_G(dz)\mathbf{E}^{\mathbf{P}^z}[\mathbf{1}_F\mathbf{E}^{\mathbf{P}^z}[C^z|\mathcal{F}_t]] \tag{B.7}$$

and $\mathbf{E}^{\mathbf{P}^z}[\mathbf{1}_F\mathbf{E}^{\mathbf{P}^z}[C^z|\mathcal{F}_t]] = \mathbf{E}^{\mathbf{P}^z}[\mathbf{1}_F C^z]$ we have that

$$\mathbf{E}[\mathbf{1}_E\left(\mathbf{E}^{\mathbf{P}^z}[C^z|\mathcal{F}_t]\right)_{|z=G}] = \int_{B\times F} C^z(\omega)\mathbf{R}(dz, dw) \tag{B.8}$$

and we have shown (B.4). Then since $\mathbf{1}_{\{T<\tau^G\}} = 1$ and for $T < \tau^z$ by Bayes' law $\mathbf{E}^{\mathbf{P}^z}[C^z|\mathcal{F}_t] = \mathbf{E}[\frac{p(\cdot,t,z)}{p(\cdot,T,z)}C^z|\mathcal{F}_t]$ we obtain (B.5) by repeating the same argument for $\tilde{C}^z(\omega) = \frac{p(\omega,t,z)}{p(\omega,T,z)}\mathbf{1}_{\{T<\tau^z\}}C^z(\omega)$.

Q.E.D.

Proof of theorem B.1

By assumption $H \in \mathbb{D}^{1,1}(\mathbb{R}^d)$ and therefore from the Clark-Ocone representation formula we have since H is $\mathcal{F}_{T_G}$- measurable that

$$H = \mathbf{E}[H] + \int_0^{T_G} \mathbf{E}[\mathcal{D}_u H|\mathcal{F}_u]d\omega(u) \tag{B.9}$$

Since for all $d\omega(t) = dW_t^z - \alpha_t^z du$ and H is by assumption $\mathcal{G}_T$- measurable we get

$$H = \mathbf{E}[H|\mathcal{F}_T] + \mathbf{E}[\int_T^{T_G} \mathbf{E}[\mathcal{D}_u H|\mathcal{F}_u]\alpha_u^G du|\mathcal{G}_T] \tag{B.10}$$

where the second integral is well defined since (B.1) holds. Since from lemma B.1 we know that

$$\left(\mathbf{E}^{\mathbf{P}^z}[\int_T^{T_G} \mathbf{E}[\mathcal{D}_t H|\mathcal{F}_t]\alpha_t^z dt|\mathcal{F}_T]\right)_{|z=G} = \mathbf{E}[\int_T^{T_G} \mathbf{E}[\mathcal{D}_u H|\mathcal{F}_u]\alpha_u^G du|\mathcal{G}_T] \tag{B.11}$$

we have established that (B.2) must hold with (B.3).

Q.E.D.

Remarks

1. Since $\mathcal{D}_t H = 0$ for $t \in [T, 1]$ whenever the contingent claim H is also $\mathcal{F}_T$- measurable such claims do not depend on the signal.

2. Theorem B.1 shows that whenever a claim is only measurable on insider information, it can be regarded as $\mathcal{F}_T \otimes \sigma(G)$- measurable random variable.

C Appendix C: The Domain of Malliavin's Derivative and its Adjoint

The domain $\mathbb{L}^{1,2}(\mathbb{R}^d)$ of the Skorohod integral, that is the adjoint operator of the Malliavin derivative or divergence operator, corresponds to the Hilbert space with norm

$$\| u \|_{L^{1,2}(R^d)} := (\mathbf{E}[\int_0^1 u_s^2 ds])^{1/2} + (\mathbf{E}[\int_0^1 ds \int_0^1 \mathcal{D}_t u_s (\mathcal{D}_t u_s)^* dt])^{1/2} \qquad \text{(C.1)}$$

The operator $\mathcal{D}_t$ in the Hilbert norm above denotes the Malliavin derivative (or divergence) operator with domain $\mathbb{D}^{1,p}(\mathbb{R}^d)$. The domain corresponds to the Banach space given by the completion of the set of smooth random variables $F \in \mathcal{S}$ on $(\Omega, \mathcal{F}_1, \mathbf{P})$, i.e. random variables of the form

$$F = f(\omega(t_1), \dots, \omega(t_m)), \quad \text{some } \; t_1, \dots t_m \in [0,1], f \in C_0^\infty(\mathbb{R}^{m \times d}) \qquad \text{(C.2)}$$

with respect to the norm

$$\| F \|_{1,p} := (\mathbf{E}[F^p])^{1/p} + (\mathbf{E}[\int_0^1 \|\mathcal{D}_t F\|^p dt])^{1/p} \qquad \text{(C.3)}$$

where for smooth random variables $F \in \mathcal{S}$ Malliavin derivatives are given by

$$\mathcal{D}_t F := \sum_{i=1}^d \frac{\partial}{\partial x_i} f(\omega(t_1), \dots \omega(t_m)), \quad t \in [0,1] \qquad \text{(C.4)}$$

For definitions concerning Malliavin Calculus we refer to Nualart (1995)

.

D Appendix D: Proofs

D.1 Proofs of section 3

D.1.1 Proof of theorem 1

By assumption 3 the Doob-Meyer decomposition of W on $(\Omega, \mathcal{F}_1, \mathbb{G}, \mathbf{P})$ is as shown in Appendix A given for $t \in [0,1]$ by

$$\omega(t) = W_t^G + \int_0^{t \wedge T_G} \alpha_s^G ds \tag{D.1}$$

Replacing this in the representation on $(\Omega, \mathcal{F}_1, \mathbb{F}, \mathbf{P})$ gives the desired result for prices, dividend rate and endowment rate processes since by assumption corresponding volatility coefficients are bounded and assumption 3 holds. Similarly the additional condition imposed on portfolio policies gives the decomposition of wealth.

Q.E.D.

