From Utility Maximization to Arbitrage Pricing, and Back

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ABSTRACT

In an incomplete market there exists a multiplicity of valuation operators. A legitimate question has been raised regarding which valuation operator should be used in such a market. To make this choice without relying on an equilibrium model and maintaining the spirit of arbitrage models in that risk attitudes do not influence the selection criteria, several authors (e.g., Rubinstein, 1994, Stutzer, 1996 and, in a slightly different context, Hansen and Jagannathan, 1997) suggest *ad hoc* criteria. This paper develops a unifying framework for choosing a valuation operator from among the multiplicity. It is shown that under mild regularity conditions every utility function induces a selection criterion and, conversely, every selection criterion is induced by a certain utility function. Utilizing our framework we show that the selection criteria in the above papers have strong links to utility maximization. In fact, the chosen valuation operator is the marginal rate of substitution of an agent maximizing his or her utility of profit emanating from selecting a portfolio subject to a zero budget constraint. This utility maximization problem is a "relaxation" of the problem of maximizing arbitrage profit as utilized in the definition of the no-arbitrage condition. It is the "relaxation" in the sense that the maximization of arbitrage profit now is solved with reference to risk, utilizing a certain utility function. Rather than require that the value of the self-financing portfolio (sought for in maximizing arbitrage profit) be nonnegative in every state of nature, the portfolio is allowed to have negative values in some states of nature while the utility of its payoff is maximized. The induced utility function must exhibit "strong enough" risk aversion to prevent the problem of maximizing utility from being unbounded. Consequently, the selection criteria in the above papers not only have a link to risk preferences, but further assume, implicitly though perhaps unintentionally, a certain degree of risk aversion.

I. Introduction

Over two decades have past since the first published paper appeared in the literature concerning the no-arbitrage condition and its essential role in valuation (Ross, 1976). In a market where the no-arbitrage condition is satisfied there exists a risk-neutral valuation operator. The need for a full-equilibrium model and assumptions regarding the risk attitude of investors in order to price securities in the market has been avoided. In an incomplete market, the valuation operator is not unique[1] and a legitimate question has been raised regarding which valuation operator should be used. Researchers have resolved this choice dilemma without relying on an equilibrium model and maintaining the essence of arbitrage models that do not permit risk attitudes to be a factor in selection criteria. Several authors (e.g., Rubinstein (1994), Stutzer (1996) and, in a slightly different context, Hansen and Jagannathan (1997)) suggest *ad hoc* criteria to select a valuation operator. We show that these selection criteria have a strong link to utility maximization. In fact, the chosen valuation operator is the marginal rate of substitution of an agent maximizing his or her utility of profit emanating from selecting a portfolio subject to a zero budget constraint. This utility maximization problem is a "relaxation" of the problem of maximizing arbitrage profit as utilized in the definition of the no-arbitrage condition. It is the "relaxation" in the sense that the maximization of arbitrage profit is solved now with reference to risk, employing a utility function. Rather than require that the value of the self-financing portfolio (sought for in maximizing arbitrage profit) be nonnegative in every state of nature, the portfolio is allowed to have negative values in some states of nature while the utility of its payoff is maximized.

Furthermore, the paper develops a unifying framework for choosing a valuation operator in the incomplete market setting. It is shown that under mild regularity conditions every utility function induces a selection criterion and conversely every selection criterion is induced by a certain utility function. Meaningful selection criteria, including those in the papers cited above, are not only linked to utility maximization but are induced by utility functions that exhibit "strong enough" risk aversion to prevent the problem from being unbounded.

The study of financial markets can be thought of as having come full circle. The finance literature (as opposed to the economics literature) moved away from consideration of investors' preferences and models of full equilibrium in valuation. Our results indicate that, when there is more than one valuation operator, the selection criteria suggested in the finance literature implicitly but perhaps unintentionally employ utility considerations. Utilizing our framework, the papers mentioned above fit into the category of valuation operator selection and, accordingly, each selection criterion can be interpreted as having a connection to a specific utility function.

[1] The effect of the multiplicity of valuation operators on pricing by replication in the debt market was examined by Dermody and Rockafellar (1991).

The literature has developed considerably since the introduction of risk-neutral valuation. The fundamental result has been put in a more general setting and used in some notable papers such as Black and Scholes (1973) for option pricing, and Harrison and Pliska (1981). Mayhew (1995) considers the estimation of a risk-neutral density function implied by observed option prices, advocating the use of spline estimators. He also provides a convenient summary of recent literature in this area including the work of Rubinstein (1994). One of the recent trends in the literature has been to increase the understanding of pricing via arbitrage arguments in incomplete markets and to consider the choice of valuation operator (Jackwerth and Rubinstein (1996)).

What is required is an "optimization method" with which to infer implied probabilities from option prices. Generally a convex minimization problem is employed to ensure that a local minimum is the global minimum. The minimization criterion chosen, a convex function, is usually justified in some manner. Jackwerth and Rubinstein demonstrate the use of several selection criteria, including minimization of the relative entropy function, also advocated by Stutzer (1996). Numerical tractability and desired smoothness are also cited as rationale for other criteria.

The stream of recent literature that includes the papers of Rubinstein, Jackwerth and Rubinstein, Stutzer, and of Buchen and Kelly (1996) is intimately related to the work of Hansen and Jagannathan (1997) and the research that has sprung from their work. In the Hansen-Jagannathan framework the set of valuation operators (in incomplete markets) is not observable and so a proxy is used. Their work focuses on the measurement of estimation errors in pricing, even though the true set of all valuation operators is not observable. The measure of estimation error is related to the distance between the proxy valuation operator and the true set of valuation operators, and the claim is made that the method of selecting the best proxy is in keeping with the spirit of arbitrage arguments and does not encompass risk attitudes. In the Hansen-Jagannathan setting, the Euclidean norm of the distance between the proxy valuation operator and the true set of valuation operators is minimized. Hence, the criterion for choosing the proxy is a convex function, and so the general framework developed here can provide an alternative interpretation of their work.

Not surprisingly, duality theory plays a central role in all of these developments. It is well known that the Lagrange multipliers of a convex problem are the optimal solutions of the dual problem.[2] Therefore, in its simplest form, the risk-neutral probability is merely a normalization of the vector of optimal solutions to the dual of the problem of maximizing arbitrage profit. Stated differently, the risk-neutral probability is the normalized shadow prices of the arbitrage profit maximization problem.

A different sort of justification for the choice of selection criteria can be made us-

[2] Duality results referred to in this paper can be found in Rockafellar (1970).

ing the results of this paper. Behind the selection criterion is its "shadow" problem. We demonstrate that the shadow (dual) problem is connected to utility maximization, as mentioned before, and restated here as the utility maximiziation problem of an "arbitrageur". The problem of maximizing arbitrage profit, utilized in defining the no-arbitrage condition, must be free of risk. However, if this problem were submitted to an "arbitrageur" who were allowed to solve the problem based on his or her risk attitudes as reflected in a utility function, the problem would be transformed into one wherein utility of profit, over the states of nature, is maximized subject to a zero budget constraint. The arbitrageur's problem is not the usual utility maximization problem because it is not subject to budget constraints and its feasible set contains only self-financing portfolios. This optimization problem is key to the central results of this paper.

The duality results used in this paper would normally require the use of the generalized convex programming scheme developed by Rockafellar. However, armed with the "trick" of introducing artificial variables and constraints, it was possible to generate our results using only what is termed ordinary Lagrangian duality. The duality results in this paper, and essentially the results in papers such as Karatzas, Lehoczky, Shreve and Xu (1991), are really just applications of duality theory that has been known in the literature of mathematical programming for quite a few decades. However, these duality arguments help to crystalize the interpretation and are the driving force behind the different criteria that are usually thought of as completely unconnected to investor risk attitudes.

We show that the dual to the arbitrageur's problem[3] is one in which a valuation operator is chosen based on a selection criterion that is the convex conjugate of the negative of the arbitrageur's utility function. The symmetry between the primal and dual problems[4] makes it possible to establish a one-to-one correspondence between each selection criteria and a particular utility function. We are thus able to classify the conditions under which the utility function in the primal problem results in a selection criterion in the dual problem. We show that, subject to mild regularity conditions, *a concave utility function induces a selection criterion*,[5] provided the no arbitrage condition holds. We also classify the convex criteria that are induced by utility maximization.

Considering specific examples from the literature, we demonstrate that one of the selection criteria advocated by Rubinstein (1994) and justified as a "smoothness" criterion

[3] This dual is found via techniques that are reminiscent of the solution concept for stochastic programming suggested by Ben-Tal (1985). The dual problem is derived with the aid of the convex conjugate of a function, the definition of which appears later in the text.

[4] The symmetry also creates potential confusion because the dual to the selection criterion problem is a utility maximization problem, and the dual to the utility maximization problem is the selection criterion problem, as we will show. Thus, the use of the words "primal" and "dual" become somewhat confusing, and the actual problem to which those labels refer depends on where you start.

[5] Sometimes the induced selection criterion is not meaningful. This point will be discussed later in the paper.

is in fact induced by a quadratic utility function. Similarly, the distance criterion advocated by Hansen and Jagannathan (1997) also can be attributed to a quadratic utility function.