D.1.2 Proof of corollary 1

Clearly if $t \in [\![0, T_G[\![$ and $\mathcal{F}_t \perp \sigma(G)$ we have for all $t \in [\![0, T_G[\![$ that $p(\omega, t, z) = q(z)$ and therefore $\alpha_t^z = 0$. That independence is also necessary follows from the fact that Malliavin derivatives are zero for all $t \in [\![0, T_G[\![$ if and only if $p(\omega, t, z)$ lies in the subspace of the zeroth Wiener chaos and is therefore constant (Nualart (1995) page 31). It follows that $\alpha_t^z = 0$ implies that G is independent of $\mathcal{F}_t$.

Q.E.D.

D.1.3 Proof of proposition 1

If we define $E := \{\frac{X_T^{\pi,e}}{e_0 B_T} \geq 1\}$ and $F := \{\frac{X_T^{\pi,e}}{e_0 B_T} > 1\}$ we have since $\mathbf{P}(E) = \mathbf{E}[\mathbf{P}(E|\mathcal{G}_0)]$ and $\mathbf{P}(F) = \mathbf{E}[\mathbf{P}(F|\mathcal{G}_0)]$ that there is an arbitrage whenever there is a conditional arbitrage.

It is possible that an arbitrage has an associated gains from trade of the form $\alpha_t \mathbf{1}_E$ some $E \in \mathcal{G}_0$ where $\mathbf{P}(E) > 0$ and $\mathcal{G}_t$- adapted process α. Since on E^c gains from trade are zero there is no arbitrage conditional on E^c. This shows that the existence of an arbitrage does not imply the existence of a conditional arbitrage.

Q.E.D.

D.1.4 Proof of theorem 2

First we characterize sets on which the density of the potential candidate for a martingale measure of the insider is zero. Then we show that if these sets are of positive probability no equivalent martingale measure can exist. Then we show that this implies the existence of FLVR but not necessarily A.

Since for $S_t^G := -\int_0^t r_s ds - \int_0^t (\theta_s + \alpha_s^G)^\star dW_s^G$ we have that

$$\left(\mathcal{E}(S^G)_t B_t\right)^2 \exp\{-\int_0^t \|\theta_s + \alpha_s^G\|^2 ds\} = \mathcal{E}(2S^G)_t B_t^2 \tag{D.2}$$

it follows that

$$\{\mathcal{E}(2S^G)_t B_t^2 > 0\} = \{\mathcal{E}(S^G)_t B_t > 0\} \bigcap \{\int_0^t \|\theta_s + \alpha_s^G\|^2 ds < \infty\} \tag{D.3}$$

Then since $\mathcal{E}(2S^G)_t B_t^2 > 0$ if and only if $\mathcal{E}(S^G)_t B_t > 0$ it follows that

$$\{\int_0^t \|\theta_s + \alpha_s^G\|^2 ds = +\infty\} = \{\mathcal{E}(-\int_0^\cdot (\theta_s + \alpha_s^G)^\star dW_s^G)_t = 0\} \tag{D.4}$$

Then since θ is bounded by assumption it follows that

$$\{\int_0^t \|\alpha_s^G\| ds = +\infty\} = \{\frac{p(t,\omega,z)}{q(z)} = 0\} \tag{D.5}$$

We therefore have from the previous properties of conditional and unconditional Wiener measures the following results:

1. If the insider information is atomic then

$$\mathbf{P}^z(\int_0^t \|\alpha_s^z\| = +\infty) = \begin{cases} \epsilon > 0 & \text{if } t \in [\![T_G, 1]\!] \\ 0 & \text{if } t \in [\![0, T_G[\![\end{cases} \tag{D.6}$$

2. If the insider information is non-atomic then

$$\mathbf{P}^z(\int_0^t \|\alpha_s^z\| = +\infty) = \begin{cases} 1 & \text{if } t \in [\![T_G, 1]\!] \\ 0 & \text{if } t \in [\![0, T_G[\![\end{cases} \tag{D.7}$$

First we consider an investment horizon which ends before resolution time $T < T_G$. In this case we have that $\mathbf{P}^z(\{\frac{d\mathbf{P}}{d\mathbf{P}^z}_{|\mathcal{F}_T} \in]0, +\infty[\}) = 1$ and therefore

$$\mathbf{E}^{\mathbf{P}^z}[\mathcal{E}(S_T^z B_T] = \mathbf{E}[\mathcal{E}(-\int_0^\cdot \theta_s^\star dW_s)_T] \tag{D.8}$$

But since by the boundedness of the coefficients of price processes for common information Novikov's condition

$$\mathbf{E}[\exp\{\frac{1}{2}\int_0^T \|\theta_s\|^2 ds\}] < +\infty \tag{D.9}$$

is satisfied and therefore it follows by proposition 1.15 p.308 of Revuz and Yor (1990) that $\mathcal{E}(S_t)B_t$ is an uniformly integrable martingale. This implies that $\mathbf{E}[\mathcal{E}(S_T)B_T] = 1$. Consequently since $\mathbf{P}^z \sim \mathbf{P}$ on $\mathcal{F}_T$ whenever $T < T_G$ we must have that

$$\mathbf{E}^{\mathbf{P}^z}[\mathcal{E}(S^z)_T B_T] = 1 \tag{D.10}$$

or equivalently

$$\mathbf{E}[\mathcal{E}(S^G)_T B_T|\mathcal{G}_0] = 1 \tag{D.11}$$

This shows that the measure $d\tilde{\mathbf{Q}} := \mathcal{E}(S_T^G)B_T d\mathbf{P}^z_{|z=G}$ defines an equivalent marmartingale measure for the insider for $T < T_G$.

We now want to show that this implies NFLVR and therefore NA.

The market price of risk on public information is by the assumptions on the spot rate, the appreciation rates and volatilities of stocks bounded. Therfore, it follows from the representation of the density process between the conditional and unconditional Wiener measure for any $t \in [\![0, T_G[\![$ that $\int_0^t \|\alpha_s^z\|^2 ds < +\infty$ $\mathbf{P}^z$ a.s.. Consequently, we have for any admissible sequence of portfolio policies π^n and positive sequence $(\delta^n)_{\in\mathbf{N}}$ with $\delta^n \geq 1$ such that $\frac{X^{\pi^n,e}}{e_0 B_t} > \delta^n$ for all $t \in [0,1]$ that

$$\mathcal{E}(S^G)_{t\wedge T_G}\frac{X^{\pi,e}_{t\wedge T_G} - \delta^n B_{t\wedge T_G}}{e_0 - \delta^n} = \mathcal{E}(\int_0^\cdot (\frac{\pi_s^\star \sigma_s}{X_s^{\pi,e} - \delta^n B_s} - (\theta_s + \alpha_s^G)^\star) dW_s^G)_{t\wedge T_G} \tag{D.12}$$

is well defined.

Since the left hand side of (D.12) is non-negative and a local martingale, it must be a super-martingale and therefore

$$\mathbf{E}^{\mathbf{P}^z}[\mathcal{E}(S^z)_T(\frac{X_T^{\pi,e} - \delta^n B_T}{e_0 - \delta^n} - \delta^n B_T)] \leq 1 - \delta^n \tag{D.13}$$

Consequently it must from Markov's inequality hold true that for any sequence of portfolio strategies such that $\frac{X_T^{\pi^n,e}}{e_0 B_T} > \delta^n$ and $\epsilon > 0$ that

$$\mathbf{E}[\mathcal{E}(S^G)_T B_T \mathbf{1}_{\{|\frac{X_T^{\pi^n,e} - \delta^n B_T}{e_0 - \delta^n} - \delta^n B_T| > \epsilon\}}|\mathcal{G}_0] \leq \frac{1}{\epsilon}(1 - \delta^n). \tag{D.14}$$

Note that $\frac{X_T^{\pi^n,e}}{e_0 B_T} - 1$ must converge in probability to zero whenever $\frac{X_T^{\pi^n,e} - \delta^n B_T}{e_0 - \delta^n} - \delta^n B_T$ does. Therefore, since $\mathbf{P}^z(\{\mathcal{E}(S^z)_T B_T > 0\}) = 1$ we have

$$\lim_{n\to+\infty} \mathbf{P}(|\frac{X_T^{\pi^n,e}}{e_0 B_T} - 1| > \epsilon | \mathcal{G}_0) = 0 \tag{D.15}$$

whenever $\delta^n \to 1$ from above for $n \to +\infty$. It follows that there are no free lunches with vanishing risk for an investment horizon shorter than resolution time.

Next we show that it is necessary for NFLVR for an investment horizon T that $\mathbf{P}(\{\mathcal{E}(S^G)_T B_T > 0\}|\mathcal{G}_0) = 1$. Clearly if $\mathbf{P}(\{\mathcal{E}(S^G)_T B_T = 0\}|\mathcal{G}_0) > 0$ we must have that $\mathbf{P}^z(\{\frac{1}{\mathcal{E}(S^z)_T B_T} = +\infty\}) > 0$ at $z = G$. But then since the sequence of mean-variance strategies

$$\tilde{\pi}_t^n := (\sigma_t \sigma_t^\star)^{-1}(b_t + \sigma_t \alpha_t^G - 1_d r_t)(X_t - \delta^n B_t)\mathbf{1}_{[\![0,T_G[\![}(t) \tag{D.16}$$

are admissible and have an associated wealth process such that

$$\frac{X_T^{\tilde{\pi}^n,e} - \delta^n B_T}{e_0 - \delta^n B_T} = \frac{\mathcal{E}(S)_T B_T}{\mathcal{E}(S)_{T_G} B_{T_G}} \frac{1}{\mathcal{E}(S^G)_T B_T}. \tag{D.17}$$

But since $\mathbf{P}^z(\{\frac{1}{\mathcal{E}(S^z)_T B_T} = +\infty\}) > 0$ at $z = G$, we therefore would have that

$$\mathbf{P}^z(\frac{X_T^{\tilde{\pi},e} - \delta^n B_T}{e_0 - \delta^n B_T} = +\infty) > 0 \tag{D.18}$$

which is impossible if for all $\epsilon > 0$ we have that $\lim_{n\to+\infty} \mathbf{P}(|\frac{X_T^{\tilde{\pi}^n,e}}{e_0 B_T}| > \epsilon) = 0$. Therefore, it cannot hold true that there is no free lunch with vanishing risk, and we have established that $\mathbf{P}^z(\{\mathcal{E}(S^z)_T B_T = 0\}) = 0$ whenever NFLVR is satisfied.

We now show that $\mathbf{P}^z(\{\mathcal{E}(S^z)_T B_T = 0\}) > 0$ when $T > T_G$. Since $\mathbf{P} \not\ll \mathbf{P}^z$ on $\mathcal{F}_{T_G}$ and therefore $\mathbf{P}^z(\{\frac{d\mathbf{P}}{d\mathbf{P}^z}|_{\mathcal{F}_{T_G}} = 0\}) > 0$ and since $\mathbf{P} \sim \mathcal{E}(S)_{T_G} B_{T_G} \cdot \mathbf{P}$ we must have that

$$\mathbf{P}(\{\mathcal{E}(S^G)_{T_G} B_{T_G} = 0\}|\mathcal{G}_0) > 0. \tag{D.19}$$

Consequently since $\mathbf{P}^z(\{\mathcal{E}(S^z)_T B_T = 0\}) = 0$ is necessary for NFLVR, it follows that whenever an insider has an investment horizon longer than the resolution time he will also have a free lunch with vanishing risk given by the sequence of mean-variance strategies given in (D.16).