It has been shown that the criterion requiring minimization of the relative entropy function is the dual of the maximization of the utility of uncertain wealth, where the utility function is exponential.[6] Thus the entropy criterion is not preference independent. The link between the entropy selection criterion and the exponential utility function is shown to be a special case of our more general result connecting utility functions and selection criteria. This may shed some doubt on Samuelson's (1990) statement "I have come over the years to have some impatience and boredom with those who try to find an analogue of the entropy ... to put into economic theory", at least in the context investigated in this paper.

The paper is geared for investigation within the environment used by both Rubinstein and Stutzer.[7] In both cases, European option prices and discretization of the possible states of nature (prices of the underlying asset) are used to impute the risk neutral probability. Thus the above ideas are developed in the simplest possible framework of Euclidean spaces as a state preference model. When attention is directed at European options, a one period model is sufficient to capture the environment: only two points in time are of importance – the initiation of the position and the maturity of the option. The discretization means that finite states are sufficient to analyze the situation, and hence a state preference model is used in the subsequent sections of this paper.

The rest of the paper is organized as follows. Section II establishes the theoretical structure that allows us to uncover the central result of this paper. That result is presented in Section III. Using a simplified framework, we show that selection criteria (under mild regularity conditions) are induced by utility functions. This simplified framework, based on the generalized convex programming framework developed by Rockafellar, is tailored to our investigation here. Section IV applies the results of Section III to examples from existing literature: the work of Jackwerth and Rubinstein and of Hansen and Jagannathan, and the special case of the relative entropy function. Concluding remarks are offered in Section V.

[6] It has been accomplished via both Lagrangian duality techniques and using results from geometric programming. See, for example, Ben-Tal (1985). Applications of this result to finance have been noted by Kapur and Kesavan, pages 191–197 (1992) and, in the context of portfolio choice, by Stutzer (1995).

[7] To the best of our knowledge, almost all papers in the literature that deal with the implied valuation operator concentrate on European option prices.

II. The Utility Maximization-Selection Criteria Link

Consider a one-period state preference model of a perfect market. Let $A = [a_{ij}]$ be the $m \times n$ security-payoff matrix for that market in which exactly one state j occurs in the future time period. In other words, security i pays a_{ij} if state j occurs for $i = 1, \dots, m$ and $j = 1, \dots, n$. Let the vector $x = (x_1, \dots, x_m)$ denote a portfolio where $x_i > 0$ is a long position and $x_i < 0$ a short position. Let $p = (p_1, \dots, p_m)$ be the vector of security prices. (Note that all vectors in this paper are column vectors unless they are labelled as row vectors by a prime $'$.)

An investor making a trade x pays $x'p$ now and receives $x'A_{.j} = x_1 a_{1j} + \cdots + x_m a_{mj}$ if state j occurs in the next time period, where $A_{.j}$ is the j^{th} columm of A. We denote by $A_{i.} = (a_{ij}, \dots, a_{in})$ the n-vector of payoffs from the i^{th} security across the j states of nature. For increased simplicity and without loss of generality, the first security in the market is assumed to be a bond and the rate of interest is assumed to be zero. Thus $A_{1.} = (1, \dots, 1)$ and $p_1 = 1$. This guarantees that a vector $d > 0$ satisfying $Ad = p$ can be interpreted as a probability measure over the n states of nature.[8] The expected value of each security, under this measure, is equal to its price. It is well known that the existence of such a measure is equivalent to the absence of arbitrage opportunities.

The problem of maximizing arbitrage profit in this market can be formulated as

$$\sup_x \Big\{ -x'p \quad | \quad x'A \geq \mathbf{0}' \Big\}. \tag{1}$$

The no-arbitrage condition (**NA**) ensures that the optimal value of this problem is zero. As a consequence of duality theory (theorem of alternatives), the existence of a vector $d > 0$ such that $Ad = p$ is equivalent to the no-arbitrage condition, i.e., to

$$\sup_x \Big\{ -x'p \quad | \quad x'A \geq \mathbf{0}' \Big\} = 0.$$

Therefore, the vector d values securities in this market: d_j is the current value of a dollar contingent on state j. Since d can be thought of as a random variable taking the value d_j in the next time period if state j occurs, d is referred to as a *stochastic discount factor*.

The problem of maximizing arbitrage profit, (1), by which the NA condition is defined, is essentially a stochastic programming problem. In fact though, we really require that the constraints of (1) be satisfied only for that state of nature that is realized in the next time period. The realized state of nature, j, in the next time period is uncertain. Therefore this problem can be formulated as the following stochastic programming problem:

$$\sup_x \Big\{ -x'p \quad | \quad x'Ab \geq 0 \Big\}. \tag{2}$$

[8] Without loss of generality, we thus avoid the need for normalization.

where b is the random vector taking the value b_j, $j = 1, \ldots, n$, if state j occurs, i.e., $(0, \ldots, 1, 0, \ldots, 0)$, the j^{th} elementary vector, with 1 at the j^{th} position.

The NA condition, by the virtue of the definition of arbitrage, must rule out any risk associated with the profit obtained, and thus it must enforce that $x'Ab_j \geq 0$ be satisfied for every state of nature j. Thus the NA condition is defined by problem (1). However if an "arbitrageur" were to solve problem (2) and were permitted implicitly to include consideration of his or her attitude towards risk, for example via his or her utility function, problem (2) would be solved in a different way than its formulation in problem (1).

Labelling the investor as an "arbitrageur" in this context is very loose. In fact, we might instead call the investor a "speculator". The pure definition of an arbitrageur describes someone who holds investment positions that impose no risk whatsoever, whereas here we will allow the risk attitude of the investor to be considered. Even so, we will continue to refer to the investor in our setting as an "arbitrageur" to enhance the link between the NA condition and the problem we formulate below. This link uncovers the risk attitude implicit in certain criteria employed for selecting a valuation operator in incomplete markets. Such criteria are presented as *ad hoc* criteria in that, in the spirit of the arbitrage arguments, they bear no connection to risk attitudes. We challenge and confound that claim, and establish the link between such selection criteria and investor risk attitudes – the key result of this paper.

Since we assume that the interest rate is zero, money received in the next time period has the same value as money received in the current time period. The constraints of (1) therefore can be summed together with current cash flows without discounting. Hence, an arbitrageur solving problem (1) will receive profit from portfolio x of $(-x'p + x'A_{.j})$ if state j occurs.

In an uncertain situation, the investor maximizes his or her utility. Such an investor ("arbitrageur"), faced with the problem of maximizing profit under uncertainty, Problem (2), would solve[9]

$$\sup_x \Big\{ U(-x'p + x'A_{.1}, -x'p + x'A_{.2}, \ldots, -x'p + x'A_{.n}) \Big\} \tag{3}$$

where U is the arbitrageur's utility function. The utility function is defined over $\Re^n$ since we would like to be able to value cash flows that are negative in some states of nature.[10]

[9] We could instead write the problem as maximization of expected utility where utility is a scalar function. Here we define the utility function in a more general way – over the states of nature. The mechanism by which we transfer problem (2) into the utility maximization problem (3) is in the spirit of the solution concepts of Ben-Tal (1985) for stochastic programming. The key to Ben-Tal's solution concept is to move the constraints to the objective function and then apply the concepts of classical economics, namely maximization of utility or expected utility, and thus arrive at a deterministic problem. This is referred to as relaxing the stochastic problem to its "deterministic equivalent".

[10] Throughout most of this paper, we will suppress the domain of U, of its convex conjugate U^* and of $(-U)$.

The optimization problem in (3) is not the usual utility maximization problem: there are no budget constraints. We refer to this problem as utility maximization with zero net investment. The arbitrageur with a portfolio position x receives a cash inflow of $-x'p$ in the current time period that can be transferred to the next time period without a future value factor. Thus the total profit from the portfolio in state j is $-x'p + x'A_{.j}$, a value that can be either positive or negative.

Essentially, the arbitrageur finances his or her portfolio by taking a short position, or by borrowing funds in the current time period, to be repaid in the next time period. The budget is zero. We assume that the NA condition is satisfied and hence we assume the absence of a portfolio position x for which the vector $(-x'p, x'A_{.1}, \ldots, x'A_{.n})$ is nonnegative and at least one of its components is positive. This guarantees the existence of a vector of discount factors, $d > 0$, such that $Ad = p$, and this is crucial to our investigation. The satisfaction of the NA condition merely rules out the existence of a direction $\hat{x}$ such that the vector $(-\hat{x}'p, \hat{x}'A_{.1}, \ldots, \hat{x}'A_{.n})$ is nonnegative with at least one positive component, and consequently also rules out the existence of an $\hat{x}$ such that $(-\hat{x}'p+\hat{x}'A_{.1}, \ldots, -\hat{x}'p+\hat{x}'A_{.n})$ is nonnegative with at least one positive component. If such a direction $\hat{x}$ did exist, a monotone utility function obviously would increase to infinity along this direction. Such a direction would violate the NA condition.