It remains to be shown that for $T \geq T_G$ there is an arbitrage if an only if $\mathcal{G}_0$ is non-atomic. If we put for all $n \in \mathbb{N}$ $\delta^n = K$ we obtain for tame portfolios from (D.12) that

$$\mathcal{E}(S^G)_{t\wedge T_G} \frac{X^{\pi,e}_{t\wedge T_G} - KB_{t\wedge T_G}}{e_0 - K} = \mathcal{E}(\int_0^{\cdot} (\frac{\pi^\star_s \sigma_s}{X^{\pi,e}_s - KB_s} - (\theta_s + \alpha^G_s)^\star) dW^G_s)_{t\wedge T_G} \tag{D.20}$$

Then suppose that $\mathcal{G}_0$ is non-atomic. As we have seen in this case we have that $\mathbf{P}^z \ll \mathbf{P}$ on $\mathcal{F}_T$ even for $T > T_G$. Consequently

$$\mathbf{E}^{\mathbf{P}^z}[\mathcal{E}(S^z)_T B_T] = \mathbf{E}[\mathbf{1}_{\{\frac{dP}{dP^z}|_{\mathcal{F}_T} > 0\}} \mathcal{E}(S)_T B_T] \tag{D.21}$$

But since $P(\{\frac{d\mathbf{P}^z}{d\mathbf{P}} = +\infty\}) = 0$ and from Novikov's condition we know that $\mathcal{E}(S)_t B_T$ is an uniformly integrable martingale starting at 1, it follows that

$$\mathbf{E}^{\mathbf{P}^z}[\mathcal{E}(S^z)_T B_T] = 1 \tag{D.22}$$

This establishes by the same argument as before that $\tilde{\mathbf{Q}}$ is a martingale measure for the insider also for $T \geq T_G$ if his anticipative information is atomic. But this time since $\mathbf{P^z}(\{\frac{d\mathbf{P^z}}{d\mathbf{P}}_{|\mathcal{F}_{T_G}} = +\infty) > 0$ the martingale measure $\tilde{\mathbf{Q}}$ can only absolutely continuous and not equivalent.

We now show that the existence of an absolutely continuous martingale measure is sufficient to rule out arbitrage opportunities.

Since the left hand side of (D.20) is non-negative local martingale and therefore a super-martingale starting at 1, we obtain by taking expectations of (D.20) with respect to the conditional measure $\mathbf{P}(\cdot|\mathcal{G}_0)$ that

$$\mathbf{E}^{\mathbf{P}}[\mathcal{E}(S)_T B_T (\frac{\frac{X^{\pi,n}_T}{B_T} - K}{e_0 - K} - 1)] \leq 0 \tag{D.23}$$

Now, if there exists a strategy $\tilde{\pi}$ such that $\mathbf{P}(\frac{X^{\tilde{\pi},e}_T}{e_0 B_T} \geq 1) = 1$ we would have that

$$\mathbf{E}^{\mathbf{P}}[\mathcal{E}(S)_T B_T (\frac{\frac{X^{\pi,n}_T}{e_0 B_T} - \frac{K}{e_0}}{1 - \frac{K}{e_0}} - 1) \mathbf{1}_{\frac{X^{\tilde{\pi},e}_T}{e_0 B_T} - 1 > 0}] \leq 0 \tag{D.24}$$

But since $\mathcal{E}(S)_T B_T > 0$ $\mathbf{P}$- a.s. we must have that $\mathbf{P}(\{\frac{X^{\pi,e}_T}{e_0 B_T} > 1\}) = 0$ in this case. Since $\mathbf{P}^z \ll \mathbf{P}$ on $\mathcal{F}_T$ for all $T \in [0,1]$ it must therefore also hold true that

$$\mathbf{P}(\{\frac{X^{\pi,e}_T}{e_0 B_T} > 1\}|\mathcal{G}_0) = 0. \tag{D.25}$$

This shows that there is no arbitrage for the insider when his anticipative signal is atomic.

Finally if $\mathcal{G}_0$ is non-atomic we have that for any that the mean variance strategy

$$\tilde{\pi}_t^G = (\sigma_t\sigma_t^\star)^{-1}(b_t + \sigma_T\alpha_{t-}^G - 1_d r_t)(X_{t-}^{\tilde{\pi}^G,e} - B_t K)\mathbf{1}_{]\!]T_G,T_G]\!]}(t) \tag{D.26}$$

has an associated wealth process such that

$$\frac{X_T^{\tilde{\pi^G},e} - KB_T}{e_0 - K} = \frac{\mathcal{E}(S^G)_{T_G-}}{\mathcal{E}(S^G)_{T_G}} \tag{D.27}$$

But since on $\mathcal{F}_{T_G}$ we have that $\mathbf{P}^z \perp \mathbf{P}$ and consequently $\frac{d\mathbf{P}}{d\mathbf{P}^z}_{|\mathcal{F}_{T_G}} = 0$ $\mathbf{P}^z$-a.s. we have at $z = G$ that $\mathcal{E}(S^z)_T = 0$ $\mathbf{P}^z$-a.s. at $z = G$ and consequently that $\frac{X_T^{\tilde{\pi},e} - KB_T}{e_0 - K} = +\infty$ $\mathbf{P}^z$- a.s. and therefore also that

$$\mathbf{P}(\frac{X_T^{\tilde{\pi},e}}{e_0 B_T} > 1|\mathcal{G}_T) = 1. \tag{D.28}$$

This shows that $\tilde{\pi}^G$ is an arbitrage.

Q.E.D.

D.1.5 Proof of corollary 2

For signals $G = G^0 + Z$ where Z is independent from G^0 we have that $\sigma(G) = \sigma(G^0)\bigvee\sigma(Z)$. And consequently that for $t \geq T_{G^0}$ that $\mathbf{P}(G \in B|\mathcal{F}_t) = \mathbf{P}(Z \in (B - x))_{|x=G^0} > 0$ for some $B \in \mathcal{B}_{\mathbf{R}^d}$, such that $T_{G^0} = 1$ $\mathbf{P}$- a.s.. It follows from theorem 2, that there are no arbitrage opportunities even for non-atomic insider information.

Q.E.D.

D.2 Proofs of section 4

D.2.1 Proof of theorem 3

For the contingent claim H define the tracking error $\phi_T^{\pi^z,c^z,z} := (X_T^{\pi^z,c^z} - C^z)$ where C^z is given by (B.3). Then since by assumption $\mathcal{E}(S)_T C^z \in \mathbb{D}^{1,1}(\mathbb{R}^d)$ we have from the Clark-Ocone formula and the fact that $\mathcal{E}(S)_T C^z$ is $\mathcal{F}_T$-measurable that

$$\mathcal{E}(S)_T C^z = \mathbf{E}[\mathcal{E}(S)_T C^z] + \int_0^T \mathbf{E}[\mathcal{D}_v\mathcal{E}(S)_T C^z|\mathcal{F}_v]d\omega(v) \tag{D.29}$$

From Itô's rule it follows that

$$\mathcal{E}(-\int_0^\cdot (\alpha_s^z)^* dW_s^z)_T \mathcal{E}(S)_T C^z = \mathbf{E}[\mathcal{E}(S)_T C^z] + \int_0^T \mathcal{D}_v[\mathcal{E}(-\int_0^\cdot (\alpha_s^z)^* dW_s^z)_v \mathbf{E}[\mathcal{E}(S)_T C^z|\mathcal{F}_v]] dW_v^z \quad \text{(D.30)}$$

where we have used the commutativity of the conditional expectation and Malliavin derivative operator to get that

$$\mathcal{D}_v[\mathcal{E}(-\int_0^\cdot (\alpha_s^z)^* dW_s^z)_v \mathbf{E}[\mathcal{E}(S)_T C^z|\mathcal{F}_v]] = \mathcal{E}(-\int_0^\cdot (\alpha_s^z)^* dW_s^z)_v \mathbf{E}[\mathcal{D}_v[\mathcal{E}(S)_T C^z]|\mathcal{F}_v] - \mathcal{E}(-\int_0^\cdot (\alpha_s^z)^* dW_s^z)_v \mathbf{E}[\mathcal{E}(S)_T C^z|\mathcal{F}_v](\alpha_v^z)^*$$

At the same time using again Itô's formula we have that

$$X_0^{\pi^z,c^z} + \int_0^T \mathcal{E}(-\int_0^\cdot (\alpha_s^z)^* dW_s^z)_v \mathcal{E}(S)_v ((\pi^z)_v^* \sigma_v - X_v^{\pi^z,c^z}(\theta_v + \alpha_v^z)^*) dW_v^z = \mathcal{E}(-\int_0^\cdot (\alpha_s^z)^* dW_s^z)_T \mathcal{E}(S)_T X_T^{\pi^z,c^z} + \int_0^T \mathcal{E}(-\int_0^\cdot (\alpha_s^z)^* dW_s^z)_v \mathcal{E}(S)_v (c_v^z - e_v) dv \quad \text{(D.31)}$$

From (D.30) and (D.31) we have for $(\hat{\pi}^z, \hat{c}^z)$ such that $\hat{c}_v^z = e_v$ and

$$(\hat{\pi}_v^z)^* \sigma_v - X_v^{\hat{\pi}^z,\hat{c}^z}(\theta_v + \alpha_v^z)^* = \frac{\mathcal{D}_v[\mathcal{E}(-\int_0^\cdot (\alpha_s^z)^* dW_s^z)_v \mathbf{E}[\mathcal{E}(S)_T C^z|\mathcal{F}_v]]}{\mathcal{E}(-\int_0^\cdot (\alpha_s^z)^* dW_s^z)_v \mathcal{E}(S)_v} \quad \text{(D.32)}$$

and initial wealth

$$X_0^{\hat{\pi}^z,\hat{c}^z} = \mathbf{E}[\mathcal{E}(S)_T C^z] \quad \text{(D.33)}$$

that $\phi_T^{\hat{\pi}^z,\hat{c}^z,z} = 0$ for all $z \in \mathbb{R}^q$ $\mathbf{P}$- a.s.. Furthermore it follows that

$$X_t^{\hat{\pi}^z,\hat{c}^z} = \mathbf{E}^{\mathbf{P}^z}[\frac{\mathcal{E}(-\int_0^\cdot (\alpha_s^z)^* dW^z)_T \mathcal{E}(S)_T}{\mathcal{E}(-\int_0^\cdot (\alpha_s^z)^* dW^z)_t \mathcal{E}(S)_t} C^z|\mathcal{F}_t] \quad \text{(D.34)}$$

or since $\frac{d\mathbf{P}^z}{d\mathbf{P}}_{|\mathcal{F}_{\cdot}} = \mathcal{E}(\int_0^{\cdot}(\alpha_s^z)^* d\omega(s))$. from Bayes' law equivalently that

$$X_t^{\hat{\pi}^z,\hat{c}^z} = \mathbf{E}[\frac{\mathcal{E}(S)_T}{\mathcal{E}(S)_t} C^z | \mathcal{F}_t] \tag{D.35}$$

Using

$$\frac{\mathcal{D}_v \mathcal{E}(S)_T}{\mathcal{E}(S)_v} = \mathcal{D}_v \left[\frac{\mathcal{E}(S)_T}{\mathcal{E}(S)_t}\right] - \frac{\mathcal{E}(S)_T}{\mathcal{E}(S)_v}\theta_v^* \tag{D.36}$$

and therefore

$$\begin{aligned}\frac{\mathcal{D}_v[\mathcal{E}(-\int_0^{\cdot}(\alpha_s^z)^* dW_s^z)_v \mathbf{E}[\mathcal{E}(S)_T B^y|\mathcal{F}_v]]}{\mathcal{E}(-\int_0^{\cdot}(\alpha_s^z)^* dW_s^z)_v \mathcal{E}(S)_v} &= \mathbf{E}[\mathcal{D}_v[\frac{\mathcal{E}(S)_T}{\mathcal{E}(S)_v} B^z]|\mathcal{F}_v] \\ &- \mathbf{E}[\frac{\mathcal{E}(S)_T}{\mathcal{E}(S)_v} B^z|\mathcal{F}_v](\theta_v + \alpha_v^z)^*\end{aligned}$$

we get from (D.32) that

$$(\hat{\pi}_t^z)^* \sigma_t = \mathbf{E}[\mathcal{D}_t \frac{\mathcal{E}(S)_T}{\mathcal{E}(S)_t} C^z | \mathcal{F}_t] \tag{D.37}$$