It can now be seen readily that the maximization problem in (3) need not be bounded, even if the NA condition is satisfied. If the arbitrageur takes a position in which one component of $-x'p + x'A_{.1}, \ldots, -x'p + x'A_{.n}$ is negative and the rest are positive, and scales this portfolio to infinity, it may be the case that the utility increases to infinity along a direction $\tilde{x}$ satisfying the above condition. This can occur if, for example, in a state with positive profit, that profit is sufficiently large to outweigh the possibility of negative profit in, for example, a highly unlikely state or states of nature. Overall along such a direction the utility maximization increases to infinity. We emphasize this notion of a direction (or the absence of such a direction) along which the function is unbounded since a selection criterion will be seen to induce a concave utility function. Every concave function is unbounded if and only if there exists a direction along which it is unbounded. In our case, the utility function is unbounded if a portfolio $\tilde{x}$ exists such that, if scaled to infinity, its associated utility increases to infinity.

Additional light is shed by considering an alternative but equivalent interpretation of the optimization problem in (3). Beginning with no initial wealth or endowment, what optimal portfolio would be chosen by an investor given his or her utility function? Note that the problem has no budget constraints and considers only self-financing portfolios and so is not the problem faced by a representative investor. If the investor cannot find a self-financing portfolio (that does not include the risk free asset), then the investor is permitted to borrow or lend in unlimited amounts. Note that here again, *a priori* there is no reason necessarily to believe that the solution to this problem is bounded. The barrier to unbounded utility of state contingent potential profit of the chosen portfolio will be the risk aversion of the investor. A sufficiently risk averse investor will not be willing to invest in a portfolio that has even a very unlikely probability of negative profit in any state, because if scaled to infinity, that state would expose the investor to unlimited liability. We will see shortly, following our examination of the dual of this utility maximization problem, that meaningful selection criteria are induced by those utility functions for which this maximization problem is bounded and possesses a solution. This point will become clear in the discussion following Theorem 2.

To facilitate the exposition of the duality framework we introduce artificial variables $v_j, j = 1 \ldots, n$, denoting profit in state j, and rewrite the optimization problem in (3) as the following constrained optimization problem

$$sup_{x,v}\Big\{U(v_1,\ldots,v_n) \quad | \quad -x'p + x'A_{.j} = v_j\Big\}. \tag{4}$$

This will allow us to use the Lagrangian of (4) to generate its dual problem. The Lagrangian of (4), $L(x, v, u^*)$ is given in equation (5) below

$$L(x, v, u^*) = U(v_1,\ldots,v_n) - \sum_{j=1}^{n}(v_j + x'p - x'A_{.j})u_j^* \tag{5}$$

where $u^* = (u_1^*, \ldots, u_n^*)$ is the vector of Lagrangian multipliers the meaning of which will become clear shortly.

Consider the economic interpretation of $L(x, v, u^*)$. Assume that in the market specified by the matrix A and the vector of prices p, it is possible to buy and sell j-state contingent dollars for the price of u_j^* where the cost of these state contingent dollars is measured in terms of utility. The arbitrageur who decides on a position x and a desired profit v_j in state j will need to supplement the cash flow from the portfolio if $v_j \neq -x'p + x'A_{.j}$ and can do so by buying (or selling) dollars contingent on state j. The net cash flow from portfolio x in state j is $-x'p + x'A_{.j}$. Thus if the arbitrageur wishes a profit of v_j in state j that differs from the net cash flow from the portfolio chosen, he or she must supplement the profit from the portfolio x by $v_j - (-x'p + x'A_{.j})$ dollars contingent on state j. The arbitrageur will have to buy $(v_j + x'p - x'A_{.j})$ dollars of profit contingent on state j (or sell, if the net payoff from the portfolio exceeds the desired level of profit) for a price u_j^* per dollar. The total cost, in terms of utility, of this supplementary profit will be $\sum_{j=1}^{n}(v_j + x'p - x'A_{.j})u_j^*$. The consequence of such a strategy is that the arbitrageur receives a total utility of $L(x, v, u^*)$. This is the utility of the profit eminating from the portfolio choice less the utility cost of the supplementary profit purchased.

It is well known that the optimal value of problem (4) is equal[11] to the *sup inf* of its

[11] This is true for every constrained optimization problem and is proven from first principles. Problems of the sup inf type are analyzed in the following way. Given a choice for x and v in the *sup* we try to find the infimum with respect to u^*. If v and x are chosen such that $(-x'p+x'A_{.j})\neq v_j$ the u_j^* will be positive if $(v_j+x'p-x'A_{.j})u_j^*\geq 0$ and negative if $(v_j+x'p-x'A_{.j})u_j^*\leq 0$. Increasing u^* by multiplying it by a positive constant α will result in a value of negative infinity for the problems. Thus v and x in the supremum will be chosen to satisfy the equality constraint and, having done so, will be chosen also to maximize utility.

Lagrangian, i.e., is equal to

$$\sup_{x,v}\inf_{u^*}\Big\{L(x,v,u^*)\Big\}. \tag{6}$$

The primal problem (4) is of utility maximization where utility is a function of the profit from the optimal portfolio held by the arbitrageur. The dual problem (its optimal value) expresses the maximized utility in terms of the prices of the contingent claims. That is, the dual problem is written in terms of the u_j^*s, the (shadow) prices of a dollar contingent on state j, $j = 1, \ldots, n$, where the price is measured in terms of utility. The dual is a minimization problem that searches for the prices u_j^* that minimize utility, where the utility is expressed in terms of u^* rather than in terms of the portfolio holdings. Under certain conditions, satisfied here assuming U is a concave function, the optimal value of the dual equals that of the primal. In these cases, the optimal vector u^* is the set of (shadow) prices for which the arbitrageur will be indifferent between buying a contingent claim at a price u^* or generating the claim by changing the holdings of the portfolio x to produce the required cash flow. Hence, these optimal u^* induce the valuation operator, or the stochastic discount factors as we shall soon see.

The dual problem is obtained by reversing the order of the *sup inf* in (6) i.e., by examining the problem

$$\inf_{u^*}\sup_{x,v}\Big\{U(v_1,\ldots,v_n) - \sum_{j=1}^{n}(v_j + x'p - x'A_{.j})u_j^*\Big\}. \tag{7}$$

The duality theory implies (under certain conditions - see our disscussion later) that solving (7) results in the same value as (6). The optimal value of (6), as per footnote 11, the optimal value of the *inf sup* of equation (5) and the value of the original problem as expressed in (3) and (4) are all equal. Thus the optimal value of (7) still measures the optimal value of the utility function, but in an indirect manner: as a function of the shadow prices u^* rather than as a function of portfolio holdings x as in equation (3). The minimum possible value assigned to the utility by the shadow prices is the maximum utility the investor can obtain from the optimal portfolio x. Hence the solution to problem (7) is directly induced by the choice of U in problem (3).

The economic meaning of (7) is most easily understood considering the *inf* and *sup* operations in two steps. First the utility is maximized via the choice of portfolio holdings and the desired profit, taking the cost, u_j^*, in terms of utility of a dollar of profit contingent on state j as given. This is the meaning of the inner *sup*, which results in the total utility obtained as a function of u^*. Then we search for the price u^* that minimizes the cost of the total utility obtained from the portfolio holdings minus (plus) the utility paid to buy (sell) profit contingent on state j. This is the meaning of the outer *inf* which results in the total cost (in utility units).

We proceed to explain the detailed mechanism for solving (7) and arriving at the dual to the arbitrageur's utility maximization problem. Begin by investigating what the optimal portfolio holdings and profits across states of nature are from the point of view of the arbitrageur if the prices of the j-state contingent dollars are given by u^*. Note, that the arbitrageur now is not constrained to have a profit of v_j such that $v_j = x'p + x'A_{.j}$ for every j. In other words, the profit need not be strictly the result of the portfolio holdings. Additional dollars of state contingent profit can be purchased. The arbitrageur can choose any profit profile, $(v_1, \ldots, v_n)$, across states of nature. If the arbitrageur desires a profit profile that is different from the net cash flow of the portfolio then the available mechanism of adjustment is to buy (sell) the supplementary state contingent dollars $(v_j + x'p - x'A_{.j})$ at the given prices u_j^*. The utility of such a policy is given by

$$\sup_{x,v}\Big\{U(v_1, \ldots, v_n) - \sum_{j=1}^{n}(v_j + x'p - x'A_{.j})u_j^*\Big\}. \tag{8}$$

It is at this point that we introduce the convex conjugate of a function f. It is a convex function defined by

$$f^*(x^*) = \sup_x\Big\{xx^* - f(x)\Big\}.$$

The dual problem derived here, using the simplified mechanism, will be given in terms of f^*, the convex conjugate of f. This mechanism is deduced from Rockafellar's work on generalized convex programming, and is tailored and simplified to meet our needs. The symmetry of the primal-dual scheme is key to the central result of this paper. The ability to uncover the utility functions implied in certain *ad hoc* criteria is a consequence of the fact that, given a mild regularity condition,[12] the relation

$$f^{**}(x) = \sup_{x^*}\Big\{x^*x - f^*(x^*)\Big\} = f(x)$$

holds. In other words, the convex conjugate of the convex conjugate of f is again f.