This establishes the results announced for an investor gambling upon the event $\{G = z\}$. If we evaluate (D.30) and (D.31) at $z = G$ we get for $(\hat{\pi}, \hat{c}) = (\hat{\pi}^G, \hat{c}^G)$ and $X_0^{\hat{\pi},\hat{c}} = X_0^{\hat{\pi}^G,\hat{c}^G}$ that $\phi_T^{\hat{\pi},\hat{c},G} = 0$ $\mathbf{P}$- a.s. and from (D.37) we get (4.5). Finally if we apply lemma B.1 to (D.35) we get since there is no tracking error the value of the claim given by (4.2)

Finally if $T \geq T_G$ and the signal reveals an event such that $B \neq \emptyset$ but $\mathbf{P}_G(B) = 0$ there is an arbitrage opportunity such as we have seen in theorem 2 that gains from trade at T_G are unbounded with probability one. Such an insider can therefore replicate any contingent claim at zero cost.

Q.E.D.

D.2.2 Proof of theorem 4

Proof

To establish the equivalence of implicit prices of insiders and outsiders it is sufficient to show that $\tilde{\mathbf{Q}} = \mathbf{Q}$ on $\mathcal{F}_T$. Since for all $E \in \mathcal{F}_T$

$$\tilde{\mathbf{Q}}(E) = \mathbf{E}^{\mathbf{P}_G}[(\mathbf{E}^{\mathbf{P}^z}[\mathbf{1}_E \frac{d\mathbf{P}}{d\mathbf{P}^z}_{|\mathcal{F}_T} \frac{d\mathbf{Q}}{d\mathbf{P}}_{|\mathcal{F}_T}])_{|z=G}] \tag{D.38}$$

and

$$\mathbf{E}^{\mathbf{P}^z}[\mathbf{1}_E \frac{d\mathbf{P}}{d\mathbf{P}^z}_{|\mathcal{F}_T} \frac{d\mathbf{Q}}{d\mathbf{P}}_{|\mathcal{F}_T}] = \mathbf{E}[\mathbf{1}_E \frac{d\mathbf{Q}}{d\mathbf{P}}_{|\mathcal{F}_T}] \tag{D.39}$$

we have that

$$\tilde{\mathbf{Q}}(E) = \mathbf{E}^{\mathbf{P}_G}[\mathbf{Q}(E)] \tag{D.40}$$

and therefore $\tilde{\mathbf{Q}}(E) = \mathbf{Q}(E)$.

Q.E.D.

D.3 Proofs of section 5

D.3.1 Proof of theorem 5

First it follows from theorem 2 that if insiders have no arbitrage opportunities investment horizons must end before resolution $T < T_G$ and/or insider information $\sigma(G)$ is completely atomic and consequently $\tilde{\mathbf{Q}} \ll \mathbf{P}$ on $\mathcal{G}_t$ for all $t \in [0,1]$ and therefore also $\mathbf{P}^z \ll \mathbf{P}$ on $\mathcal{F}_t$ for all $t \in [0,1]$. It follows that in the absence of arbitrage we have

$$\mathbf{P}^z(\{\frac{q(z)}{p(\omega,t,z)} \in]0,+\infty[\}) = 1 \tag{D.41}$$

Then if we define $e^z := e_0 + \mathbf{E}^{\mathbf{P}^z}[\int_0^T \mathcal{E}(S^z)_t e_t dt]$ and $\mathcal{E}(S^z) := \mathcal{E}(S)_t \frac{q(z)}{p(\omega,t,z)}$ we have for fixed $z \in \mathbb{R}^q$ that the value function

$$J(e^z) := \inf_{y^z>0} \mathbf{E}^{\mathbf{P}^z}[\int_0^T u(t, I(t, y^z\mathcal{E}(S^z)_t)) - y^z\mathcal{E}(S^z)_t I(t, y^z\mathcal{E}(S^z)_t)dt + y^z e^z] \tag{D.42}$$

satisfies

$$J(e^z) = \mathbf{E}^{\mathbf{P}^z}[\int_0^T u(t, I(t, \hat{y}^z\mathcal{E}(S^z)_t))dt] \tag{D.43}$$

where $\hat{y}^z$ is such that

$$\mathbf{E}^{\mathbf{P}^z}[\int_0^T \mathcal{E}(S^z)_t I(t, \hat{y}^z\mathcal{E}(S^z)_t))dt] = e^z \tag{D.44}$$

and where the existence of $\hat{y}^z$ follows from (5.2). Now since for $\mathcal{F}_t$ adapted non-negative consumption processes c

$$\mathbf{E}^{\mathbf{P}^z}[\int_0^T \sup_{c\geq 0}[u(t,c_t) - \hat{y}^z c_t]dt] \geq \sup_{c^z\geq 0} \mathbf{E}^{\mathbf{P}^z}[\int_0^T [u(t,c_t) - \hat{y}^z c_t]dt] \tag{D.45}$$

and the convex conjugate function in the first integral is

$$\sup_{c^z \geq 0}[u(t, c_t) - \hat{y}^z \mathcal{E}(S^z)_t c_t] = u(t, \hat{c}_t^z) \tag{D.46}$$

we have established that the consumption policy of a "gambler"

$$\hat{c}_t^z = I(t, \hat{y}^z \mathcal{E}(S^z)_t) \tag{D.47}$$

is optimal for the problem

$$\sup_{c \geq 0} \mathbf{E}^{\mathbf{P}^z}[\int_0^T u(t, c_t) dt] \tag{D.48}$$

subject to the static budget constraint

$$\mathbf{E}^{\mathbf{P}^z}[\int_0^T \mathcal{E}(S^z)_t c_t dt] = e^z \tag{D.49}$$

where $\hat{y}^z$ is the multiplier associated to the constraint. To establish that these strategies are optimal for the dynamic problem

$$\sup_{(\pi, c) \in \mathcal{A}(\mathbf{P}^z, \mathbb{F}, e)} \mathbf{E}^{\mathbf{P}^z}[\int_0^T u(t, c_t) dt] \tag{D.50}$$

it is sufficient to show that optimal wealth satisfies $X^{\hat{\pi}^z, \hat{c}^z} > -K$ some $K > 0$ and $X_T^{\hat{\pi}^z, \hat{c}^z} \geq 0$ both $\mathbf{P}^z$- a.s.. Since preferences are strictly monotone and there is no incentive for a bequest we have that $X_T^{\hat{\pi}^z, \hat{c}^z} = 0$ $\mathbf{P}^z$-a.s.. Then since

$$\mathbf{E}^{\mathbf{P}^z}[\mathcal{E}(S^z)_T X_T^{\pi, c} | \mathcal{F}_t] - \mathcal{E}(S^z)_t X_t^{\pi, c} = \mathbf{E}^{\mathbf{P}^z}[\int_0^T \mathcal{E}(S^z)_t (c_t - e_t) dt | \mathcal{F}_t] \tag{D.51}$$

it follows from the fact that the endowment process e is bounded from below that the optimal wealth $X^{\hat{\pi}^z, \hat{c}^z}$ must be bounded from below. This establishes the optimality of $\hat{c}^z$ for an investor "gambling" upon the event $\{G = z\}$.

Next we have to show that $\hat{c}^G$ is optimal for an investor having beliefs $\mathbf{P}$ and additional information $\sigma(G)$.

Since all $c_t \in \mathcal{A}(\mathbf{P}, \mathbb{G}, e)$ are also in $\mathbb{L}^{1,2}(\mathbb{R}+)$ we have from theorem B.1 that there exists $c^z = (c_t^z; t \in [0, T])$ such that

$$c_t = \tilde{c}_t^G \tag{D.52}$$

and where $c^z \in \mathcal{A}(\mathbf{P}^z, \mathbb{F}, e)$. From the optimality of $\hat{c}^z$ in $\mathcal{A}(\mathbf{P}^z, \mathbb{F}, e)$ we must have that

$$\mathbf{E}^{\mathbf{P}^z}[\int_0^T u(t, \hat{c}_t^z) dt] \geq \mathbf{E}^{\mathbf{P}^z}[\int_0^T u(t, c_t^z) dt] \tag{D.53}$$

for fixed $z \in \mathbb{R}^q$. This remains true at $z = G$ and it follows from the lemma B.1 that

$$\mathbf{E}[\int_0^T u(v, \hat{c}_v^G) dv | \mathcal{G}_0] \geq \mathbf{E}[\int_0^T u(v, c_v) dv | \mathcal{G}_0] \tag{D.54}$$

for all $c_t \in \mathcal{A}(\mathbf{P}, \mathbb{G}, e)$.

It remains to proof that the cumulative net consumption process $\int_0^T (\hat{c}_t - e_t) dt$ can be financed by $\mathcal{G}_t$- measurable portfolio strategy. For $T < T_G$ this was established in theorem 3. We have to show that the same kind of argument remains true for discrete signals if $T \geq T_G$. We have already established that $\frac{q}{p(\omega,t,z)} < \infty$ for all $t \in [0,1]$ $\mathbf{P}^z$- a.s.. In fact we have on $\{G = z\}$ for $t \in [\![T_G, 1]\!]$

$$\mathcal{E}(S^z)_t = \mathbf{P}_G(\{z\}) \mathcal{E}(S)_t \tag{D.55}$$

$\mathbf{P}^z$-a.s. and therefore that the density process of the absolutely continuous local martingale measure $\tilde{\mathbf{Q}}$ is finite and positive on $\{G = z\}$. We therefore have on $\{G = z\}$ from the comparison of

$$\int_0^T \mathcal{E}(s^z)_t (\hat{c}_t^z - e_t) dt = e_0 + \int_0^T \mathcal{E}(S^z)_t (\pi_t^* \sigma_t - (\theta_t + \alpha_t^z)^*) dW_t^z \tag{D.56}$$

where $\alpha_t^z = 0$ and correspondingly $dW_t^z = d\omega(t)$ for $t \in [\![T_G, 1]\!]$ $\mathbf{P}^z$-a.s. with

$$\int_0^T \mathcal{E}(S^z)_t (\hat{c}_t^z - e_t) dt = e_0 + \int_0^T \mathcal{D}_t \left\{ \frac{q(z)}{p(\omega, t, z)} \mathbf{E}[\int_t^T \mathcal{E}(S)_t (\hat{c}_t^z - e_t) dt | \mathcal{F}_t] \right\} dW_t^z$$

that

$$\mathcal{D}_t \left\{ \frac{q(z)}{p(\omega, t, z)} \mathbf{E}[\int_0^T \mathcal{E}(S)_t (\hat{c}_t^z - e_t) dt | \mathcal{F}_t] \right\} = \mathcal{E}(S^z)_t (\hat{\pi}_t^z)^* \sigma_t - (\theta_t + \alpha_t^z)^*) \tag{D.57}$$

Since $\mathbf{P}^z(\{G = z\}) = 1$ solving for $\hat{\pi}_t^z$ as in the proof of theorem 3 gives the results on $\mathbf{P}^z \otimes \lambda$-a.e.. As for consumption expressions for optimal strategies of a gambler "evaluated" at $z = G$ are optimal for the consumption-investment problem of an insider since they replicate $\mathcal{G}_t$ adapted cumulative net-consumption without tracking error.

Finally if there are arbitrage opportunities then as we have seen in theorem 2 we can finance any cumulative consumption process with zero cost. This implies that the static budget constraint will never bind and therefore the marginal value of wealth $\hat{y}^z$ is zero. It follows that the first order condition $\partial_2 u(t, \hat{c}_t^z) = 0$. But this is by the Inada conditions only possible if consumption is unbounded with positive $\mathbf{P}^z$- probability. Since $\mathbf{P}^z \not\ll \mathbf{P}$ such states have non-zero $\mathbf{P}^z$- probability and consequently the value of the problem for fixed outcome of the signal $G = z$ explodes. This remains true if we "evaluate" at $z = G$.