We can now proceed, given the price u^*, to analyze the optimal level of profit, v, and the optimal portfolio holdings, x, in (8) by noticing that (8) can be rewritten as (9) below.

$$sup_v\Big\{v'(-u)^* - (-U)(v_1, \ldots, v_n)\Big\} - sup_x\Big\{\sum_{j=1}^{n}(x'p - x'A_{.j})u_j^*\Big\} \tag{9}$$

Equation (9) demonstrates that we calculate the supremum first with respect to v and then with respect to x. We can take advantage of the fact that the expression is separable in x and v and of the definition of f^* to write (9) as

$$(-U)^*(-u_1^*, \ldots, -u_n^*) + sup_x\Big\{\sum_{j=1}^{n}(x'p - x'A_{.j})u_j^*\Big\}. \tag{10}$$

[12] The condition is that f be a convex and lower semi-continuous function, or in the parlance of convex analysis, a closed convex function.

A careful examination of $\sum_{j=1}^{n}(x'p - x'A_{.j})u_j^*$ reveals that it is linear in x for a given u^* and hence, except when its slope is zero, i.e., when[13]

$$p \sum_{j=1}^{n} u_j^* = Au^*, \tag{11}$$

the supremum will be (positive) infinity. When state contingent claims can be bought for u_j^*, unless the set of prices u^* satisfies equation (11), the supremum in (8) will be unbounded. We now know the characteristics of the supremum of (8) for given shadow prices u^*, and proceed to consider the infimum of (10) with respect to u^*.

The dual problem is one of minimization of the cost of utility, considering only those u^* that satisfy equation (11). Thus the *inf sup* of the Lagrangian and the dual problem will be given by

$$\inf_{u^*}\Big\{(-U)^*(-u_1^*, \ldots, -u_n^*) \quad | \quad \sum_{j=1}^{n} a_{ij}u_j^* = p_i \sum_{j=1}^{n} u_j^*; \quad i = 1, \ldots, m\Big\}. \tag{12}$$

The value of $(-U)^*$ at the optimal u^* is the maximum utility the arbitrageur can achieve, expressed in terms of the shadow prices u^* rather than in terms of the portfolio holdings x. The optimal u_j^* are shadow prices of elementary contingent claims. The arbitrageur is indifferent between a cash flow that is the direct result of his or her optimal portfolio selection x^* or obtaining the same cash flow by engaging in the purchase (or sale) of state contingent dollar payoffs each at a price u^*.

It is as if the arbitrageur investigates the optimal portfolio and its consequent cash flow $x^{*\prime}A$ and then "strips" it to its components $x^{*\prime}A_{.j}$. The arbitrageur is indifferent between buying the portfolio x^* for $x^{*\prime}p$ or buying each state contingent cash flow $x^{*\prime}A_{.j}$ for a price u_j^*, the total cost being $\sum_{j=1}^{n} u_j^* x^{*\prime}A_{.j}$. It is in this sense that the stochastic discount factors, induced by u_j^*, as we shall show, are implicit in the structure of the market as specified by A and p and in the utility function of the arbitrageur. It is thus apparent that these shadow prices and the induced valuation operators are very closely linked to the utility function of the arbitrageur.

The optimization problem in (12) has a very intimate link to the process by which a selection criteria is employed to choose a stochastic discount factor when multiple valid such factors exist, e.g., in an incomplete market.[14] It is apparent from the structure of

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[13] This result is obtained simply by differentiating the linear expression with respect to x.

[14] Sometimes the structure of the market is such that it is actually incomplete. It may also be the case that it is actually complete but only limited observation of security prices is possible. In this latter instance the market is artificially incomplete, but it is the case that a unique valuation operator exists. As such, the interpretation of the problem is slightly different, but still follows the main thrust of the interpretation presented in this paper. Suppose only limited observation of security prices is possible because of cost restrictions or other barriers to

the optimization problem that a certain convex[15] function, $(-U)^*$, is minimized in order to choose a u^* from the set $\{u^* \mid \sum_{j=1}^n a_{ij}u_j^* = p_i \sum_{j=1}^n u_j^*\}$. The next subsection elaborates on the interpretation of this problem and its link to utility maximization and the selection of stochastic discount factors.

Before we investigate that interpretation, leading to the central ideas of this paper, we digress slightly to present the framework to this point in a slightly different manner. We wish to link the optimization problem being considered to the situation of a marginal investor optimizing his or her utility subject to a budget constraint.

$$\max_{\mathbf{x}}\Big\{U(x'A_{.1}, x'A_{.2}, \ldots, x'A_{.n}) \quad | \quad x'p = W\Big\}$$

where W denotes the investor's total wealth. The investor finds the optimum and we label this $\mathbf{x}^*$.

Suppose now that this investor desires a change, $\mathbf{x}$, from the optimal (equilibrium) portfolio already found.

$$\max_{\mathbf{x}}\Big\{U(x^{*\prime}A_{.1} - x'p + x'A_{.1}, x^{*\prime}A_{.2} - x'p + x'A_{.2}, \ldots, x^{*\prime}A_{.n} - x'p + x'A_{.n} \quad | \quad x'p = W\Big\}$$

Since $\mathbf{x}^*$ was optimal, the budget constraint must already have been exhausted. Since the budget constraint has already been exhausted, the change, $\mathbf{x}$, must be a self-financed portfolio. Furthermore, it should be clear that the optimal $\mathbf{x}$ from this problem will be equal to zero since the $\mathbf{x}^*$ within this second problem was already optimal. The second problem above, though, is very similar to the one that we analyze throughout this paper.

If we now define a new utility function

$$V(x_1, x_2, \ldots) = U(x^{*\prime}A_{.1} - x'p + x'A_{.1}, x^{*\prime}A_{.2} - x'p + x'A_{.2}, \ldots, x^{*\prime}A_{.n} - x'p + x - A_{.n})$$

then the new utility function V is just a shift of the previous function U. It maintains the properties of U and the analysis as presented in this paper remains unaltered using such an interpretation.

completeness. Even though a unique valuation operator actually exists, in this artificially incomplete setting, the researcher (econometrician) must use some criteria to choose a valuation operator. This selection or use of "second best" is linked to risk attitudes. The econometrician in this case employs a criterion that coincides with the marginal rate of substitution of an arbitrageur whose utility function is defined by the convex conjugate of the selection criterion. The remainder of our properties and theorems will apply as in the main body of the paper.

[15] The function U^* is assumed to be concave and thus $-U$ is convex and $(-U)^*$ is convex since the convex conjugate operator always results in a convex function.

III. Selection Criteria are Induced by Utility Functions: Theory

The previous section investigated the problem of maximization of utility of profit, subject to portfolio choices being self-financing, and its dual. We began to investigate the economic intuition of those two problems. This section continues that investigation and allows us to draw conclusions that link utility functions to selection criteria in incomplete market settings. The central results of this paper are contained in the theorems of this section.

Consider again the utility maximization problem that would be solved by the arbitrageur if he or she had been permitted to include his or her attitude toward risk via a utility function. The arbitrageur would solve problem (3), repeated here for convenience and labelled as problem (P).

$$\sup_x \Big\{ U(-x'p + x'A_{.1}, -x'p + x'A_{.2}, \ldots, -x'p + x'A_{.n}) \Big\}. \qquad (P)$$

Note again that (P) may be unbounded. This will depend on the market structure (the matrix A and the vector p) and on the risk preferences of the arbitrageur embodied in the utility function. Recall the problem dual to (P) noted below for convenience, labelled as problem (D).

$$\inf_{u^*} \Big\{ (-U)^*(-u_1^*, \ldots, -u_n^*) \quad \Big| \quad \sum_{j=1}^n a_{ij} u_j^* = p_i \sum_{j=1}^n u_j^*; \quad i = 1, \ldots, m \Big\}. \qquad (D)$$

The constraints of the minimization problem in (D) can be interpreted in the following way. The shadow price, in terms of utility, of a dollar contingent on state j is u_j^*. The unit of u_j^* is thus utility per dollar increase[16] in the cash flow that will occur if state j is realized next period. Security i will pay a_{ij} dollars in state j. Thus the left side of the constraints of (D) represents the value, expressed in terms of utility, of the state contingent payoff eminating from security i. The units of the left hand side are dollars multiplied by utility divided by dollars, and thus the units of the left side are utility. On the right hand side we have the price of security i, p_i, multiplied by $\sum_{j=1}^n u_j^*$. In terms of units, the right hand side is measured in terms of utility also.

What is the meaning of the equality constraints? If the arbitrageur chooses to pocket a dollar rather than investing it, he or she will have a (sure) dollar in the next time period, regardless of which state of nature is realized. The value of a dollar contingent on state j is u_j^*. Having a risk free dollar today is worth $\sum_{j=1}^n u_j^*$ in terms of utility, so the value in terms of utility of p_i is $p_i \sum_{j=1}^n u_j^*$. Whether the arbitrageur purchases security i or simply pockets the funds, in the next time period, the utility associated with either option will be the same when the constraints of (D) are satisfied. Valuing the two possibilities with

[16] In fact a more rigourous statement is in order: it is the infinitesimal increase in utility per infinitesimal increase in the cash flow in state j.

the optimal u^* of problem (D) results in the arbitrageur being indifferent between them. Indeed these notions are familiar, and to be a candidate for the shadow price in the market defined by the matrix of cash flows A and the vector of security prices p for an arbitrageur with a utility function U, u^* must "correctly" price the payoff from each security.

The same ideas can be emphasized by examining the primal problem and recognizing that the constraints of problem (D) are actually the first order conditions for the maximization of the primal problem. The equality constraints are equivalent to the set of equations requiring the gradients of the utility function be equal to zero. When problem (D) has an optimal solution $(u_1^*, \ldots, u_n^*)$ it can be written[17] as (D_e)

$$\inf_d \Big\{ (-U)^*(\alpha^* d_1, \ldots, \alpha^* d_n) \quad | \quad Ad = p \Big\} \tag{D_e}$$

where

$$\alpha^* = \sum_{j=1}^{n} u_j^* \tag{13a}$$

and

$$d_j = \frac{u_j^*}{\sum_{k=1}^{n} u_k^*;} = 1; \quad j = 1, \ldots, n. \tag{13b}$$

The vector d essentially behaves as u^* in our setting. The units in which things are measured deserve some attention. We have u_j^* as the price in terms of utility of a dollar contingent on state j and so $\sum_j u_j^*$ is the price in terms of utility of a current dollar (i.e. a risk free dollar in the current period). The factor d_j as defined by equation (13b) is the price of a dollar contingent on state j in terms of current dollars. That is $d = (d_1, \ldots, d_n)$ is a stochastic discount factor or a valuation operator. A valuation operator must price the primary securities in the market and thus must satisfy $Ad = p$. In an incomplete market, such a d will not be unique.

Under mild regularity conditions (see Corollary 29.1.4 of Rockafellar), if the optimal value of the primal problem is finite then there exists a solution to the dual. The utility function U is defined on all of $\Re^n$ and thus the primal problem is always feasible, i.e., for every x, $U(Cx)$ is finite, where C is the matrix that defines the linear transformation $x: x \rightarrow (-x'p + x'A_{.j}, \ldots, -x'p + x'A_{.n})$. Thus a utility function U will induce a meaningful selection criterion provided the infimum of $(-U)$ is finite and attained.

Problem (D_e) models the situation wherein a stochastic discount factor is chosen from the feasible set of discount factors, namely from $\Big\{ d \quad | \quad Ad = p \Big\}$. The element d_j is interpreted as the partial derivative of U, with respect to the j^{th} argument, divided by

[17] The constraints of (D_e) are obtained from (D) divided by $\sum_{j=1}^{n} u_j$ and the objective function as it appears in (12) in terms of d is as defined in (D_e).

the sum of the partial derivatives at the optimal point. It is well known[18] that the d_j here are the marginal rates of substitution between consumption today and in the future. Problem (D_e) is dual to the problem of maximizing utility[19] of uncertain wealth. It is the minimization of the negative of the convex conjugate of the negative of a utility function. In an incomplete financial market there exists a multiplicity of valid valuation operators, vectors d that satisfy $Ad = p$. The one that is chosen via a particular selection criterion will be the one that coincides with the marginal rate of substitution of a particular arbitrageur's utility function (at the optimal point, x^*, of the primal problem).

We have demonstrated the connection between the problem of maximizing utility of uncertain wealth and the problem of choosing a valuation operator in an incomplete market. The function that is minimized in the selection criterion is the convex conjugate of the negative of the utility function of the investor making the selection. This is the central result of this paper as stated in Theorem 1. The chosen d corresponds to the marginal rate of substitution of the arbitrageur at the optimal point of the utility maximization problem. Furthermore, u^* is the shadow price, in terms of utility, for the problem of maximizing profit subject to a zero budget constraint for the given utility function U. Hence d, by virtue of being a function of u^* as defined by (13), is induced by the utility function U.

Theorem 1: *The problems* $\sup_x U(-x'p + x'A_{.j})$ *and* $\inf_d\{(-U)^*(\alpha d) \quad | \quad Ad = p\}$, *for* α *as defined in* (13), *are dual to one another.*

Proof:

The derivation presented in the text proves that (D) is dual to (P). The converse is demonstrated via usual Lagrangian duality in the discussion leading to (D') preceeding Theorem 2.

QED

Theorem 1 facilitates uncovering the utility function implicit in a certain selection criterion. Thus, given an optimization problem for selecting a valuation operator and

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[18] See, for example, the exposition in Duffie (1992).

[19] If the utility function U is defined to be monotone increasing and differentiable, then $(-U)^*$ is defined to be ∞ outside the negative orthant. The familiar arbitrage restriction on the optimization problem that $d>0$ is, by a change of variable, implicit. This point can be seen as follows. By Theorem 24.3 of Rockafellar, the connection between the (sub)gradients of U and the domain of its conjugate is known. Let $h=-U$. The domain of h^* must be non-positive vectors since h is monotone decreasing. Recall the definition of the convex conjugate: $h^*=\sup_x\{xx^*-h(x)\}$. Assume that x^* is not less than or equal to zero. If the vector x^* has even one positive component, let the corresponding component of x go to infinity and set the other components of the vector equal to zero. Thus the term xx^* will go to infinity, and $h(x)$ will decrease since it is monotone decreasing. Increase one component of x and the rest stays constant. It follows that the domain of h^* is a subset of the negative orthant. However, in making the change from $(-U)^*$ as a function of $-u_j^*$ to being a function of d, the restriction that $d>0$ is obtained. Thus, if the utility function is assumed to be monotone increasing, the restriction that $d>0$ is also implicit in Theorem 1 below.

applying to the symmetry of the convex conjugacy operator, the utility function that induces that selection criterion can be recovered. Note that the criterion actually specifies (via duality arguments) the utility function, and not merely its gradient at the optimal solution. For a lower semi-continuous convex function, the conjugate of the conjugate is the original function, i.e.,

$$f(x) = f^{**}(x) = \sup_{x^*}\Big\{x^*x - f^*(x^*)\Big\}.$$

Hence, based on Theorem 1, given a problem of the type

$$\inf_d\Big\{g(d) \quad | \quad Ad = p\Big\}$$

the utility function that induces this selection criteria is given by

$$U_{g(x)}(x) = -\sup_{x^*}\Big\{x'x^* - g(x^*)\Big\} = \inf_{x^*}\Big\{g(x^*) - xx^*\Big\}.$$

It is not always the case that the induced utility function possesses the required properties, e.g., non-satiation and monotonicity. In view of the symmetry between the primal and dual problems, it is possible to identify the properties of the selection criterion that are the mirror image of the properties of the utility function. Hence, we can determine if the induced utility function possesses the required properties without actually calculating it. This may help in deciding which selection criterion to employ. Conversely, given a utility function, U, one can discover a selection criterion that is induced by that utility function. The criterion f is given by

$$f(d) = (-U)^*(d) = \sup_x\Big\{x'd - (-U)(x)\Big\}$$

The dual problem generated above, (D), was obtained by reversing the order of the *sup* and *inf* of the Lagrangian in (6). Working backwards and calculating the dual of the optimization problem (D) by reversing the *sup* and *inf* of its Lagrangian results in the optimization problem (P). Following these steps and writing the Lagrangian of (D) as

$$L(x, d) = (-U)^*(d) + x(Ad - p\sum_{j=1}^{n} d_j), \tag{14}$$

it is possible to proceed in the following manner. The $\inf_d \sup_x$ of $L(x, d)$ equals the optimal value of problem (D) for reasons explained in footnote 9. Reversing the *inf* and *sup* yields

$$\sup_x \inf_d\Big\{(-U)^*(d) + x(Ad - p\sum_{j=1}^{n} d_j)\Big\}$$

which is equal to

$$\sup_x -\Big\{\sup_d \Big\{d_1(\sum a_{i1}x_i - \sum x_i p_i) + \ldots + d_n(\sum a_{in}x_i - \sum x_i p_i) - (-U)^*(d)\Big\}\Big\}$$

that can be also written as

$$\sup_x \Big\{-(-U)(-x'p + x'A_{.1}, \ldots, -x'p + x'A_{.n})\Big\} \qquad (D')$$

which is problem (P).

The symmetric derivation above, shows that the dual of the dual is the primal problem. The dual to the problem of minimizing a function of the valuation operator subject to the constraints $Ad = p$ (a valuation operator selection criterion) is the problem of maximization of utility of uncertain wealth. This allows us to state properties of the selection criteria in terms of properties of the utility function, and *vice versa*.

Theorem 2: *The utility function U does not exhibit non-satiation if and only if $(-U)^*$ has a subgradient at zero. This condition is satisfied if zero is in the relative interior of the domain of $(-U)^*$.*

Proof:

The proof follows from statements (a) and (b) of Theorem 27.1 of Rockafellar. If zero is in the domain of $(-U)^*$ then $(-U)$ is bounded below. The infimum of $(-U)$ is attained if $(-U)^*$ has a subgradient at zero.

Note that the arguments of U are defined at the point $-x'p + x'A_{.j}$, $j = 1, \ldots, n$. Thus, when we refer to the domain of U it is the image of x under the linear transformation $C : x \to -x'p + x'A_{.j}$ as defined above.