Q.E.D.

D.3.2 Proof of proposition 2 and corollaries 3, 4 and 5

Again we are first conditioning on $\{G = z\}$ in a first step and get results by "evaluating" at $z = G$ in a second step. Since the marginal rate of substitution of a gambler can be written for the state dependent Bernoulli indicator $v(\omega, \cdot, \cdot)$ in (5.20) as

$$\frac{\partial_2 u(s, \hat{c}_s^z)}{\partial_2 u(t, \hat{c}_t^z)} = \left(\frac{\left(\frac{d\mathbf{P}}{d\mathbf{P}^z}\right)_t}{\left(\frac{d\mathbf{P}}{d\mathbf{P}^z}\right)_s} \right)(\omega) \frac{\partial_2 v(\omega, s, \hat{c}_s^z)}{\partial_2 v(\omega, s, \hat{c}_s^z)} \tag{D.58}$$

we can on $\{G = z\}$ equivalently write (5.14) as

$$(\hat{\pi}_t^z)^* \sigma_t = \mathbf{E}[\int_t^T \frac{\partial_2 v(\cdot, u, \hat{c}_u^z)}{\partial_2 v(\cdot, t, \hat{c}_t^z)} (\hat{c}_u^z - e_u) du | \mathcal{F}_t] A(t, \hat{c}_t^z) \mathcal{D}_t \hat{c}_t^z +$$
$$\mathbf{E}[\int_t^T \frac{\partial_2 v(u, \hat{c}_u^z, \cdot, z)}{\partial_2 v(t, \hat{c}_t^z, \cdot, z)} [1 - (\hat{c}_u^z - e_u) A(u, \hat{c}_u^z)] \mathcal{D}_t \hat{c}_u du | \mathcal{F}_t] -$$
$$\mathbf{E}[\int_t^T \frac{\partial_2 v(u, \hat{c}_u^z, \cdot, z)}{\partial_2 v(t, \hat{c}_t^z, \cdot, z)} \mathcal{D}_t e_u) du | \mathcal{F}_t] \tag{D.59}$$

But from the expression for "gambler's" optimal consumption policy (D.47) taking instantaneous Malliavin derivatives we get

$$\mathcal{D}_t \hat{c}_t^z = \frac{1}{A(t, \hat{c}_t^z)} (\alpha^z + \theta)^* \tag{D.60}$$

and for $t < u$

$$\mathcal{D}_t \hat{c}_u^z = \frac{1}{A(u, \hat{c}_u^z)} \mathcal{D}_t [\log \frac{q(z) \mathcal{E}(S)_u}{p(\omega, t, z)}] \tag{D.61}$$

Finally it follows from theorem 2.1.1 page 102 of Nualart (1995) that the Malliavin derivative of the endowment rate process $\mathcal{D}_t e_u$ is found as solution of the linearized stochastic differential equation(5.26). If we replace this expression together with (D.60) and (D.61) in (D.59) we get the result announced by arranging terms.

Q.E.D.

D.3.3 Proof of proposition 3

In the proofs of corollaries 1 we have shown that (5.35) implies $\alpha_t^z = 0$ $\mathbf{P}\otimes\lambda$-a.e.. It follows from the expressions of optimal consumption in theorem 5 that in this case $\hat{c}_t^G = \hat{c}_t^z = \tilde{c}_t$ for all $t \in [0,T]$ where $\tilde{c}$ denotes the optimal consumption policy in $\mathcal{A}(\mathbf{P}, \mathbb{F}, e)$. This is sufficient to establish the equivalence of the value functions.

Q.E.D.

D.3.4 Proof of proposition 4

Since admissible strategies in the consumption-investment problem of an insider have to be $\mathcal{G}_t$- adapted we clearly have that

$$\mathcal{F}_t^{\hat{c}^G} \bigvee \mathcal{F}_t^{\hat{\pi}^G} \subset \mathcal{G}_t \tag{D.62}$$

for all $t \in [0,T]$ and it remains to find conditions for which we have the reversed inclusion. From (5.10) we see that $\hat{\pi}_0^G = 0$ and therefore necessary conditions for which the information generated by optimal strategies corresponds to all individual private information must also guarantee that

$$\sigma(\hat{c}_0^G) = \sigma(G) \tag{D.63}$$

But to establish this is equivalent to establish the existence of a Borel function such that

$$G = h(\hat{c}_0^G) \tag{D.64}$$

Since $\hat{c}_0^G$ satisfies the state by state first order conditions

$$\partial_2 u(0, \hat{c}_0^G) = \mathcal{Y}(e^G, G) \tag{D.65}$$

where $\mathcal{Y}(x,z)$ is such that $\mathcal{X}(\mathcal{Y}(x,z),z) = x$ it is by the definition of $\mathcal{X}(y,z)$ sufficient for the existence of the Borel measurable function h to show that the mapping

$$z \mapsto \mathbf{E}[\int_0^T \mathcal{E}(S)_u e_u \frac{q(z)}{p(\omega,u,z)})du] \tag{D.66}$$

is bijective. It follows that for h to exist the mapping

$$z \mapsto \frac{q(z)}{p(\omega,u,z)} \tag{D.67}$$

must be bijective $\mathbf{P} \otimes \lambda$- a.e.. Then suppose such a mapping exists and is given by k. Consequently it must be true that $k^{-1}(z)p(\omega, t, z) = q(z)$ $\mathbf{P} \otimes \lambda$-a.e.. This would imply since $\mathbf{E}[p(\omega, t, z)] = q(z)$ that $k^{-1}(z) = 1$ which proofs that such a bijection does not exist unless the signal is independent of the public information. Consequently the optimal insider strategies will never be fully revealing if it is non-redundant.

Q.E.D.