QED

Theorem 2 aids with determination of those instances in which a given utility function will imply a meaningless selection criterion. Under what conditions is the problem of maximizing utility not bounded? We considered this situation in section II.1. Satisfaction of the NA condition rules out the existence of a portfolio $\hat{x}$ that results in all elements of the vector $(-\hat{x}'p + \hat{x}'A_{.1}, \ldots, -\hat{x}'p + \hat{x}'A_{.n})$ being nonnegative with at least one element positive. Such a portfolio can be scaled so that the utility function is increasing along this direction. The utility maximization problem may still be unbounded if a portfolio $\tilde{x}$ such that $(-\tilde{x}'p + \tilde{x}'A_{.1}, \ldots, -\tilde{x}'p + \tilde{x}'A_{.n})$ contains some positive and some negative elements presents the investor with the possibility of gain in some states of nature and loss in others. The resolution of the boundedness will depend on the risk aversion of the investor, as discussed earlier. If the problem is unbounded, the dual problem is not feasible and the induced selection criterion will fail to identify a d in the set $\{d \mid Ad = p\}$, It therefore follows that the advocated selection criterion must corespond to a utility function exhibiting "strong enough" risk aversion.

The economic interpretation of our central result also could be understood in the following way. The existence of a valuation operator d such that $Ad = p$ is ensured by the no arbitrage condition. In a complete market, a unique valuation exists and thus the NA points to the valuation operator. In an incomplete market we resort to a selection criterion that will help us to select one of the possible ds satisfying $Ad = p$. In essence we do that by looking at the NA condition and searching for a utility function for which the marginal rate of substitution will be one of the valuation operators satisfying $Ad = p$. It is as if we are now employing a second criterion in order to narrow down the set of valuation operators that satisfy $Ad = p$, itself obtained by assuming the NA condition is satisfied. We do so by re-examining the NA condition defined by the optimization problem (1) and determining how this problem would be solved by a utility maximization agent if he or she could input his or her risk preferences. The result is the optimization problem (P) that identifies a set of marginal rates of substitution (at the optimal solution) that is a member of the set of all feasible NA valuation operators. In this sense, each utility function induces a particular d that is member of the set of ds satisfying $Ad = p$. The selection criteria thought of as *ad hoc* are simply modeling the above process in terms of u^* rather than in terms of solving the optimization problem defining the NA when risk attitudes are allowed to enter the process via a utility function.

A crucial element for generating the symmetry between the two problems is "the knowledge" that the dual should be analyzed recognizing that its constraints are $Ad = p(\sum_j d_j)$, and not omitting the $\sum_j d_j$. In fact, given our assumptions regarding the rate of interest and the nature of the first security in the matrix A, the term $\sum_j d_j$ is a 1. Without our assumptions, though, the results uncovered here still hold.

If the $\sum_j d_j$ term were omitted from the right hand side of the contraints, the transition from the dual back to the primal problem would not necessarily restore a primal problem in the same general form as (P). Beginning with

$$L(x, d) = (-U)^*(d) - x(Ad - p)$$

and calculating the *sup inf* yields

$$\sup_x \inf_d \Big\{ (-U)^*(d) + x(Ad - p) \Big\}$$

which is equal to

$$\sup_x - \sup_d \Big\{ x(p - Ad) - (-U)^*(d) \Big\} = \sup_x - \Big\{ x'p - (-U)(-x'A) \Big\}$$

that can also be written as

$$\sup_x \Big\{ -x'p - (-U)(-x'A) \Big\} \qquad (P_a)$$

which is not of the same general form as problem (P). It is the symmetry of the framework that establishes the linkage between utility functions and selection criteria and facilitates the analysis of examples from the finance literature in the nest section.

There is a case, though, where the omission of the $\sum_j d_j$ term from the Lagrangian of the dual problem will generate a version of the primal problem that has a recognizable interpretation. The relation between the relative entropy criterion and the exponential utility function can be seen using this method of moving from the dual to the primal problem, even with the omitted term. Indeed, this perhaps explains why the relation between utility functions and selection criteria is familiar **only for** this particular case (Kesavan and Kapur, 1996 and Stutzer, 1996) and the reason the literature is not aware that it represents a specific example of a generalized property as stated in Theorem 1. The key to the convenience for recognizing the relationship lies in the following. The optimal solution of a maximization problem with $f(x)$ as its objective function will obtain whether $f(x)$ or its natural logarithm is maximized. If f is is the exponential utility function, since ln of f is the inverse of f, the above steps yield the cited result. We shall return and take a closer look at the case of the relative entropy-exponential utility relation in Section IV and show more fully that the stated connection can be uncovered.

IV. Selection Criteria are Induced by Utility Functions: Examples

This section applies the methodology developed in the former section to several existing studies in the finance literature. It uncovers the utility function implicitly assumed in each of these studies, though the selection criteria employed there are justified either as *ad hoc* criteria with no reference to risk attitudes embodied in a particular utility function, or via other means. We first focus our attention on the smoothing criteria suggested by Rubinstein (1994) and Jackwerth and Rubinstein (1996).

Implied Binomial Trees and the Smoothing Criteria

Rubinstein (1994) develops a theoretical model for implying the binomial trees in which the risk neutral distribution of an underlying asset's price is imputed from the prices of associated options. An empirical application of that model to S&P 500 index options is done by Jackwerth and Rubinstein (1996). (We refer to this work of Jackwerth and Rubinstein hereafter simply as JR.) In both papers, the environment is the familiar binomial tree.

It is easy to see that in the environment of these papers, the states of nature are associated with the final values of the underlying asset, which, for the sake of consistency, we continue to denote as in the former section by j, $j = 1, \ldots, n$. Given a set of European options that mature at the same time (the last period of the binomial tree) we can associate the possible realizations of the underlying asset at that time with the states of nature. To be consistent in our presentation, we will continue to assume that the rate of interest is equal to zero. Thus given this general environment, JR try to impute values of d_j such that if S_j is the value of the underlying asset in state j at the end of the period, the price of, for example, a European call option on S, O_p, should satisfy

$$O_{p_k} = \sum_{j=1}^{n} d_j \max\Big\{S_j - K_k, 0\Big\}, k = 1, \ldots, l$$

where K_k is the exercise price. The system of equations $Ad = p$ in our presetation is thus equivalent to the system of equations in the JR world. Having established this system of equations for the options written on S maturing at the same time, we proceed to solve for the implied d_j and discover that there are many solutions.

Jackwerth and Rubinstein examine the question of smoothness of the posterior distributions. They suggest choosing the implied distribution that maximizes smoothness, i.e., minimization of the following function:[20]

$$\sum_{j=1}^{n} (d_{j-1} - 2d_j + d_{j+1})^2 \tag{15}$$

where $d_{-1} = d_{n+1} = 0$. Rewriting $(d_{j-1} - 2d_j + d_{j+1})$ as $(d_{j+1} - d_j) - (d_j - d_{j-1})$ and interpreting, as they do, $(d_{j+1} - d_j)$ as the "derivative" of d with respect to j, $(d_{j-1} - 2d_j + d_{j+1})$ becomes the approximation to the second derivative. Thus the expression in (15) is a proxy for the absolute magnitude of the second derivative which they try to minimize in order to obtain a "smooth" stochastic discount factor d_j.

Consider the JR selection criterion for the case of three states of nature. It can be verified that when $n = 3$ the objective function in (15) can be written in the quadratic form $d'Md$ where M is a symmetric positive definite (and thus non-singular) matrix given by:

$$\begin{pmatrix} 12 & -12 & 2 \\ -12 & 18 & -4 \\ 2 & -4 & 2 \end{pmatrix}$$

The JR criterion for selecting a valuation operator can be written as

$$\inf_{d}\Big\{f(d) \quad | \quad Ad = p\Big\} \tag{16}$$

where

$$f(d) = \frac{1}{2}d'Md,$$

and where we multiplied $d'Md$ by $\frac{1}{2}$ to simplify the calculation since it does not alter the optimal solution.

[20] Jackwerth and Rubinstein, page 13, (1996). Their notation uses P that we have replaced here by d to correspond to the notation used throughout the rest of this paper.

We calculate the convex conjugate of f,

$$f^*(d^*) = \sup_d \Big\{ d'd^* - f(d) \Big\}, \tag{17}$$

and equate the dervative of $d'd^* - f(d)$, with respect to d, to zero, obtaining

$$d^* - Md = 0.$$

Since M is invertible we can solve for $d = M^{-1}d^* = 0$ and substitute it back into (17) to arrive at the quadratic utility function that induces JR's selection criterion. Namely, the quadratic utility function is given by $-\frac{1}{2}(d^{*\prime}M^{-1}d^*)$ where M^{-1} is given by

$$\begin{pmatrix} \frac{5}{18} & \frac{2}{9} & \frac{1}{6} \\ \frac{2}{9} & \frac{5}{18} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{3} & 1 \end{pmatrix}.$$

Using our framework, we thus uncover an alternative interpretation for the JR smoothness criterion: this criterion is induced by a quadratic utility function. We have demonstrated this for an example with three states of nature.

Both Rubinstein, and Jackwerth and Rubinstein estimate ending risk-neutral probability distributions. In our notation, but using the JR terminology, d_j is an implied (posterior) ending nodal risk-neutral probability and $\hat{d}$ is a corresponding prespecified (prior) ending nodal lognormal risk-neutral probability. What is required is an "optimization method" with which to infer implied probabilities from option prices. The 1996 paper examines several possible objective functions including minimization of the sum of squared differences between the d_j and $\hat{d}_j$, i.e.,

$$\min \Big\{ (d - \hat{d})'(d - \hat{d}) \quad | \quad Ad = p, d \geq 0 \Big\},$$

that is induced by a quadratic utility maximization problem. We shall return to an investigation of this sort of problem when we discuss the Hansen-Jagannathan framework. Jackwerth and Rubinstein also examine the minimization of the relative entropy function which is the topic of our next subsection.

The Entropy Criterion is Induced by Exponential Utility

The link between the entropy criterion and the exponential utility function has been established elsewhere.[21] In this subsection we take a closer look at the entropy criterion in light of the methodology of this paper. This criterion has been employed in existing

[21] See the reference mentioned in the context of this discussion at end of Section II.

finance literature such as Jackwerth and Rubinstein (1996) and Stutzer (1996). Stutzer presents an example of empirically employing this methodology that fits the framework used in this paper. A histogram of possible realizations of the price of the underlying asset at time T, a few periods ahead is constructed, and the prices of options written on the asset and maturing at time T are recorded. Interpreting $a_{i,j}$ as the payoff from option i in state j, the possible prices of the underlying asset at time T as the states of nature, $j = 1, \ldots, n$, and the vector p such that p_i is the price of the ith option, the system $Ad = P$ describes the set of possible valuation operators (stochastic discount factors). Recalling that we assume that the first asset is the risk free rate of interest (a bond) and its price is \$1, then a d satisfying $Ad = p$ will also be a risk neutral probability measure, provided $d_j > 0$. This last inequality is guaranteed by the chosen objective function (that we try to minimize and that we define formally as $+\infty$ if $d_i \leq 0$ for at least one i).

The (relative) entropy criterion is defined as

$$I(d, \hat{d}) = \sum_{j=1}^{n} d_j \log \frac{d_j}{\hat{d}_j},$$

where $\hat{d}$ is a posterior probability. It seeks to choose a valuation operator (probability distribution), d, by minimizing the "distance" between the chosen d and a given probability distribution $\hat{d}$, perhaps a subjective measure. The given probability distribution $\hat{d}$ can also be the lognormal distribution as in Jackwerth and Rubinstein. The Kullback-Leibler relative entropy measure is also known as a measure of directed divergence. It shares many properties of a norm and thus has the interpretation of a distance function. The term "directed divergence" is used rather than "distance" since it does not possess the property that $I(a, b)$ necessarily equals $I(b, a)$. The relative entropy function has a value of zero only if the two distributions are identical. This can be thought of as the "distance" between a given probability measure $\hat{d}$ (the subjective probability measure of the investor) and the one for which we are searching, d.

The optimization problem by which a valuation operator is chosen based on the entropy selection criterion can be stated as

$$\inf_{d} \Big\{ \sum_{j=1}^{n} d_j \log \frac{d_j}{\hat{d}_j} \quad | \quad Ad = p \Big\} \tag{18}$$

The entropy criterion, based on Theorem 2, is induced by a utility function that is both concave and monotone increasing and that exhibits non-satiation. Application of Theorem 1 derives the dual of the minimization problem in (18). We begin by calculating the negative convex conjugate of the relative entropy function. The function is separable in d_i and therefore, by Theorem 16.4 of Rockafellar, the convex conjugate of the relative entropy function will be the sum of the convex conjugates of each function making up the sum.

We calculate the convex conjugate of $d_j \log(d_j/\hat{d}_j)$:

$$\sup_{d_j}\Big\{d_j x_j - d_j \log \frac{d_j}{\hat{d}_j}\Big\} \tag{19}$$

Letting $y_j = \frac{d_j}{\hat{d}_j}$, (19) becomes

$$\begin{aligned} &= \sup_{y_j}\Big\{d_j x_j - y_j \hat{d}_j \log(y_j)\Big\} \\ &= \hat{d}_j \Big\{\sup_{y_j}\Big\{y_j x_j - y_j \log(y_j)\Big\}\Big\} \end{aligned} \tag{20}$$

Calculating the supremum (differentiating with respect to y_j and equating to zero) we obtain the convex conjugate of $d_j \log(d_j/\hat{d}_j)$:

$$\hat{d}_j e^{x_j - 1}.$$

Therefore, at the point $(-x'p + x'A_{.1}, \ldots, -x'p + x'A_{.n})$, the convex conjugate of the sum of these functions is

$$\sum_{j=1}^{n} \hat{d}_j e^{-x'p + x'A_{.j} - 1}$$

which can be written as

$$\frac{1}{e}\sum_{j=1}^{n} \hat{d}_j e^{-x'p + x'A_{.j}}. \tag{21}$$

By Theorem 1, the negative convex conjugate of the function in the valuation operator selection criterion is the function to be maximized in the dual problem. The convex conjugate of the relative entropy function is given in (21). Thus, the selection criterion that minimizes the relative entropy function is dual to the problem of

$$\sup_{x} E_{\hat{d}}\Big\{-e^{-(-x'p + x'Ab)}\Big\}. \tag{22}$$

where $E_{\hat{d}}$ is the expected value of the random variable b, as defined following equation (2), with respect to the probability measure $\hat{d}$. The utility function being maximized is the exponential, which does indeed satisfy the guidelines of Theorem 2.

We can rearrange (22) as follows

$$\begin{aligned} &\sup_{x} E_{\hat{d}}\{-e^{-(-x'p + x'Ab)}\} \\ &= \sup_{x} -\{E_{\hat{d}} e^{x'p - x'Ab}\} \\ &= -\inf_{x}\{E_{\hat{d}} e^{x'p - x'Ab}\}. \end{aligned} \tag{23}$$

Since $x'p$ is non-stochastic, (23) can be further modified

$$-\inf_x e^{x'p} E_{\hat{d}} e^{-x'Ab}. \tag{24}$$

Taking a monotonic transformation of (24), we have

$$\begin{aligned} &-\inf_x \log^{x'p} + \log E_{\hat{d}} e^{-x'Ab} \\ &\qquad = -\Big\{ -\sup_x -x'p - \log E_{\hat{d}} e^{-x'Ab} \Big\}. \end{aligned} \tag{25}$$

Finally, simplifying (25) we have

$$\sup_x -x'p - \log E_{\hat{d}} e^{-x'Ab}. \tag{26}$$

Thus the chosen portfolio x that maximizes (22), dual to the entropy criterion, will also maximize (26). The optimization problem in (26) is in the same general form as the alternative primal (P_a) at the end of section III, i.e., in the form of $-x'p - H(-x'A)$, for some function H. It results, though, from transforming the objective function via the natural logarithm, and then recognizing that the natural logarithm and the exponential functions are inverse to one another.

The valuation operator chosen by minimizing the relative entropy function (choosing the one closest to a given density) is, via[22] Theorem 1, equivalent to selecting a portfolio that maximizes expected utility, (22) or alternatively (26). The valuation operator so chosen is the marginal rate of substitution induced by the exponential utility function at the optimal solution. The expression in (26) demonstrates how the risk aversion of the utility function bounds the expected arbitrage profit. The term $-\log E_{\hat{d}} e^{-x'Ab}$ acts like a penalty term to bound the profit $-x'p$. Such a term, reflecting the risk aversion implicit in the assumed utility function, will be evident again in our analysis of the Hansen-Jagannathan approach.

Direct application of Theorem 1 demonstrates the link between the relative entropy criterion and the exponential utility function. This selection criterion is familiar from the Finance literature and is one special case of the general result of Theorem 1. The preceding interpretation may provide some answers, at least in this context, to the questions Samuelson pondered and with which he lost patience.

The Hansen-Jagannathan Framework

We suggest an alternative interpretation of the work of Hansen and Jagannathan in light of the results of Sections II and III. We connect the notions of risk aversion and penalty terms as we did for the case of the entropy.

[22] This result has been shown elsewhere, as previously mentioned, using standard duality techniques. The presentation here is a direct application of Theorem 1 of this paper.

Recent work by Hansen and Jagannathan also considers valuation when markets are incomplete and a multiplicity of valuation operators exists. In the Hansen-Jagannathan framework there is a set of true valuation operators, **M**, that has more than one element. They seek to obtain a measure of mispricing by comparing a valuation operator from their set **M** to a proxy valuation operator. Thus they compare a true d to a proxy $\hat{d}$. They select the proxy valuation operator by minimizing the mispricing, that is by minimizing the norm (the distance) between the set of true valuation operators, $\{d \quad | \quad Ad = p\} = \mathbf{M}$, and a proxy valuation operator, $\hat{d}$ and solve:

$$\min_{d>0}\Big\{\|\hat{d} - d\| \quad | \quad Ad = p\Big\}. \tag{27}$$

The similarity between the optimization problem in (27) and the one suggested by JR is now more apparent. In the JR case $\hat{d}$ is interpreted as a posterior probability based on the lognormal distribution. Problem (27) is mathematically equivalent to that suggested by JR and in order to investigate it using our framework we alter it. As before, we rewrite the problem as

$$\min_{d>0}\Big\{\|\hat{d} - d\| \quad | \quad Ad = p\sum_{j=1}^{n} d_j\Big\} \tag{28}$$

Assume now that the norm used above can be defined as $\frac{1}{2}(\hat{d}-d)'Q(\hat{d}-d)$ where Q is a positive definite symmetric matrix. We wish to apply Theorem 1 to the Hansen-Jagannathan framework in order to demonstrate another example of using our methodology.

We begin to calculate the problem that is dual to (28), knowing from Theorem 1 that it will be a utility maximization problem, and knowing something about the inherent risk characteristics. With the above notation the criterion for selecting a proxy valuation operator $\hat{d}$ can be written as:

$$\inf_{d}\Big\{f(d) \quad | \quad Ad = p\Big\}, \tag{29}$$

where

$$f(d) = \frac{1}{2}(\hat{d} - d)'Q(\hat{d} - d).$$

In order to calculate the convex conjugate of f we determine the supremum in (30)

$$f^*(x^*) = \sup_{d}\Big\{d'x^* - f(d)\Big\} \tag{30}$$

by taking the derivative of f with respect to x and equate it to zero, obtaining

$$x^* - Q(\hat{d} - d) = 0.$$

Consider the case where the matrix Q is invertible.[21] We solve for the value of d at the maximum and substitute $d = Q^{-1}x^* + \hat{d}$ into $f^*(d^*)$, arriving at

$$f^*(x^*) = \frac{1}{2}x^{*\prime}Q^{-1}x^* + \hat{d}'x^*.$$

By Theorem 1, the dual to the selection criterion in problem (29) is the maximization of the negative of the convex conjugate of the function f. Here, f is a convex and lower semi-continuous function, and so its $f^{**} = f$. The selection criterion in the Hansen-Jagannathan framework is, therefore, the dual to the following problem of maximizing this concave function

$$\sup_x -f^*(-x'p + x'A_{.1}, \ldots, -x'p + x'A_{.n}). \tag{31}$$

The utility function $-f^*$ is quadratic. Hansen-Jagannathan choose to work with the Euclidian norm, hence the Q in our general disscussion above is actually the identity matrix. Therefore, the dual problem (31), for which the optimal value is equal to that of (31) can be written as

$$\sup_x \Big\{ -\sum_{j=1}^{n}(-x'p + x'A_{.j})^2 + \sum_{j=1}^{n}(-x'p + x'A_{.j}\hat{d}_j) \Big\}. \tag{32}$$

Thus, using our methodology, one interpretation of the measure of mispricing in the Hansen-Jagannathan framework is that it is induced by a quadratic utility function. It is generally agreed that, in spite of its computational convenience, the quadratic utility function possesses unfortunate properties that do not correspond with economic intuition. These undesirable properties include both increasing absolute and relative risk aversion.

The quadratic utility function does exhibit risk aversion. As already mention in the entropy case, the link between a risk averse utility function and the inclusion of a penalty term should be noted. Risk aversion in the utility function and a penalty term are each mechanisms for bounding potential arbitrage profit as becomes apparent below. The optimization problem in (32) can be restated as

$$\sup_x \sum_{j=1}^{n} \hat{d}_j(-x'p + A'\hat{d}) - \sum_{j=1}^{n}(-x'p + x'A_{.j})^2. \tag{33}$$

[21] If the matrix Q is not invertible, the problem is not economically meaningful. However, we know that there exists a unique symmetric positive semi-definite matrix Q' (which can be calculated from Q) such that $Q'Q = QQ' = P$ where P is the matrix of the linear transformation which projects $\Re$ orthogonally onto the orthogonal complement L of the subspace $\{x|Qx=0\}$. For this Q' it is the case that:

$$f^*(d^*) = \begin{cases} \frac{1}{2}d^{*\prime}Q'd^*, & \text{if } d^* \in L \\ \infty, & \text{otherwise.} \end{cases}$$

The reader is referred to Rockafellar (1970), page 108, for the details in the quadratic case.

Denoting $A\hat{d}$ by $\hat{p}$, i.e. the set of prices that are induced by the proxy operator, $\hat{d}$, and assuming that $\sum_{j=1}^{n} \hat{d}_j = 1$, problem (33) can be written as

$$\sup_x -x'(p - \hat{p}) - \sum_{j=1}^{n} (-x'p + x'A_{.j})^2. \tag{34}$$

The first term in equation (34) is the difference between the portfolio priced using a true valuation operator, d, and when it is priced using the proxy valuation operator, $\hat{d}$. Were it possible to trade in two markets, i.e., at both sets of prices, then this difference would represent arbitrage profit. The Hansen-Jagannathan measure of mispricing is thus explained in the following way. Potential arbitrage profit is unlimited. One mechanism used to bound arbitrage profit is the inclusion of a penalty term in the objective function. The last term of (34) is a penalty term, limiting the potential arbitrage profit away from infinity, since if the chosen portfolio position x is scaled to infinity this term increases and in turn decreases the value of the objective function. This penalty term is a reflection of the risk aversion in the quadratic utility function. If the risk aversion of the utility function is not sufficient to bound the problem, then the induced selection criterion will not be meaningful. Furthermore, the expression of the penalty term is induced by the type of utility function.

We know from our investigation of the entropy criterion that it, too, demonstrates the connection between risk aversion in the utility function and the inclusion of a penalty term. We also know that the relative entropy measure has a distance interpretation. It, too, like the Hansen-Jagannathan measure of mispricing, can be employed in an analogous manner. The measure of mispricing using relative entropy will be

$$\inf d \geq 0 \Big\{ I(d, \hat{d}) \quad | \quad Ad = p \Big\}. \tag{35}$$

If $\hat{d}$ is in the set such that $\mathbf{M}= \{d \quad | \quad Ad = p\}$ then (35) will have an optimal value of zero. Otherwise, the measurement of error based on the "distance" between $\hat{d}$ and the closest element of the set $\mathbf{M}$ will be based on the relative entropy function.

The Hansen-Jagannathan measure of mispricing, although developed with a different goal in mind than choosing a valuation operato,r can be interpreted in this guise and is shown here to be implicitly linked to a utility function. Indeed, this section has demonstrated that three *ad hoc* selection criteria are not only related to risk attitudes, but, in order to be effective, must actually be induced by the utility function of a risk averse "arbitrageur".

V. Conclusions

In a complete market the valuation operator is unique; its existence is a consequence of the absence of arbitrage opportunities. It is characterized (in the framework of this paper) by $Ad = p, d > 0$, which is a direct implication of the maximization problem (1) being bounded. Recall that this problem is bounded if and only if its optimal solution is zero, which means no arbitrage opportunities are available. In an incomplete market the characterization $Ad = p, d > 0$ is not sufficient to identify a valuation operator. Consequently, a selection criterion is employed to choose one d. These criteria are usually expressed as a convex optimization problems in which a concave function of d is minimized over the convex set $\{d|Ad = p, d > 0\}$. Using duality results, it is shown that the dual of this problem is an unconstrained maximization problem involving a concave objective function wherein the arguments are the profit from the portfolio x in state j, i.e. $(-x'p + x'A_{.j})$. This gives the first clue of the intimate relation between utility maximization and *ad hoc* criteria for choosing valuation operators in incomplete markets.

Furthermore, in light of Theorem 2, investigating the optimization problem in which a stochastic discount factor is chosen reveals that these *ad hoc* criteria that are minimized in the dual problem possess certain properties that imply their duals (the primal) have an objective function with properties of a utility function. It is in this guise that we interpret the relationship in the following way. Since the absence of arbitrage opportunities modeled via the problem of maximizing arbitrage profit results in more than one correct valuation operator, we examine whether a particular relaxation of the problem will aid in selecting one. Relaxing the arbitrage problem in the sense that risk attitudes are allowed to contribute to the selection of the optimal portfolio, via a utility function, we examine the optimal portfolio choice given a particular utility function. The fact that the dual problem of maximizing a concave function is feasible means that the risk attitudes implicit in the solution of this problem are those of a risk averse "arbitrageur". If the selection process identifies a stochastic discount factor then, from duality results, the problem of maximizing the utility of profit is bounded. In other words, the risk aversion implicit in the revealed preferences of the arbitrageur was strong enough to bound this problem, as explained in the text. Furthermore, the chosen stochastic discount factor coincides with the marginal rate of substitution of the arbitrageur at the optimal solution. We conclude then that the "*ad hoc*" selection criteria involving minimization of a convex function (if there is a solution) correspond to the maximization of utility functions subject to no constraints (i.e. self-financing portfolios) that have optimal solutions although *a priori* they can be unbounded. We conclude then that these criteria are not *ad hoc* and that each is induced by a particular utility function. This paper reveals the precise connection between the utility function and its corresponding selection criterion and uses it to analyze a few examples from the existing literature.

It would appear that finance theory has come full circle. Pricing via the no-arbitrage condition allowed for valuation without reference to preference ordering or utility considerations. Given the results of this paper, utility functions induce the selection criteria necessary to choose among the elements of the set of valid valuation operators in incomplete markets. Furthermore, the utility functions correspond to those of “very” risk averse arbitrageurs since otherwise it would not be possible to identify a d from within the set $\mathbf{M}$ as explained above. Thus in incomplete markets we are not so free from utility considerations after all.

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